Lectures Références

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Numerical Methods for PDE: Finite Differences and Finites Volumes

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Plan

1. Finite Difference(FD) and Finite volume(FV) : Overview
2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions

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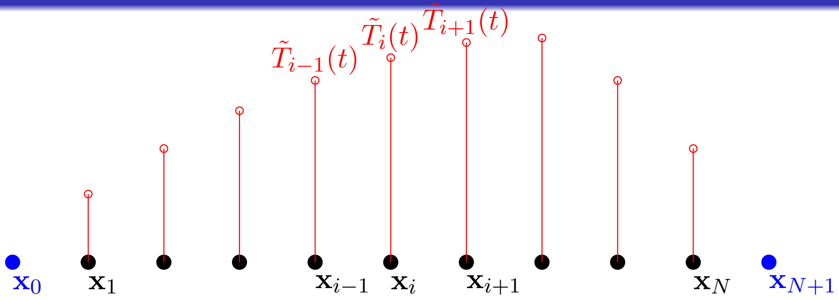
1D Scalar diffusion Equation

$$\begin{cases} \dfrac{\partial T}{\partial t} = \lambda \dfrac{\partial^2 T}{\partial \mathbf{x}^2}, & \forall \mathbf{x} \in (0,1), \quad t > 0, \\ T(t,\mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in (0,1), \quad t = 0, \\ T(t,\mathbf{x}=0) = \alpha_0 & \forall t \geq 0 \\ T(t,\mathbf{x}=1) = \alpha_1 & \forall t \geq 0 \end{cases}$$

Properties

- well poseness : bb.
- Existence : The solution $T(\mathbf{x})$ always exists and is unique.
- Regularity : The solution $T(\mathbf{x})$ is regular and uniformly bounded
- Positivity :
- Maximum principle : .

1D mesh for the discretization (approximated solution)



$$\mathbf{x}_0 = 0, \quad \mathbf{x}_{N+1} = 1, \quad \mathbf{x}_i = i\delta x, \quad \delta x = \frac{1}{N+1}, \quad \tilde{T}_i = \tilde{T}(\mathbf{x}_i)$$

Numerical Scheme : Reduced the initial PDE to the computation of a ODE system with the unknowns $\tilde{T}_i(t)$ for $i = 1, ..., N$. And then solve the ODE system using a "pertinent" time integration.

Question : How !

Finite difference strategy

- Finite difference scheme (space) : At the mesh point x_i and for any time t :

$$\frac{\partial^2 \tilde{T}}{\partial \mathbf{x}^2}(\mathbf{x}) \simeq \frac{\tilde{T}_{i+1}(t) - 2\tilde{T}_i(t) + \tilde{T}_{i-1}(t)}{\delta \mathbf{x}^2}$$

- Defined the solution at a given time by the set of unknowns $\tilde{T}_i(t)$ for $i = 1, ..., N$.

$$\frac{d\tilde{\boldsymbol{T}}}{dt} - \underline{\mathcal{A}}\tilde{\boldsymbol{T}} = \boldsymbol{S}$$

- Solve this system by using extension of methods proposed for ODE's : "Euler Forward", "ABp", "RKp",

Finite difference strategy : Semi-discretized Scheme

$$\frac{d\tilde{\boldsymbol{T}}}{dt} - \underline{\mathcal{A}}\tilde{\boldsymbol{T}} = \boldsymbol{S}$$

where

$$\underline{\mathcal{A}} = \frac{\lambda}{\delta \mathbf{x}^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}, \quad \tilde{\boldsymbol{T}} = \begin{pmatrix} \tilde{T}_1 \\ \vdots \\ \tilde{T}_i \\ \vdots \\ \tilde{T}_N \end{pmatrix}, \quad \boldsymbol{S} = \begin{pmatrix} S_1 \\ \vdots \\ S_i \\ \vdots \\ S_N \end{pmatrix}$$

$S_1 = \frac{\lambda\alpha_0}{\delta\mathbf{x}^2}$ and $S_N = \frac{\lambda\alpha_1}{\delta\mathbf{x}^2}$ all the other S_i are zero.

FD : some discretized schemes

$$\frac{\partial T}{\partial t} - \mathcal{L}(\partial_{\mathbf{x}})T = 0 + BC$$

$$\frac{d\tilde{\boldsymbol{T}}}{dt} - \underline{\mathcal{A}}\tilde{\boldsymbol{T}} = \boldsymbol{S}$$

- Euler-Forward (Explicit) :

$$\tilde{\boldsymbol{T}}^{n+1} = \tilde{\boldsymbol{T}}^{n} + \delta t \boldsymbol{S} + \delta t \underline{\mathcal{A}}\tilde{\boldsymbol{T}}^{n}$$

- Euler-Backward (Implicit) :

$$(Id - \delta t \underline{\mathcal{A}})\,\tilde{\boldsymbol{T}}^{n+1} = \tilde{\boldsymbol{T}}^{n} + \delta t \boldsymbol{S}$$

- Crank-Nicholson :

$$\left(Id - \frac{\delta t}{2}\underline{\mathcal{A}}\right)\tilde{\boldsymbol{T}}^{n+1} = \tilde{\boldsymbol{T}}^{n} + \delta t \boldsymbol{S} + \frac{\delta t}{2}\underline{\mathcal{A}}\tilde{\boldsymbol{T}}^{n}$$

FD (parabolic case) : Semi-discretized stability analysis

$$\frac{d\tilde{\boldsymbol{T}}}{dt} - \underline{\mathcal{A}}\tilde{\boldsymbol{T}} = \boldsymbol{S}$$

The matrix $\underline{\mathcal{A}}$ is is diagonalizable. Let us denote $\underline{P}$ the matrix of eigenvectors and $\underline{\Lambda}$ the matrix (diagonal) of eigenvalues :

$$\underline{\mathcal{A}} = \underline{P}\,\underline{\Lambda}\,\underline{P}^{-1}$$

Therefore, using $\tilde{\boldsymbol{\omega}} = \underline{P}^{-1}\tilde{\boldsymbol{T}}$ and $\boldsymbol{B} = \underline{P}^{-1}\boldsymbol{S}$,

$$\underline{P}^{-1}\frac{d\tilde{\boldsymbol{T}}}{dt} - \underline{P}^{-1}\underline{\mathcal{A}}\tilde{\boldsymbol{T}} = \underline{P}^{-1}\boldsymbol{S} \Longrightarrow \frac{d\tilde{\boldsymbol{\omega}}}{dt} - \underline{\Lambda}\tilde{\boldsymbol{\omega}} = \boldsymbol{B}$$

Set of independants ODE's that can be solve separately : $\tilde{\boldsymbol{T}} = \underline{P}\tilde{\boldsymbol{\omega}}$

$$\frac{d\tilde{\omega}_j}{dt} - \lambda_j\tilde{\omega}_j = B_j, \quad j = 1, \cdots, N$$

FD : Semi-discretized stability analysis

In this case $\boldsymbol{B}$ is independant of the time. Then

$$\frac{d\tilde{\boldsymbol{\omega}}}{dt} - \underline{\Lambda}\tilde{\boldsymbol{\omega}} = \boldsymbol{B} \Longrightarrow \tilde{\boldsymbol{\omega}}(t) = \exp\left(\underline{\Lambda}t\right)\tilde{\boldsymbol{\omega}}(0) - \underline{\Lambda}^{-1}\boldsymbol{B}$$

$$\begin{aligned}\tilde{\boldsymbol{T}} = \underline{P}\tilde{\boldsymbol{\omega}} &= \underline{P}\exp\left(\underline{\Lambda}t\right)\underline{P}^{-1}\tilde{\boldsymbol{T}}(0) - \underline{P\Lambda}^{-1}\underline{P}^{-1}\boldsymbol{S} \\ &= \left(\sum_{m\geq 0}\frac{t^m}{m!}\underline{A}^m\right)\tilde{\boldsymbol{T}}(0) - \underline{P\Lambda}^{-1}\underline{P}^{-1}\boldsymbol{S} \\ &= \text{Transient solution} + \text{Particular solution}\end{aligned}$$

semi-discretized stability criterion (space stability)

The transient solution will decay with time if

$$Real(\lambda_j) \leq 0, \qquad \text{for all } j = 1, \cdots, N$$

FD : One step scheme

$$\frac{\partial T}{\partial t} - \mathcal{L}(\partial_{\mathbf{x}})T = 0 + BC.$$

One step numerical scheme

$$\left(\underline{\mathcal{M}}(\delta t, \delta \mathbf{x})\right)^{-1} \tilde{\boldsymbol{T}}^{n+1} - \underline{\mathcal{N}}(\delta t, \delta \mathbf{x})\tilde{\boldsymbol{T}}^{n} = \boldsymbol{S}$$

In a condensed form :

$$\tilde{\boldsymbol{T}}(t^n + \delta t) = \mathcal{G}(\delta t, \delta \mathbf{x})\tilde{\boldsymbol{T}}(t^n)$$

Discrete representation of the exact solution $T(\mathbf{x}, t)$

For a given space and time steps (resp. $\delta \mathbf{x}$ and δt) we define $\boldsymbol{T}^n$ as a vector of size $N = \frac{1}{\delta \mathbf{x}} - 1$ that components are :

$$\boldsymbol{T}_j^n = T(\mathbf{x}_j, t^n) \equiv T(j\delta \mathbf{x}, n\delta t), \quad j = 1, \cdots, N$$

One step scheme : Definitions

Discrete norms :

$$\|\tilde{\boldsymbol{T}}\|_\infty = \max_{j=1}^{N} |\tilde{T}_j|, \quad \|\tilde{\boldsymbol{T}}\|_1 = \delta \mathbf{x} \sum_{j=1}^{N} |\tilde{T}_j|, \quad \|\tilde{\boldsymbol{T}}\|_2^2 = \delta \mathbf{x} \sum_{j=1}^{N} |\tilde{T}_j|^2$$

- Truncation error : it is a vector $\boldsymbol{\mathcal{E}}$ of size N defined by

$$\boldsymbol{\mathcal{E}}^{n+1} = \frac{\boldsymbol{T}(t^n + \delta t) - \mathcal{G}(\delta t, \delta \mathbf{x}) \boldsymbol{T}(t^n)}{\delta t}$$

- Approximation error : it is a vector $\boldsymbol{e}$ of size N defined by

$$\boldsymbol{e}^{n+1} = \boldsymbol{T}(t^n + \delta t) - \tilde{\boldsymbol{T}}(t^n + \delta t) = \boldsymbol{T}(t^n + \delta t) - \mathcal{G}(\delta t, \delta \mathbf{x}) \tilde{\boldsymbol{T}}(t^n)$$

- Consistency and Accurecy A numerical scheme is consistent of p'th order accurate in time and k'th order accurate in space if

$$\|\boldsymbol{\mathcal{E}}^{n+1}\| = O(\delta t^p) + O(\delta \mathbf{x}^k) \quad \text{with } p > 0 \text{ and } k > 0$$

multi step scheme : Definitions

$$\tilde{\boldsymbol{T}}^{n+1} = \sum_{m=0}^{s} \mathcal{G}_m \left(\delta t, \delta \mathbf{x}, \tilde{\boldsymbol{T}}^{n+1-m}\right)$$

- Truncation error : it is a vector $\boldsymbol{\mathcal{E}}$ of size N defined by

$$\boldsymbol{\mathcal{E}}^{n+1} = \frac{\boldsymbol{T}^{n+1} - \displaystyle\sum_{m=0}^{s} \mathcal{G}_m \left(\delta t, \delta \mathbf{x}, \boldsymbol{T}^{n+1-m}\right)}{\delta t}$$

- Approximation error : it is a vector $\boldsymbol{e}$ of size N defined by

$$\boldsymbol{e}^{n+1} = \boldsymbol{T}(t^n+\delta t) - \tilde{\boldsymbol{T}}^{n+1} = \boldsymbol{T}(t^n+\delta t) - \sum_{m=0}^{s} \mathcal{G}_m \left(\delta t, \delta \mathbf{x}, \tilde{\boldsymbol{T}}^{n+1-m}\right)$$

- Consistency and Accurecy A numerical scheme is consistent of p'th order accurate in time and k'th order accurate in space if

$$\|\boldsymbol{\mathcal{E}}^{n+1}\| = O(\delta t^p) + O(\delta \mathbf{x}^k) \quad \text{with } p > 0 \text{ and } k > 0$$

Example of the Euler-Forward/CD 0(2)

$$s = 1, \quad \mathcal{G}_0 \equiv 0, \quad \mathcal{G}_1 = (Id + \delta t \underline{\mathcal{A}})\,\tilde{\boldsymbol{T}}^n + \delta t \boldsymbol{S}$$

- Truncation error : $j = 2, \cdots, N-1$

$$\begin{aligned}
\mathcal{E}_j^{n+1} &= \frac{T(x_j,t^n+\delta t) - T(x_j,t^n)}{\delta t} \\
&\quad -\lambda \left(\frac{T(x_j+\delta \mathbf{x},t^n) - 2T(x_j,t^n) + T(x_j-\delta \mathbf{x},t^n)}{\delta \mathbf{x}^2} \right) \\
&= \left(\frac{\partial T}{\partial t}\right)_j^n + O(\delta t) - \lambda \left(\frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^n + O(\delta \mathbf{x}^2) \\
&= \left(\frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^n + O(\delta t) + O(\delta \mathbf{x}^2) \\
&= O(\delta t) + O(\delta \mathbf{x}^2)
\end{aligned}$$

  Do it for $j = 1$ and $j = N$.

- Consistency and Accurecy ($N\delta \mathbf{x} \equiv O(1)$)

$$\|\boldsymbol{\mathcal{E}}^{n+1}\|_1 = N\delta \mathbf{x}\left[O(\delta t) + O(\delta \mathbf{x}^2)\right] = O(\delta t) + O(\delta \mathbf{x}^2)$$

Example of the Euler-Backward/CD 0(2)

$$s = 1, \quad \mathcal{G}_0 = \delta t \underline{\mathcal{A}} \tilde{\boldsymbol{T}}^{n+1}, \quad \mathcal{G}_1 = \tilde{\boldsymbol{T}}^n + \delta t \boldsymbol{S}$$

- Truncation error : $j = 2, \cdots, N-1$

$$\begin{aligned} \mathcal{E}_j^{n+1} &= \frac{T(x_j, t^n + \delta t) - T(x_j, t^n)}{\delta t} \\ &\quad -\lambda \left(\frac{T(x_j + \delta \mathbf{x}, t^n + \delta t) - 2T(x_j, t^n + \delta t) + T(x_j - \delta \mathbf{x}, t^n + \delta t)}{\delta \mathbf{x}^2} \right) \\ &= \left(\frac{\partial T}{\partial t}\right)_j^n + O(\delta t) - \lambda \left(\frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^{n+1} + O(\delta \mathbf{x}^2) \\ &= \left(\frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^n + O(\delta t) + O(\delta \mathbf{x}^2) \\ &= O(\delta t) + O(\delta \mathbf{x}^2) \end{aligned}$$

Do it for $j = 1$ and $j = N$.

- Consistency and Accurecy ($N \delta \mathbf{x} \equiv O(1)$)

$$\|\boldsymbol{\mathcal{E}}^{n+1}\|_1 = N \delta \mathbf{x} \left[O(\delta t) + O(\delta \mathbf{x}^2) \right] = O(\delta t) + O(\delta \mathbf{x}^2)$$

Example of the Crank Nicholson/CD 0(2)

$$s = 1, \quad \mathcal{G}_0 = \frac{\delta t}{2} \underline{\mathcal{A}} \tilde{\boldsymbol{T}}^{n+1}, \quad \mathcal{G}_1 = \left(Id + \frac{\delta t}{2} \underline{\mathcal{A}} \right) \tilde{\boldsymbol{T}}^n + \delta t \boldsymbol{S}$$

- Truncation error : $j = 2, \cdots, N-1$

$$\begin{aligned}
\mathcal{E}_j^{n+1} = & \tfrac{T(x_j, t^n + \delta t) - T(x_j, t^n)}{\delta t} \\
& - \tfrac{\lambda}{2} \left(\tfrac{T(x_j + \delta \mathbf{x}, t^n + \delta t) - 2T(x_j, t^n + \delta t) + T(x_j - \delta \mathbf{x}, t^n + \delta t)}{\delta \mathbf{x}^2} \right) \\
& - \tfrac{\lambda}{2} \left(\tfrac{T(x_j + \delta \mathbf{x}, t^n) - 2T(x_j, t^n) + T(x_j - \delta \mathbf{x}, t^n)}{\delta \mathbf{x}^2} \right) \\
= & \left(\tfrac{\partial T}{\partial t} \right)_j^n + \tfrac{\delta t}{2} \left(\tfrac{\partial^2 T}{\partial t^2} \right)_j^n + O(\delta t^2) \\
& - \tfrac{\lambda}{2} \left(\tfrac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^{n+1} + O(\delta \mathbf{x}^2) - \tfrac{\lambda}{2} \left(\tfrac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^n + O(\delta \mathbf{x}^2) \\
= & \left(\tfrac{\partial T}{\partial t} \right)_j^n + \tfrac{\delta t}{2} \left(\tfrac{\partial^2 T}{\partial t^2} \right)_j^n + O(\delta t^2) - \tfrac{\lambda}{2} \left(\tfrac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^n + O(\delta \mathbf{x}^2) \\
& - \tfrac{\lambda}{2} \left(\tfrac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^n - \tfrac{\lambda \delta t}{2} \left(\tfrac{\partial}{\partial t} \tfrac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^n + O(\delta t^2) + O(\delta \mathbf{x}^2)
\end{aligned}$$

Example of the Crank Nicholson/CD 0(2)

$$s = 1, \quad \mathcal{G}_0 = \frac{\delta t}{2} \underline{\mathcal{A}} \tilde{\boldsymbol{T}}^{n+1}, \quad \mathcal{G}_1 = \left(Id + \frac{\delta t}{2} \underline{\mathcal{A}} \right) \tilde{\boldsymbol{T}}^n + \delta t \boldsymbol{S}$$

- Truncation error : $j = 2, \cdots, N-1$

$$\begin{aligned} \mathcal{E}_j^{n+1} = & \left(\frac{\partial T}{\partial t}\right)_j^n + \frac{\delta t}{2} \left(\frac{\partial^2 T}{\partial t^2}\right)_j^n + O(\delta t^2) - \frac{\lambda}{2} \left(\frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^n + O(\delta \mathbf{x}^2) \\ & - \frac{\lambda}{2} \left(\frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^n - \frac{\lambda \delta t}{2} \left(\frac{\partial}{\partial t} \frac{\partial^2 T}{\partial \mathbf{x}^2}\right)_j^n + O(\delta t^2) + O(\delta \mathbf{x}^2) \\ = & \left[\frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial \mathbf{x}^2} + \frac{\delta t}{2} \frac{\partial}{\partial t} \left(\frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial \mathbf{x}^2} \right) \right]_j^n + O(\delta t^2) + O(\delta \mathbf{x}^2) \\ = & \; O(\delta t^2) + O(\delta \mathbf{x}^2) \end{aligned}$$

Do it for $j = 1$ and $j = N$.

- Consistency and Accurecy ($N \delta \mathbf{x} \equiv O(1)$)

$$\| \boldsymbol{\mathcal{E}}^{n+1} \|_1 = N \delta \mathbf{x} \left[O(\delta t^2) + O(\delta \mathbf{x}^2) \right] = O(\delta t^2) + O(\delta \mathbf{x}^2)$$

Example of the leap-frog/CD 0(2)

$$s = 2, \quad \mathcal{G}_0 \equiv 0, \quad \mathcal{G}_1 = 2\delta t \underline{\mathcal{A}} \tilde{\boldsymbol{T}}^n + 2\delta t \boldsymbol{S} \quad \mathcal{G}_2 = \tilde{\boldsymbol{T}}^{n-1}$$

- Truncation error : $j = 2, \cdots, N-1$

$$\begin{aligned}
\mathcal{E}_j^{n+1} = & \ \frac{T(x_j, t^n + \delta t) - T(x_j, t^n - \delta t)}{\delta t} \\
& -2\lambda \left(\frac{T(x_j + \delta \mathbf{x}, t^n) - 2T(x_j, t^n) + T(x_j - \delta \mathbf{x}, t^n)}{\delta \mathbf{x}^2} \right) \\
= & \ 2 \left(\frac{\partial T}{\partial t} \right)_j^n + O(\delta t^2) - 2\lambda \left(\frac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^n + O(\delta \mathbf{x}^2) \\
= & \ 2 \left(\frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial \mathbf{x}^2} \right)_j^n + O(\delta t^2) + O(\delta \mathbf{x}^2) \\
= & \ O(\delta t^2) + O(\delta \mathbf{x}^2)
\end{aligned}$$

One step scheme : Definitions

Consistensy : for any t_* and $\mathbf{x}_*$

$n_* = \frac{t_*}{\delta t}$, $j_* = \frac{\mathbf{x}_*}{\delta x}$, The numerical scheme is consistent (space and time) if

$$\lim_{\substack{\delta t \longrightarrow 0 \\ \delta \mathbf{x} \longrightarrow 0}} \left((\underline{\mathcal{M}}(\delta t, \delta \mathbf{x}))^{-1} \boldsymbol{T}^{n+1} - \underline{\mathcal{N}}(\delta t, \delta \mathbf{x}) \boldsymbol{T}^{n} - \boldsymbol{S} \right) = 0$$

FD : Discretized stability analysis

For the one step FD scheme we have :

$$\Longrightarrow \tilde{\boldsymbol{T}}^{n+1} = (\underline{\mathcal{M}}\,\underline{\mathcal{N}})^{n+1}\,\tilde{\boldsymbol{T}}^{0} + \delta t \sum_{m=1}^{n+1} \underline{\mathcal{M}}^{n+1-p}\boldsymbol{S}$$

Stability condition

A necessary stability condition is that eigenvalues of $(\underline{\mathcal{M}}\,\underline{\mathcal{N}})$ lie in the unit disk. Therefore, only eigenvalues on the unit circle can cause instabilities :

- if eigenvalues on the circle are simple, then the scheme is stable
- if some eigenvalues on the circle are multiple, the scheme is stable if and only if the associated subspaces are of dimension one.

Stability criterion for the Explicit scheme

Euler-Forward (Explicit) $\underline{\mathcal{M}} = Id, \quad \underline{\mathcal{N}} = Id + \delta t \underline{\mathcal{A}}$

In this case $\underline{\mathcal{M}}\,\underline{\mathcal{N}}$ is diagonalisable and eigenvalues are (amplification factors)

$$g(\theta_j) = \lambda_j = 1 - \frac{4\lambda\delta t}{\delta \mathbf{x}^2} \sin^2\left(\frac{\pi}{2} j\delta\mathbf{x}\right) \quad \text{for } j = 1, \cdots, N.$$

Note that $0 < j\delta\mathbf{x} < 1$ for $j = 1, \cdots, N$, such that egeinvalues are all differents. Therefore the stability is obtained when

$$|g(\theta)| = \left|1 - \frac{4\lambda\delta t}{\delta\mathbf{x}^2} \sin^2\theta\right| \leq 1 \quad \forall\theta$$

This condition is achieved when $1 - \frac{4\lambda\delta t}{\delta\mathbf{x}^2} \geq -1$ and this gives

Stability criterion for the Euler-Forward / central difference scheme

$$\frac{\lambda\delta t}{\delta\mathbf{x}^2} \leq \frac{1}{2}$$

Stability criterion for the Implicit scheme

Euler-Backward (Implicit) $\underline{\mathcal{M}}^{-1} = Id - \delta t \underline{\mathcal{A}}, \quad \underline{\mathcal{N}} = Id$

In this case $\underline{\mathcal{M}}\,\underline{\mathcal{N}}$ is diagonalisable and eigenvalues are

$$\lambda_j = \frac{1}{1 + \frac{4\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\left(\frac{\pi}{2} j\delta\mathbf{x}\right)} \quad \text{for } j = 1, \cdots, N.$$

Therefore

$$g(\theta) = \frac{1}{1 + \frac{4\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\theta} \quad \text{and } |g(\theta)| \leq 1 \quad \forall\theta$$

and the implicit Euler scheme is unconditionally stable.

Stability criterion for the CN scheme

CN Scheme $\quad \underline{\mathcal{M}}^{-1} = Id - \frac{\delta t}{2}\underline{\mathcal{A}}, \quad \underline{\mathcal{N}} = Id + \frac{\delta t}{2}\underline{\mathcal{A}}$

In this case $\underline{\mathcal{M}}\,\underline{\mathcal{N}}$ is diagonalisable and eigenvalues are

$$\lambda_j = \frac{1 - \frac{2\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\left(\frac{\pi}{2}j\delta\mathbf{x}\right)}{1 + \frac{2\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\left(\frac{\pi}{2}j\delta\mathbf{x}\right)} \quad \text{for } j = 1, \cdots, N.$$

Therefore the implicit CN scheme is unconditionally stable. Indeed

$$g(\theta) = \frac{1 - \frac{2\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\theta}{1 + \frac{2\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\theta} \quad \text{and} \quad |g(\theta)| \le 1 \quad \forall\theta$$

Stability : Von Neumann criterion (general case)

Discrete Fourier mode : : $\hat{\boldsymbol{T}}(t,\theta)$, note $\imath^2 = -1$.

$$\hat{\boldsymbol{T}}(t,\theta) = \beta(t,\theta)\hat{\boldsymbol{T}}(\theta) \quad \text{with} \quad \hat{T}_j(\theta) = \exp(\imath j\theta) \quad \text{for } 1 \leq j \leq N$$

Amplification factor $g(\theta)$: $\quad g(\theta) = \dfrac{\beta(t+\delta t,\theta)}{\beta(t,\theta)}$

Von Neumann stability criterion : $|g(\theta)| \leq 1 \quad \forall\theta$.

A finite difference sceme is stable (Von Neumann) if $|g(\theta)| \leq 1\ \forall\theta$.

In this case the behavior at the boundaries are ignored ! ! !

Von Neumann stability criterion : Applications

- Euler-Forward : $\tilde{T}_j^{n+1} = \tilde{T}_j^n + \frac{\delta t \lambda}{\delta \mathbf{x}^2}\left(\tilde{T}_{j+1}^n - 2\tilde{T}_j^n + \tilde{T}_{j-1}^n\right)$

$$\begin{aligned}\hat{T}_j^{n+1} &= \beta(t)\left[1 + \frac{\delta t \lambda}{\delta \mathbf{x}^2}\left(\exp(\imath\theta) - 2 + \exp(-\imath\theta)\right)\right]\hat{T}_j^n \\ &= \beta(t)\left[1 + \frac{\delta t \lambda}{\delta \mathbf{x}^2}\left(2\cos(\theta) - 2\right)\right]\hat{T}_j^n\end{aligned}$$

$$\Longrightarrow g(\theta) = 1 + \frac{\delta t \lambda}{\delta \mathbf{x}^2}\left(2\cos(\theta) - 2\right) = 1 - \frac{4\lambda\delta t}{\delta \mathbf{x}^2}\sin^2\frac{\theta}{2}$$

FD : Discretized stability analysis

FD : Discretized stability analysis

Stability of a FD scheme (uniformlly bounded solution)

A finite diffefrence scheme, is stable (iterativelly) for a given norm $\|\|$ if, for any time $T > 0$, $\exists M \in \mathbb{N}^*$, $\exists \delta t_0 \in \mathbb{R}^+$, $\exists \delta \mathbf{x}_0 \in \mathbb{R}^+$ and a constant C_T such that

$$\|\tilde{\boldsymbol{T}}^n\| \le C_T \sum_{m=0}^{M} \|\tilde{\boldsymbol{T}}^m\|$$

$$\text{for} \quad 0 \le n\delta t \le T, \quad 0 < \delta \mathbf{x} \le \delta \mathbf{x}_0, \quad 0 < \delta t \le \delta t_0$$

The Stability sufficient condition

A finite diffefrence scheme, is stable if

$$\|\tilde{\boldsymbol{T}}^n\| \le (1 + O(\delta t)) \, \|\tilde{\boldsymbol{T}}^n\|$$

Explicit/Implicit schemes

- Explicit schemes :
  Advantages
  - very easy calculations,
  - simply step ahead.

  Disadvantage
  - low accuracy, in time
  - subject to instabilities, then need small time steps and requires many time steps to reach a given time.
- Imlicit schemes :
  Advantages
  - Unconditionally stable (in general). The choice of the time step is governed by overall accuracy.
  - May be able to take larger time steps and then require fewer steps to reach a given time.

  Disadvantage
  - More difficult calculations, especially for 2D and 3D.

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