Lectures Références

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Numerical Methods for PDE: Finite Differences and Finites Volumes

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1. Finite Difference(FD) and Finite volume(FV) : Overview
2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions

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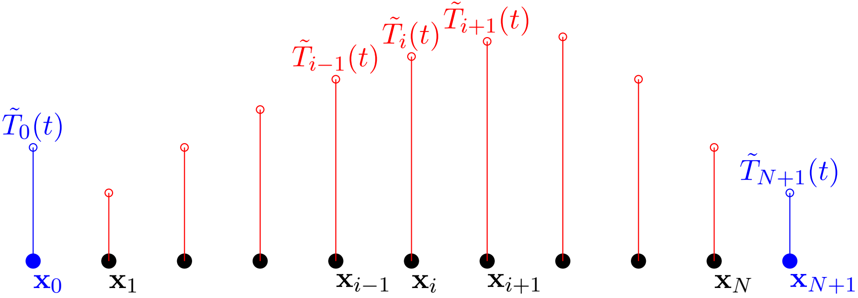
1D Scalar advection-diffusion Equation (periodic)

$$\begin{cases} \dfrac{\partial T}{\partial t} + c\dfrac{\partial T}{\partial x} = \lambda\dfrac{\partial^2 T}{\partial \mathbf{x}^2}, & \forall \mathbf{x} \in (0,1), \quad t > 0, \\ T(t,\mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in (0,1), \quad t = 0, \\ T(t,\mathbf{x}=0) \equiv T(t,\mathbf{x}=1) & \forall t \geq 0 \end{cases}$$

Properties

- well poseness : bb.
- Existence : The solution $T(\mathbf{x})$ always exists and is unique.
- Regularity : The solution $T(\mathbf{x})$ is regular and uniformly bounded
- Positivity :
- Maximum principle : .

1D mesh for the discretization (approximated solution)



$$\mathbf{x}_1 = 0, \quad \mathbf{x}_{N+1} = 1, \quad \mathbf{x}_i = (i-1)\delta x, \quad \delta x = \frac{1}{N}$$

$\tilde{T}_i = \tilde{T}(\mathbf{x}_i)$ for $1 \le i \le N$, $\quad \tilde{T}_0 \equiv \tilde{T}_N, \quad \tilde{T}_{N+1} \equiv \tilde{T}_1$

Finite difference strategy : advection operator

Taylor's expansions :

$$T(t,\mathbf{x}+\beta\delta\mathbf{x}) = T(t,\mathbf{x})+\beta\delta\mathbf{x}\frac{\partial T}{\partial \mathbf{x}}(t,\mathbf{x})+\beta^2\frac{\delta\mathbf{x}^2}{2}\frac{\partial^2 T}{\partial \mathbf{x}^2}(t,\mathbf{x})+...+R_m(t,\mathbf{x})$$

Discrete approximations of $\frac{\partial T}{\partial x}$:

- Central Finite Difference scheme : $\vartheta = \{-1, 1\}$.

$$\frac{\partial T}{\partial x} \simeq \frac{T(t,\mathbf{x}+\delta\mathbf{x}) - T(t,\mathbf{x}-\delta\mathbf{x})}{2\delta\mathbf{x}}$$

- Upwind Finite Difference scheme : $\vartheta \subset \{-1, 0, 1\}$.

$$\frac{\partial T}{\partial x} \simeq \frac{T(t,\mathbf{x}+\delta\mathbf{x}) - T(t,\mathbf{x}-\delta\mathbf{x})}{2\delta\mathbf{x}} - \frac{|c|}{c}\frac{T(t,\mathbf{x}+\delta\mathbf{x}) - 2T(t,\mathbf{x}) + T(t,\mathbf{x}-\delta\mathbf{x})}{2\delta\mathbf{x}}$$

Finite difference strategy : some Difference operators

First order derivatives :

$$\delta_\xi^\beta \tilde{T}(\xi) = \frac{\tilde{T}(\xi + \beta\delta\xi) - \tilde{T}(\xi)}{\beta\delta\xi}, \quad \bar{\delta}_\xi^\beta \tilde{T}(\xi) = \frac{\delta_\xi^\beta + \delta_\xi^{-\beta}}{2} \tilde{T}(\xi)$$

- Forward finite difference :
  $\delta_{\mathbf{x}}^+ \tilde{T}_j = \frac{\tilde{T}_{j+1} - \tilde{T}_j}{\delta\mathbf{x}} = \frac{\partial T}{\partial \mathbf{x}}\Big|_j + R_1^+(\delta\mathbf{x})$
- Backward finite difference :
  $\delta_{\mathbf{x}}^- \tilde{T}_j = \frac{\tilde{T}_j - \tilde{T}_{j-1}}{\delta\mathbf{x}} = \frac{\partial T}{\partial \mathbf{x}}\Big|_j + R_1^-(\delta\mathbf{x})$
- Centered finite difference :
  $\delta_{\mathbf{x}} \tilde{T}_j = \frac{\tilde{T}_{j+1} - \tilde{T}_{j-1}}{2\delta\mathbf{x}} = \left(\frac{\delta_{\mathbf{x}}^+ + \delta_{\mathbf{x}}^-}{2}\right) \tilde{T}_j = \frac{\partial T}{\partial \mathbf{x}}\Big|_j + R_1^0(\delta\mathbf{x}^2)$

Second order derivatives :

- $\delta_{\mathbf{xx}} \tilde{T}_j = \frac{\tilde{T}_{j+1} - 2\tilde{T}_j + \tilde{T}_{j-1}}{\delta\mathbf{x}^2} = \frac{\partial^2 T}{\partial \mathbf{x}^2}\Big|_j + R_2^0(\delta\mathbf{x}^2)$

One step Finite difference scheme for advection/diffusion

$$\frac{\tilde{T}_j^{n+1} - \tilde{T}_j^n}{\delta t} + c\left(\theta_c \delta_{\mathbf{x}} \tilde{T}_j^{n+1} + (1-\theta_c)\delta_{\mathbf{x}} \tilde{T}_j^n\right)$$
$$-\zeta \delta \mathbf{x}\left(\theta_c \delta_{\mathbf{xx}} \tilde{T}_j^{n+1} + (1-\theta_c)\delta_{\mathbf{xx}} \tilde{T}_j^n\right)$$
$$= \lambda\left(\theta_d \delta_{\mathbf{xx}} \tilde{T}_j^{n+1} + (1-\theta_d)\delta_{\mathbf{xx}} \tilde{T}_j^n\right)$$

ζ is the numerical diffusion coeficient.

- $\zeta = 0$ for the centered scheme
- $\zeta = \frac{|c|}{2}$ for the upwind scheme.

The parameters $\theta \equiv \theta_c$ or θ_d can takes the values :

- $\theta = 0$ for explicit approximation of space derivatives.
- $\theta = 1$ for implicit approximation of space derivatives.
- $\theta = \frac{1}{2}$ for Crank-Nicholson approximation of space derivatives.

Truncation Error

$$
\begin{aligned}
\mathcal{E}_j^{n+1} = & \frac{T_j^{n+1} - T_j^n}{\delta t} + c\left(\theta_c \delta_{\mathbf{x}} T_j^{n+1} + (1-\theta_c)\delta_{\mathbf{x}} T_j^n\right) \\
& -\zeta \delta \mathbf{x} \left(\theta_c \delta_{\mathbf{xx}} T_j^{n+1} + (1-\theta_c)\delta_{\mathbf{xx}} T_j^n\right) \\
& -\lambda \left(\theta_d \delta_{\mathbf{xx}} T_j^{n+1} + (1-\theta_d)\delta_{\mathbf{xx}} T_j^n\right) \\
= & (\partial_t T)_j^n + \frac{\delta t}{2}\left(\partial_t^2 T\right)_j^n + O(\delta t^2 \partial_t^3) \\
& + c\theta_c \left(\partial_{\mathbf{x}} T\right)_j^{n+1} + c(1-\theta_c)\left(\partial_{\mathbf{x}} T\right)_j^n + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^3) \\
& -\zeta \delta \mathbf{x} \left[\theta_c \left(\partial_{\mathbf{x}}^2 T\right)_j^{n+1} + (1-\theta_c)\left(\partial_{\mathbf{x}}^2 T\right)_j^n\right] + \zeta \delta \mathbf{x} O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4) \\
& -\lambda \left[\theta_d \left(\partial_{\mathbf{x}}^2 T\right)_j^{n+1} + (1-\theta_d)\left(\partial_{\mathbf{x}}^2 T\right)_j^n\right] + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4)
\end{aligned}
$$

Truncation Error

$$
\begin{aligned}
\mathcal{E}_j^{n+1} = \; & (\partial_t T)_j^n + \frac{\delta t}{2}\left(\partial_t^2 T\right)_j^n + O(\delta t^2 \partial_t^3) \\
& + c\,(\partial_{\mathbf{x}} T)_j^n + c\theta_c \delta t\,(\partial_t \partial_{\mathbf{x}} T)_j^n + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^3, \delta t^2 \partial_t^2 \partial_{\mathbf{x}}) \\
& + \zeta \delta \mathbf{x}\left[-\partial_{\mathbf{x}}^2 T + \theta_c \delta t \partial_t \partial_{\mathbf{x}}^2 T\right]_j^n + \zeta \delta \mathbf{x} O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4 + \delta t^2 \partial_t^2 \partial_{\mathbf{x}}^2) \\
& - \lambda\left(\partial_{\mathbf{x}}^2 T\right)_j^n + \theta_d \lambda \delta t\left(\partial_t \partial_{\mathbf{x}}^2 T\right)_j^n + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4 + \delta t^2 \partial_t^2 \partial_{\mathbf{x}}^2) \\
= \; & \left(\partial_t T + c\partial_{\mathbf{x}} T - \lambda \partial_{\mathbf{x}}^2 T\right)_j^n \\
& + \left[\delta t \partial_t\left(\frac{1}{2}\partial_t T + \theta_c c \partial_{\mathbf{x}} T - \theta_d \lambda \partial_{\mathbf{x}}^2 T\right) - \zeta \delta \mathbf{x}\left(\partial_{\mathbf{x}}^2 T\right)\right]_j^n \\
& + \left[\partial_t^3 + \partial_t^2 \partial_{\mathbf{x}} + \partial_t^2 \partial_{\mathbf{x}}^2 + \zeta \delta \mathbf{x} \partial_t^2 \partial_{\mathbf{x}}^2\right] O\left(\delta t^2\right) \\
& + \left[\partial_{\mathbf{x}}^3 + \partial_{\mathbf{x}}^4 + \zeta \delta \mathbf{x} \partial_{\mathbf{x}}^4\right] O\left(\delta \mathbf{x}^2\right)
\end{aligned}
$$

$$
\mathcal{E}_j^{n+1} = \delta t\left[\partial_t\left(\frac{1}{2}\partial_t T + \theta_c c \partial_{\mathbf{x}} T - \theta_d \lambda \partial_{\mathbf{x}}^2 T\right)\right]_j^n - \zeta \delta \mathbf{x}\left(\partial_{\mathbf{x}}^2 T\right)_j^n + O(\delta h^2 \partial^3 +
$$

blue terms are principal component of the truncation error.

Truncation Error : Conclusions for regular solutions.

$$\mathcal{E}_j^{n+1} = \delta t \left[\partial_t \left(\frac{1}{2} \partial_t T + \theta_c c \partial_{\mathbf{x}} T - \theta_d \lambda \partial_{\mathbf{x}}^2 T \right) \right]_j^n - \zeta \delta \mathbf{x} \left(\partial_{\mathbf{x}}^2 T \right)_j^n + O(\delta h^2 \partial^3 +$$

- These schemes are always consistent : $\lim\limits_{\substack{\delta t \longrightarrow 0 \\ \delta \mathbf{x} \longrightarrow 0}} \|\boldsymbol{\mathcal{E}}^{n+1}\| = 0.$
- Upwind schemes (for advection : $\zeta \neq 0$) are always first order accurate in space.
- Crank-Nicholson schemes for both advection and diffusion ($\theta_c = \theta_d = \frac{1}{2}$) are second order accurate in time.
- Centered(for advection)/Crank-Nicholson scheme is second order accurate both in time and space.

Modified equation

We assume now that we can define a smooth approximated solution $V(x,t)$ such as $V(x_j,t^n) = \tilde{T}_j^n$ for all j and n.
Therefore,

$$\frac{V(x_j,t^{n+1}) - V(x_j,t^n)}{\delta t}$$

$$+c\theta_c \frac{V(x_{j+1},t^{n+1}) - V(x_{j-1},t^{n+1})}{2\delta x} + c(1-\theta_c)\frac{V(x_{j+1},t^n) - V(x_{j-1},t^n)}{2\delta x}$$

$$-\left[\zeta\delta\mathbf{x}\theta_c + \lambda\theta_d\right] \frac{V(x_{j+1},t^{n+1}) - 2V(x_j,t^{n+1}) + V(x_{j-1},t^{n+1})}{\delta x^2}$$

$$-\left[\zeta\delta\mathbf{x}(1-\theta_c) + \lambda(1-\theta_d)\right] \frac{V(x_{j+1},t^n) - 2V(x_j,t^n) + V(x_{j-1},t^n)}{\delta x^2} = 0$$

We can applied the Taylor expansions for $V(x,t)$ to obtained the modified equation statisfied by the approximate solution.

Modified equation

$$
\begin{aligned}
&(\partial_t V)_j^n + \frac{\delta t}{2}\left(\partial_t^2 V\right)_j^n + \frac{\delta t^2}{6}\left(\partial_t^3 V\right)_j^n + O(\delta t^3 \partial_t^4) \\
&+c\theta_c \left[(\partial_{\mathbf{x}} V)_j^{n+1} + \frac{\delta x^2}{6}\left(\partial_x^3 V\right)_j^{n+1}\right] + O(\delta \mathbf{x}^4 \partial_{\mathbf{x}}^5) \\
&+c(1-\theta_c)\left[(\partial_{\mathbf{x}} V)_j^n + \frac{\delta x^2}{6}\left(\partial_x^3 V\right)_j^n\right] + O(\delta \mathbf{x}^4 \partial_{\mathbf{x}}^5) \\
&-\left[\zeta \delta \mathbf{x} \theta_c + \lambda \theta_d\right]\left[\left(\partial_{\mathbf{x}}^2 V\right)_j^{n+1} + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4)\right] \\
&-\left[\zeta \delta \mathbf{x}(1-\theta_c) + \lambda(1-\theta_d)\right]\left[\left(\partial_{\mathbf{x}}^2 V\right)_j^n + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4)\right] = 0
\end{aligned}
$$

Modified equation

$$
\begin{aligned}
&\left[\partial_t V + \frac{\delta t}{2}\partial_t^2 V + \frac{\delta t^2}{6}\partial_t^3 V + O(\delta t^3 \partial_t^4)\right]_j^n \\
&+c\theta_c\left[\delta t \partial_{\mathbf{x}}\partial_t V + \frac{\delta t^2}{2}\partial_{\mathbf{x}}\partial_t^2 V + O(\delta h^3 \partial^4)\right]_j^n \\
&+c\left[\partial_{\mathbf{x}} V + \frac{\delta x^2}{6}\partial_x^3 V + O(\delta h^4 \partial^5)\right]_j^n \\
&-\left[\zeta \delta \mathbf{x} \theta_c + \lambda \theta_d\right]\left[\delta t \partial_{\mathbf{x}}^2 \partial_t V + O(\delta h^2 \partial^4)\right]_j^n \\
&-\left[\zeta \delta \mathbf{x} + \lambda\right]\left[\partial_{\mathbf{x}}^2 V + O(\delta \mathbf{x}^2 \partial_{\mathbf{x}}^4)\right]_j^n = 0
\end{aligned}
$$

$$
\partial_t V + c\partial_{\mathbf{x}} V - \lambda \partial_{\mathbf{x}}^2 V =
$$

$$
-\frac{\delta t}{2}\partial_t^2 V - c\theta_c \delta t \partial_{\mathbf{x}}\partial_t V + \zeta \delta \mathbf{x} \partial_{\mathbf{x}}^2 V + \lambda \theta_d \delta t \partial_{\mathbf{x}}^2 \partial_t V + O(\delta h^2)
$$

$$
\partial_t^2 V = c^2 \partial_{\mathbf{x}}^2 V - 2c\lambda \partial_{\mathbf{x}}^3 V + \lambda^2 \partial_{\mathbf{x}}^4 V + O(\delta h)
$$

$$
\partial_{\mathbf{x}}\partial_t V = -c\partial_{\mathbf{x}}^2 V + \lambda \partial_{\mathbf{x}}^3 V + O(\delta h)
$$

modified equation

$$\partial_t V + c\partial_{\mathbf{x}} V - \lambda \partial_{\mathbf{x}}^2 V =$$
$$-\frac{\delta t}{2}\partial_t^2 V - c\theta_c \delta t \partial_{\mathbf{x}}\partial_t V + \zeta \delta \mathbf{x} \partial_{\mathbf{x}}^2 V + \lambda\theta_d \delta t \partial_{\mathbf{x}}^2 \partial_t V + O(\delta h^2)$$
$$\partial_t^2 V = c^2 \partial_{\mathbf{x}}^2 V - 2c\lambda \partial_{\mathbf{x}}^3 V + \lambda^2 \partial_{\mathbf{x}}^4 V + O(\delta h)$$
$$\partial_{\mathbf{x}}\partial_t V = -c\partial_{\mathbf{x}}^2 V + \lambda \partial_{\mathbf{x}}^3 V + O(\delta h)$$

The modified equation is then writen as

$$\partial_t V + c\partial_{\mathbf{x}} V - \lambda \partial_{\mathbf{x}}^2 V = \mathcal{T}(\partial_{\mathbf{x}}) V$$

$$\mathcal{T}(\partial_{\mathbf{x}}) \equiv \beta_2 \partial_{\mathbf{x}}^2 + \beta_3 \partial_{\mathbf{x}}^3 + \beta_4 \partial_{\mathbf{x}}^4 - \cdots \quad \text{where}$$

- $\beta_{2m}\partial_{\mathbf{x}}^{2m} V$ are associated to ampliude error $\equiv$ dissipation.
- $\beta_{2m+1}\partial_{\mathbf{x}}^{2m+1} V$ are associated to phase error $\equiv$ dispersion..

Modified Equation

$$\partial_t V + c\partial_{\mathbf{x}} V - \lambda \partial^2_{\mathbf{x}} V = \sum_{k\geq 2} \beta_k \partial^k_{\mathbf{x}} V$$

Solving this equation, by using Fourrier transform in space :

$$\widehat{V}(\theta,t) = \exp(\mu(\theta)t)\widehat{V}(\theta,0) = e^{\mu_a t} e^{\imath \mu_\phi t}\widehat{V}(\theta,0)$$

$$\text{where} \quad \mu(\theta) = -\imath c\theta - \lambda\theta^2 + \sum_{k\geq 2} \beta_k (\imath\theta)^k$$

$$\mu_a = Real\left[\mu(\theta)\right] = -\lambda\theta^2 + \sum_{m\geq 1} \beta_{2m}(-1)^m \theta^{2m}$$

$$\mu_\phi = Imag\left[\mu(\theta)\right] = -c\theta + \sum_{m\geq 1} \beta_{2m+1}(-1)^m \theta^{2m+1}$$

- $-\lambda\theta^2 - \mu_a$ is the ampliude error $\equiv$ dissipation.
- $-c\theta - \mu_\phi$ is the phase error $\equiv$ dispersion..

Equivalent equation (regular solution) : Explicit scheme

$$\mathcal{T}(\partial_{\mathbf{x}}) \equiv -\frac{c^2\delta t}{2}\partial_{\mathbf{x}}^2 + \zeta\delta\mathbf{x}\partial_{\mathbf{x}}^2 + O(\delta h^2 \partial^3 \cdots)$$

Appropriate choise of ζ leads to the elimination of the second order dissipation vanish.

The Lax-Wendroff scheme :: $\quad \zeta = \dfrac{c^2\delta t}{2\delta\mathbf{x}}$

This scheme is second order in space and time.

VN Stability (L_2) : Discrete Fourier mode & transformations.

$$\hat{\boldsymbol{T}}(t,\theta) = \beta(t,\theta)\hat{\boldsymbol{T}}(\theta) \quad \text{with} \quad \hat{T}_j(\theta) = \exp\left(\imath j\theta\right) \quad \text{for } 1 \le j \le N$$

$$\frac{\hat{T}_j^{n+1} - \hat{T}_j^n}{\delta t} = \frac{\beta^{n+1} - \beta^n}{\delta t}\hat{T}_j(\theta),$$

$$\delta_{\mathbf{x}}\hat{T}_j(t) = \frac{\hat{T}_{j+1}(t) - \hat{T}_{j-1}(t)}{2\delta\mathbf{x}} = \beta(t)\frac{e^{\imath\theta} - e^{-\imath\theta}}{2\delta\mathbf{x}}\hat{T}_j = \imath\beta(t)\frac{\sin\theta}{\delta\mathbf{x}}\hat{T}_j(\theta)$$

$$\delta_{\mathbf{xx}}\hat{T}_j(t) = -\beta(t)\frac{4\sin^2\frac{\theta}{2}}{\delta\mathbf{x}^2}\hat{T}_j(\theta)$$

Von Neumann Stability Analaysis : Discrete Fourier mode

$$\frac{\tilde{T}_j^{n+1} - \tilde{T}_j^{n}}{\delta t} + c\left(\theta_c \delta_{\mathbf{x}} \tilde{T}_j^{n+1} + (1-\theta_c)\delta_{\mathbf{x}} \tilde{T}_j^{n}\right)$$
$$-\zeta \delta \mathbf{x}\left(\theta_c \delta_{\mathbf{xx}} \tilde{T}_j^{n+1} + (1-\theta_c)\delta_{\mathbf{xx}} \tilde{T}_j^{n}\right)$$
$$= \lambda\left(\theta_d \delta_{\mathbf{xx}} \tilde{T}_j^{n+1} + (1-\theta_d)\delta_{\mathbf{xx}} \tilde{T}_j^{n}\right)$$

$$\Longrightarrow \frac{\beta^{n+1} - \beta^{n}}{\delta t} + \left[\theta_c \beta^{n+1} + (1-\theta_c)\beta^{n}\right]\frac{\imath c \sin\theta}{\delta \mathbf{x}}$$
$$+ \left[\theta_c \beta^{n+1} + (1-\theta_c)\beta^{n}\right]\frac{4\zeta \sin^2\frac{\theta}{2}}{\delta \mathbf{x}}$$
$$= -\left[\theta_d \beta^{n+1} + (1-\theta_d)\beta^{n}\right]\frac{4\lambda \sin^2\frac{\theta}{2}}{\delta \mathbf{x}^2}$$

$$g(\theta) = \frac{1 - (1-\theta_c)\left(\frac{\imath c\delta t \sin\theta}{\delta \mathbf{x}} + \frac{4\zeta\delta t \sin^2\frac{\theta}{2}}{\delta \mathbf{x}}\right) - (1-\theta_d)\frac{4\lambda\delta t \sin^2\frac{\theta}{2}}{\delta \mathbf{x}^2}}{1 + \theta_c\left(\frac{\imath c\delta t \sin\theta}{\delta \mathbf{x}} + \frac{4\zeta\delta t \sin^2\frac{\theta}{2}}{\delta \mathbf{x}}\right) + \theta_d\frac{4\delta t\lambda \sin^2\frac{\theta}{2}}{\delta \mathbf{x}^2}}$$

Von Neumann Stability Analaysis

$$g(\theta) \equiv g(X) = \frac{1 - (1-\theta_c)\frac{4\sigma\zeta}{c}X - 4\nu(1-\theta_d)X - \imath\sigma(1-\theta_c)\sin\theta}{1 + \theta_c\frac{4\sigma\zeta}{c}X + 4\nu\theta_d X + \imath\sigma\theta_c\sin\theta}$$

$$\text{where} \quad \sigma = \frac{c\delta t}{\delta \mathbf{x}}, \quad 0 \le X = \sin^2\frac{\theta}{2} \le 1, \quad \nu = \frac{\delta t \lambda}{\delta \mathbf{x}^2}$$

Using the fact that $\sin^2\theta = 4(X - X^2)$ and setting $\alpha_i = 2\sigma(1-\theta_c)$, $\overline{\alpha}_i = 2\sigma\theta_c$, $\alpha_r = (1-\theta_c)\frac{4\sigma\zeta}{c} + 4\nu(1-\theta_d)$ and $\overline{\alpha}_r = \theta_c\frac{4\sigma\zeta}{c} + 4\nu\theta_d$, we obtain

$$|g(X)|^2 = \frac{(1-\alpha_r X)^2 + \alpha_i^2(X - X^2)}{(1+\overline{\alpha}_r X)^2 + \overline{\alpha}_i^2(X - X^2)} = \frac{1 + X\left[(\alpha_i^2 - 2\alpha_r) + (\alpha_r^2 - \alpha_i^2)X\right]}{1 + X\left[(\overline{\alpha}_i^2 + 2\overline{\alpha}_r) + (\overline{\alpha}_r^2 - \overline{\alpha}_i^2)X\right]}$$

Von Neumann Stability Analaysis

$$|g(\theta)|^2 \equiv |g(X)|^2 = \frac{1 + X\left[(\alpha_i^2 - 2\alpha_r) + (\alpha_r^2 - \alpha_i^2)X\right]}{1 + X\left[(\overline{\alpha}_i^2 + 2\overline{\alpha}_r) + (\overline{\alpha}_r^2 - \overline{\alpha}_i^2)X\right]}$$

Von Neumann Stability condition : $|g(\theta)| \leq 1, \ \forall\theta$

Here it is equivalent, for $0 \leq X \leq 1$, to

$$(\alpha_i^2 - 2\alpha_r) + (\alpha_r^2 - \alpha_i^2)X \leq (\overline{\alpha}_i^2 + 2\overline{\alpha}_r) + (\overline{\alpha}_r^2 - \overline{\alpha}_i^2)X$$

$$\Leftrightarrow \begin{cases} \alpha_i^2 - 2\alpha_r & \leq \overline{\alpha}_i^2 + 2\overline{\alpha}_r \quad :: X = 0 \\ \alpha_r^2 - 2\alpha_r & \leq \overline{\alpha}_r^2 + 2\overline{\alpha}_r \quad :: X = 1 \end{cases}$$

VN Stability : Explicit centered (advection) scheme

VN condition : $\alpha_i^2 - 2\alpha_r \leq \overline{\alpha}_i^2 + 2\overline{\alpha}_r$ and $\alpha_r^2 - 2\alpha_r \leq \overline{\alpha}_r^2 + 2\overline{\alpha}_r$

Explicit centered scheme : $\alpha_i = 2\dfrac{\delta t c}{\delta \mathbf{x}}, \quad \alpha_r = 4\dfrac{\delta t \lambda}{\delta \mathbf{x}^2}, \quad \overline{\alpha}_i = \overline{\alpha}_r = 0.$

Then the VN Stability condition is satisfyed when δt and $\delta \mathbf{x}$ satisfies the following conditions :

$$\nu = \frac{\delta t \lambda}{\delta \mathbf{x}^2} \leq \frac{1}{2} \quad \text{and} \quad Pe = \frac{c^2 \delta t}{\lambda} \leq 2$$

- advection dominated asymptotic when $\lambda \to 0$: the VN condition require a time step $\delta t \to 0$: unstable ! ! !
- diffusion dominated asymptotic when $c \to 0$: the VN condition require a time step $\frac{\delta t \lambda}{\delta \mathbf{x}^2} \leq \frac{1}{2}$. see previous lectures.

L_∞ Stability

L_∞ Stability condition

Let us consider a numerical scheme given by

$$\tilde{T}_j^{n+1} = \sum_k \alpha_k \tilde{T}_{j(k)}^n$$

This numerical scheme is L_∞ when (sufficient condition) :

$$\sum_k \alpha_k = 1 \quad \text{and} \quad \alpha_k \geq 0, \ \forall k$$

Indeed, These conditions insure that

$$\|\tilde{\boldsymbol{T}}^{n+1}\|_\infty \leq \|\tilde{\boldsymbol{T}}^n\|_\infty$$

L_∞ Stability of Explicit schemes ($\|\tilde{T}\|_\infty = \max_j |\tilde{T}_j|$)

$$\tilde{T}_j^{n+1} - \tilde{T}_j^n + \sigma\left(\tilde{T}_{j+1}^n - \tilde{T}_{j-1}^n\right) = (\nu + \xi)\left(\tilde{T}_{j+1}^n - 2\tilde{T}_j^n + \tilde{T}_{j-1}^n\right)$$

where $\sigma = \frac{c\delta t}{\delta \mathbf{x}}$, $\xi = \frac{\zeta \delta t}{\delta \mathbf{x}}$ and $\nu = \frac{\lambda \delta t}{\delta \mathbf{x}^2}$

$$\tilde{T}_j^{n+1} = \Big[\nu + \xi - \sigma\Big]\,\tilde{T}_{j+1}^n + \Big[1 - 2(\nu + \xi)\Big]\,\tilde{T}_j^n + \Big[\nu + \xi + \sigma\Big]\,\tilde{T}_{j-1}^n$$

The scheme is L_∞ stable if $\nu + \xi \pm \sigma \geq 0$ and $\nu + \xi \leq \frac{1}{2}$.

$$\nu + \xi \leq \frac{1}{2} \quad \Longleftrightarrow \quad \frac{c\delta t}{\delta \mathbf{x}} + \frac{\lambda \delta t}{\delta \mathbf{x}^2} \leq \frac{1}{2}$$

Indeed (in this case)

$$|\tilde{T}_j^{n+1}| \leq \Big[\nu + \xi - \sigma\Big]\,|\tilde{T}_{j+1}^n| + \Big[1 - 2(\nu + \xi)\Big]\,|\tilde{T}_j^n| + \Big[\nu + \xi + \sigma\Big]\,|\tilde{T}_{j-1}^n|$$

$$\|\tilde{T}^{n+1}\|_\infty = \|\mathcal{G}(\tilde{T}^n)\|_\infty \leq \|\tilde{T}^n\|_\infty$$

L_1 Stability of Explicit schemes ($\|\tilde{T}\|_1 = \delta x \sum_j |\tilde{T}_j|$)

$$\tilde{T}_j^{n+1} = \Big[\nu + \xi - \sigma\Big]\,\tilde{T}_{j+1}^n + \Big[1 - 2(\nu + \xi)\Big]\,\tilde{T}_j^n + \Big[\nu + \xi + \sigma\Big]\,\tilde{T}_{j-1}^n$$

The scheme is L_1 stable if $\nu + \xi \pm \sigma \geq 0$ and $\nu + \xi \leq \frac{1}{2}$. Indeed (in this case) ($\|\tilde{T}\|_\infty$)

$$|\tilde{T}_j^{n+1}| \leq \Big[\nu + \xi - \sigma\Big]\,|\tilde{T}_{j+1}^n| + \Big[1 - 2(\nu + \xi)\Big]\,|\tilde{T}_j^n| + \Big[\nu + \xi + \sigma\Big]\,|\tilde{T}_{j-1}^n|$$

$$\|\tilde{T}^{n+1}\|_1 = \|\mathcal{G}(\tilde{T}^n)\|_1 \leq \|\tilde{T}^n\|_1$$

Then

$$\|\mathcal{G}\|_1 = \sup_{\tilde{T}^n \neq 0} \left(\frac{\|\mathcal{G}(\tilde{T}^n)\|_1}{\|\tilde{T}^n\|_1} \right) \leq 1$$

L_2 Stability of Explicit schemes ($\|\tilde{T}\|_2^2 = \delta x^2 \sum_j |\tilde{T}_j|^2$)

$$\tilde{T}_j^{n+1} = \alpha_{+1}\tilde{T}_{j+1}^n + \alpha_0\tilde{T}_j^n + \alpha_{-1}\tilde{T}_{j-1}^n$$

with $\alpha_{+1} = \nu + \xi - \sigma, \quad \alpha_0 = 1 - 2(\nu + \xi), \quad \alpha_{-1} = \nu + \xi + \sigma,$
Extension of the numerical scheme to functions

$$\tilde{T}^{n+1}(\mathbf{x}) = \alpha_{+1}\tilde{T}^n(\mathbf{x} + \delta\mathbf{x}) + \alpha_0\tilde{T}^n(\mathbf{x}) + \alpha_{-1}\tilde{T}^n(\mathbf{x} - \delta\mathbf{x})$$

Fourier transform of this extension (assumed periodic) :

$$\begin{aligned} \hat{T}^{n+1}(k) &= \Big[\alpha_{+1}\exp\left(\imath k\delta\mathbf{x}\right) + \alpha_0 + \alpha_{-1}\exp\left(-\imath \xi\delta\mathbf{x}\right)\Big]\hat{T}^n(k) \\ &= g(\theta_k, \delta\mathbf{x}, \delta t)\hat{T}^n(k) \end{aligned}$$

$\theta_k = k\delta\mathbf{x}$

$$g(\theta, \delta\mathbf{x}, \delta t) = 1 + (\nu + \xi)\Big[-2 + 2\cos(\theta)\Big] - 2\imath\sigma\sin(\theta)$$

L_2 Stability of Explicit schemes ($\|\tilde{T}\|_2^2 = \delta x^2 \sum_j |\tilde{T}_j|^2$)

$$\hat{T}^{n+1}(k) = g(\theta_k, \delta \mathbf{x}, \delta t)\hat{T}^n(k)$$

where the amplification factor is

$$\begin{aligned} g(\theta, \delta \mathbf{x}, \delta t) = & \quad 1 + (\nu + \xi)\left[-2 + 2\cos(\theta)\right] - 2\imath\sigma\sin(\theta) \\ = & \quad 1 - 4(\nu+\xi)\sin^2\left(\frac{\theta}{2}\right) - 2\imath\sigma\sin(\theta) \end{aligned}$$

The scheme if L_2 stable if $\forall\theta$ we have $|g(\theta, \delta \mathbf{x}, \delta t)| \leq 1$ (VN Condition). Indeed, in this case (using Plancherel relation)

$$\begin{aligned} \|\tilde{T}^{n+1}\|_2^2 = & \quad \|\hat{T}^{n+1}\|_2^2 = \delta x^2 \sum_k |\hat{T}^{n+1}(k)|^2 = \delta x^2 \sum_k |\hat{T}^{n+1}(k)|^2 \\ \leq & \quad \max_k \left[|g(\theta_k, \delta \mathbf{x}, \delta t)|\right] \delta x^2 \sum_k |\hat{T}(k)|^2 \\ \leq & \quad \max_k \left[|g(\theta_k, \delta \mathbf{x}, \delta t)|\right] \|\tilde{T}\|_2^2 \end{aligned}$$

L_2 Stability of Explicit schemes ($\|\tilde{T}\|_2^2 = \delta x^2 \sum_j |\tilde{T}_j|^2$)

$$\|\tilde{T}^{n+1}\|_2^2 \leq \max_k \Big[|g(\theta_k, \delta \mathbf{x}, \delta t)|\Big] \|\tilde{T}\|_2^2$$

$$\|\mathcal{G}\|_2^2 = \sup_{\tilde{T}^n \neq 0} \left(\frac{\|\mathcal{G}(\tilde{T}^n)\|_2^2}{\|\tilde{T}^n\|_2^2} \right) \leq \max_k \Big[|g(\theta_k, \delta \mathbf{x}, \delta t)|\Big]$$

$$g(\theta_k, \delta \mathbf{x}, \delta t) = \Big[\alpha_{+1} \exp(\imath \theta_k) + \alpha_0 + \alpha_{-1} \exp(-\imath \theta_k)\Big]$$

If $\alpha_{+1} \geq 0$, $\alpha_0 \geq 0$ and $\alpha_{-1} \geq 0$ (Courant-Friedrichs-Lewy condition) then the scheme is L_2 stable.Indeed, in this case

$$|g(\theta_k, \delta \mathbf{x}, \delta t)| \leq \alpha_{+1} |\exp(\imath \theta_k)| + \alpha_0 + \alpha_{-1} |\exp(-\imath \theta_k)| = 1$$

L_∞ Stability of Implicit schemes ($\|\tilde{T}\|_\infty = \max_j |\tilde{T}_j|$)

$$\tilde{T}_j^{n+1} - \tilde{T}_j^n + \sigma\left(\tilde{T}_{j+1}^{n+1} - \tilde{T}_{j-1}^{n+1}\right) = (\nu+\xi)\left(\tilde{T}_{j+1}^{n+1} - 2\tilde{T}_j^{n+1} + \tilde{T}_{j-1}^{n+1}\right)$$

If we assume $\sigma = \frac{c\delta t}{\delta \mathbf{x}} \geq 0$, $\xi = \frac{\zeta\delta t}{\delta \mathbf{x}}$, $\nu = \frac{\lambda\delta t}{\delta \mathbf{x}^2}$ and $\nu + \xi \geq 0$

$$\tilde{T}_j^{n+1} + \sigma\tilde{T}_{j+1}^{n+1} + (\nu+\xi)\left(\tilde{T}_{j+1}^{n+1} + \tilde{T}_{j-1}^{n+1}\right) = \tilde{T}_j^n + \sigma\tilde{T}_{j+1}^{n+1} + 2(\nu+\xi)\tilde{T}_j^{n+1}$$

Therefore

$$\|\tilde{T}^{n+1}\|_\infty\left(1+\sigma+2(\nu+\xi)\right) \leq \|\tilde{T}^n\|_\infty + \left(\sigma+2(\nu+\xi)\right)\|\tilde{T}^{n+1}\|_\infty$$

Do it for

- a) $\sigma \leq 0$ and $\nu + \xi \leq 0$.
- b) $\sigma \leq 0$ and $\nu + \xi \geq 0$.
- c) $\sigma \geq 0$ and $\nu + \xi \leq 0$.

$$\Longrightarrow \|\tilde{T}^{n+1}\|_\infty \leq \|\tilde{T}^n\|_\infty$$

unconditionnal stability for implicit schemes.

Plan

1. Finite Difference(FD) and Finite volume(FV) : Overview
2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions

Plan

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