Lectures Références

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Numerical Methods for PDE: Finite Differences and Finites Volumes

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Scalar Conservation Laws

General Integral form :

$$\frac{d}{dt}\int_{\mathbf{x}_l}^{x_r} \boldsymbol{u} d\mathbf{x} = -\left(f(\boldsymbol{u}_r) - f(\boldsymbol{u}_l)\right)$$

$\boldsymbol{u}$ is the unknown conserved quantity (scalar) and f is the flux function(given).

- Conservative EDP form : $\partial_t \boldsymbol{u} + \partial_x \Big[f(\boldsymbol{u}) \Big] = 0$
- Primitive form (Non conservative) : regular solutions of the conservation law are also solutions of the following equation :

$$\partial_t \boldsymbol{u} + f'(\boldsymbol{u})\partial_x \boldsymbol{u} = 0$$

- Viscous profile. Question : can we find a solution $\boldsymbol{u}^\varepsilon$ of the following system

$$\partial_t \boldsymbol{u}^\varepsilon + \partial_x f\left(\boldsymbol{u}^\varepsilon\right) = \varepsilon \partial_x^2 \boldsymbol{u}^\varepsilon$$

  such that $\boldsymbol{u} = \lim_{\varepsilon \longrightarrow 0} \boldsymbol{u}^\varepsilon$. This will help for numerical design.

Examples

- Linear Advection equation : $\boldsymbol{u}$ the density and

$$f(\boldsymbol{u}) = c\boldsymbol{u}$$

  where c is constant.

- Invicid Burgers equation : $\boldsymbol{u}$ is the velocity and

$$f(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^2.$$

- Traffic flow : $\boldsymbol{u}$ is the density of cars and

$$f(\boldsymbol{u}) = \alpha\left(\boldsymbol{u} - \frac{\boldsymbol{u}^2}{\beta}\right).$$

- Buckley-Leverett equation : $\boldsymbol{u}$ is the saturation of water and

$$f(\boldsymbol{u}) = \frac{\boldsymbol{u}^2}{\boldsymbol{u}^2 + \alpha(1-\boldsymbol{u})^2}.$$

Characteristic curves

The curves, parametrized as $\mathbf{x} \equiv \mathbf{x}(t)$ and such that

$$\frac{d\mathbf{x}(t)}{dt} = f'\left(\boldsymbol{u}(\mathbf{x}(t), t)\right)$$

are called Characteristics.

The solution $\boldsymbol{u}$ is constant along the characteristis

$$\frac{d\boldsymbol{u}(\mathbf{x}(t), t)}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \frac{d\mathbf{x}(t)}{dt}\frac{\partial \boldsymbol{u}}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{u}}{\partial t} + f'\left(\boldsymbol{u}(\mathbf{x}(t), t)\right)\frac{\partial \boldsymbol{u}}{\partial \mathbf{x}} = 0$$

Therefore, Characteristics are straight lines : $\boldsymbol{u}(\mathbf{x}(t), t) = \boldsymbol{u}(\mathbf{x}(t_0), t_0) = \boldsymbol{u}_0(\mathbf{x}(t_0))$ and

$$\mathbf{x}(t) = \mathbf{x}(t_0) + (t - t_0)\, f'\left(\boldsymbol{u}(\mathbf{x}(t_0))\right)$$

Existence of smooth Solutions

When the function $\xi \longrightarrow f'\Big(\boldsymbol{u}_0(\xi)\Big)$ is increasing and $\boldsymbol{u}_0(\xi)$ is regular and bounded, There exists a unique solution of the Cauchy problem. This solution is defined as

$$\boldsymbol{u}(x,t) = \boldsymbol{u}_0\left(\xi(x,t)\right) \qquad \text{where} \qquad \xi(x,t) = x - tf'\left(\boldsymbol{u}_0(\xi(x,t))\right)$$

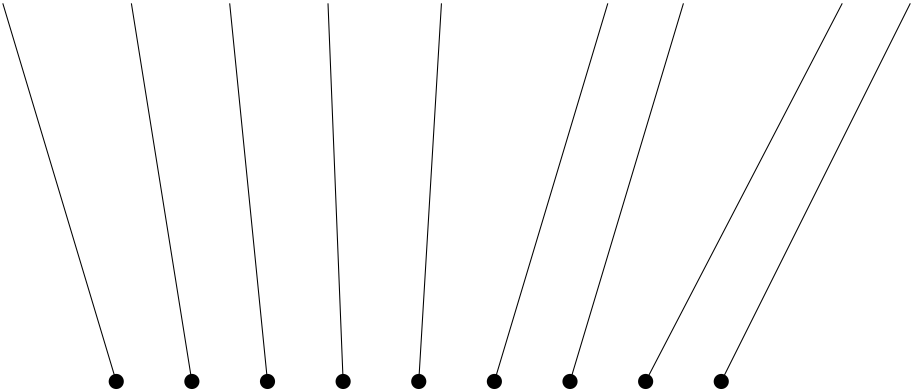
Existence is OK : by using Characteristic curves.
Uniqness : Let $\mathcal{F}(\xi) = \xi - x + tf'\left(\boldsymbol{u}_0(\xi)\right)$.

$$\frac{d}{d\xi}f'\left(\boldsymbol{u}_0(\xi(t))\right) > 0 \Longrightarrow \mathcal{F}' = 1 + t\frac{d}{d\xi}f'\left(\boldsymbol{u}_0(\xi(t))\right) > 0$$

$$\mathcal{F}(-\infty) < 0, \quad \mathcal{F}(\infty) > 0, \Longrightarrow \exists!\ \xi(x,t) \quad \mathcal{F}(\xi(x,t)) = 0$$

Therefore, we have the uniqness of $\boldsymbol{u}(x,t) = \boldsymbol{u}_0\left(\xi(x,t)\right)$.

Existence of smooth Solutions



When the initial solution is smooth and Characteristic cures do not cross : $f'(\boldsymbol{u}_0(\xi))$ is an increasing function.

Discontinuous Solutions : Schock formation

Schock formation : Let $\boldsymbol{u}_0$ and f regular functions.

When $\frac{d}{d\xi} f'\left(\boldsymbol{u}_0(\xi(t))\right) < 0$, it exists a time $t = t^* = -\frac{1}{\frac{d}{d\xi} f'(\boldsymbol{u}_0(\xi(t)))}$ such that $Y(t) = \partial_x \boldsymbol{u}(x(t), t)$ is no longuer bounded.

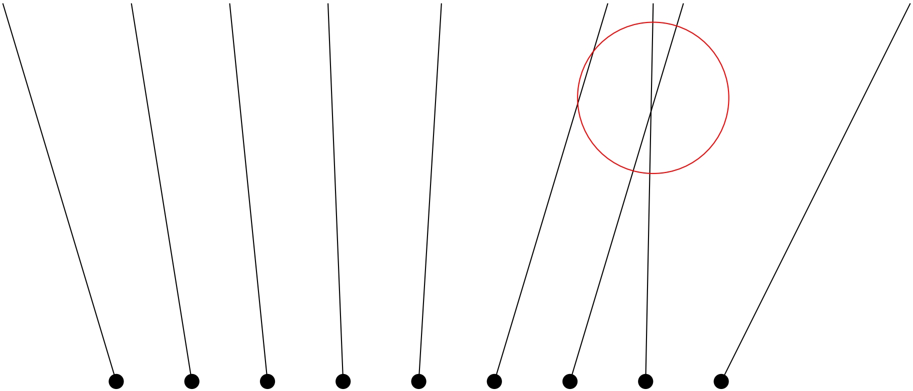
$$\begin{aligned} \frac{dY}{dt} &= \partial_x \partial_t \boldsymbol{u}(x(t), t) + \frac{dx(t)}{dt} \partial_x \left(\partial_x \boldsymbol{u}(x(t), t)\right) \\ &= -\partial_x \left(f'(\boldsymbol{u}) \partial_x \boldsymbol{u}\right) + f'(\boldsymbol{u}) \partial_x \left(\partial_x \boldsymbol{u}(x(t), t)\right) \\ &= -f''\left(\boldsymbol{u}(x(t), t)\right) Y^2 = -f''\left(\boldsymbol{u}_0(\xi)\right) Y^2 \end{aligned}$$

$Y(t)$ satisfied a Riccati ODE, and the solution is :

$$Y(t) = \frac{Y(0)}{1 + tY(0) f''\left(\boldsymbol{u}_0(\xi)\right)} = \frac{Y(0)}{1 + t \frac{d}{d\xi} f'\left(\boldsymbol{u}_0(\xi(t))\right)}$$

Therefore, when $\frac{d}{d\xi} f'\left(\boldsymbol{u}_0(\xi(t))\right) < 0$ the solution cannot be regular after the time $t = t^* = -\frac{1}{\frac{d}{d\xi} f'(\boldsymbol{u}_0(\xi(t)))} > 0$

Discontinuous Solutions : Schock formation



The initial solution can be smooth but characteristic cures cross after a given time : $f'\left(\boldsymbol{u}_0(\xi)\right)$ is a non increasing function.

Weak Solutions : are often non unique

Cauchy Problem :

$$\begin{cases} \partial_t \boldsymbol{u} + \partial_x \left(f(\boldsymbol{u}) \right) = 0 & \forall x \in \mathbb{R}, \quad \forall t > 0 \\ \boldsymbol{u}(x, t = 0) = \boldsymbol{u}_0(x) & \forall x \in \mathbb{R} \end{cases}$$

Definition (Weak Solution)

$\boldsymbol{u}$ is a weak solution of the Cauchy problem if the following statements are satisfied :

- $\boldsymbol{u} \in L^1_{loc}(\mathbb{R} \times]0, T[)$.
- $\forall \phi(x,t)$ regular function with compact support in $\mathbb{R} \times \mathbb{R}$ $(C^1_0(\mathbb{R} \times \mathbb{R}))$:

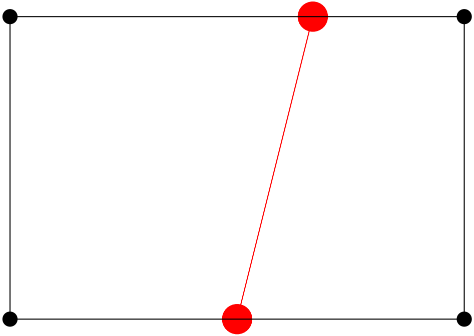
$$\int_0^{\infty} dt \int_{-\infty}^{\infty} dx \; (\boldsymbol{u} \partial_t \phi + f(\boldsymbol{u}) \partial_x \phi) = - \int_{-\infty}^{\infty} dx \; u_0(x) \phi(x, 0)$$

Caracterization of Discontinuities

Let consider the constant states $\boldsymbol{u}_l$ and $\boldsymbol{u}_r$ separeted by a curve of discontinuity $x = X(t)$

$$\mathcal{Q} = [X(t) - \delta x,\ X(t) + \delta x] \times [t,\ t + \delta t]$$

with $\quad X(t+s) \in \,]X(t) - \delta x,\ X(t) + \delta x[\quad$ pour $\quad 0 \leq s \leq \delta t$



Rankine-Hugoniot Jump condition

Integral form of the conservation law integrated over $(t, t+\delta t)$ gives

$$\int_{X(t)-\delta x}^{X(t+\delta t)} \boldsymbol{u}_l dx + \int_{X(t+\delta t)}^{X(t)+\delta x} \boldsymbol{u}_r dx = \int_{X(t)-\delta x}^{X(t)} \boldsymbol{u}_l dx + \int_{X(t)}^{X(t)+\delta x} \boldsymbol{u}_r dx - \int_t^{t+\delta t} f(\boldsymbol{u}_r) dt + \int_t^{t+\delta t} f(\boldsymbol{u}_l) dt$$

$$\frac{X(t+\delta t) - X(t)}{\delta t} (\boldsymbol{u}_l - \boldsymbol{u}_r) = f(\boldsymbol{u}_l) - f(\boldsymbol{u}_r) \qquad \forall \delta t > 0$$

$$\delta t \to 0 \Longrightarrow f(\boldsymbol{u}_r) - f(\boldsymbol{u}_l) = s(t)(\boldsymbol{u}_r - \boldsymbol{u}_l) \tag{1}$$

where $s(t) = X'(t)$ is the shock speed.

Jump condition : define the wave speed s

$$f(\boldsymbol{u}_r) - f(\boldsymbol{u}_l) = s(\boldsymbol{u}_r - \boldsymbol{u}_l)$$

Caracterization of discontinuities : contact and shocks

$$m = f(\boldsymbol{u}_r) - s\boldsymbol{u}_r = f(\boldsymbol{u}_l) - s\boldsymbol{u}_l \Longleftrightarrow \Big[m \Big] = 0$$

where m is the flux crossing the discontinuity.

- When $m = 0$: the associated wave is linearly degenerated and is named Contact Discontinuity (CD)
- When $m \neq 0$: the associated wave is linearly genuinely nonlinear and is named Shock.

For linear advection $f(\boldsymbol{u}) = a\boldsymbol{u} \Longrightarrow s = a$ and $m = 0$. Waves are Contact discontinuities.
Burgers Equation : $f(\boldsymbol{u}) = \frac{\boldsymbol{u}^2}{2} \Longrightarrow s = \frac{\boldsymbol{u}_r + \boldsymbol{u}_l}{2}$

$$m = \frac{\boldsymbol{u}_r^2}{2} - \boldsymbol{u}_r \frac{\boldsymbol{u}_r + \boldsymbol{u}_l}{2} = -\frac{\boldsymbol{u}_r \boldsymbol{u}_l}{2}$$

Weak solutions piecewise regular

A piecewise function $\boldsymbol{u}(\mathbf{x}, t)$ is a weak solution of the Cauchy problem if and only if :

- $\boldsymbol{u}(\mathbf{x}, t = 0) = \boldsymbol{u}_0(\mathbf{x})$
- Where $\boldsymbol{u}$ is regular it satisfy the partial differential equation.
- across discontinuities the Rankine-Hugoniot relation is satisfied.

Manipulating equations has impact on weak solutions

Let us consider functions $\phi(X)$ and $\Psi(X)$ such that

$$\phi'\left(\Psi(\boldsymbol{u})\right) = f'(\boldsymbol{u}) \quad \text{with} \quad \Psi'(\boldsymbol{u}) \neq 0$$

where $\boldsymbol{u}$ is a solution (regular) of

$$\partial_t \boldsymbol{u} + \partial_x f(\boldsymbol{u}) = 0$$

Then $v = \Psi(\boldsymbol{u})$ satisfy the following conservation law.

$$\partial_t v + \partial_x \phi(v) = 0$$

Jump condition for this new conservation law is not equivalent to the one associated to $\partial_t \boldsymbol{u} + \partial_x f(\boldsymbol{u}) = 0$. Indeed, in general,

$$s_u = \frac{f(\boldsymbol{u}_r) - f(\boldsymbol{u}_l)}{\boldsymbol{u}_r - \boldsymbol{u}_l} \neq s_v = \frac{\phi\left(\Psi(\boldsymbol{u}_r)\right) - \phi\left(\Psi(\boldsymbol{u}_l)\right)}{\Psi(\boldsymbol{u}_r) - \Psi(\boldsymbol{u}_l)}$$

Example : For the Burgers Equation

$$\partial_t \boldsymbol{u} + \partial_x \left(\frac{\boldsymbol{u}^2}{2} \right) = 0 \Longrightarrow f(\boldsymbol{u}) = \frac{\boldsymbol{u}^2}{2}$$

Then let us define $\Psi(X)$ and $\phi(X)$ as

$$\Psi(X) = X^2 \quad \text{and} \quad \phi(X) = \frac{2}{3} X^{\frac{3}{2}}$$

Then we find that $\phi'(X) = \sqrt{X}$ and $\Psi'(X) = 2X$ and finally, for positive $\boldsymbol{u} > 0$,

$$\phi'(\Psi(\boldsymbol{u})) = \boldsymbol{u} \quad \text{and} \quad \Psi'(\boldsymbol{u}) \neq 0$$

$$\Longrightarrow \partial_t \left(\boldsymbol{u}^2 \right) + \partial_x \left(\frac{2}{3} \boldsymbol{u}^3 \right) = 0$$

But

$$\frac{\boldsymbol{u}_r + \boldsymbol{u}_l}{2} \neq \frac{2}{3} \frac{(\boldsymbol{u}_l^2 + \boldsymbol{u}_l \boldsymbol{u}_r + \boldsymbol{u}_r^2)}{\boldsymbol{u}_l + \boldsymbol{u}_r}$$

Weak solutions : Non-uniqueness

Definition of the Riemann Problem

A Riemann problem is a Cauchy problem with the following initial data

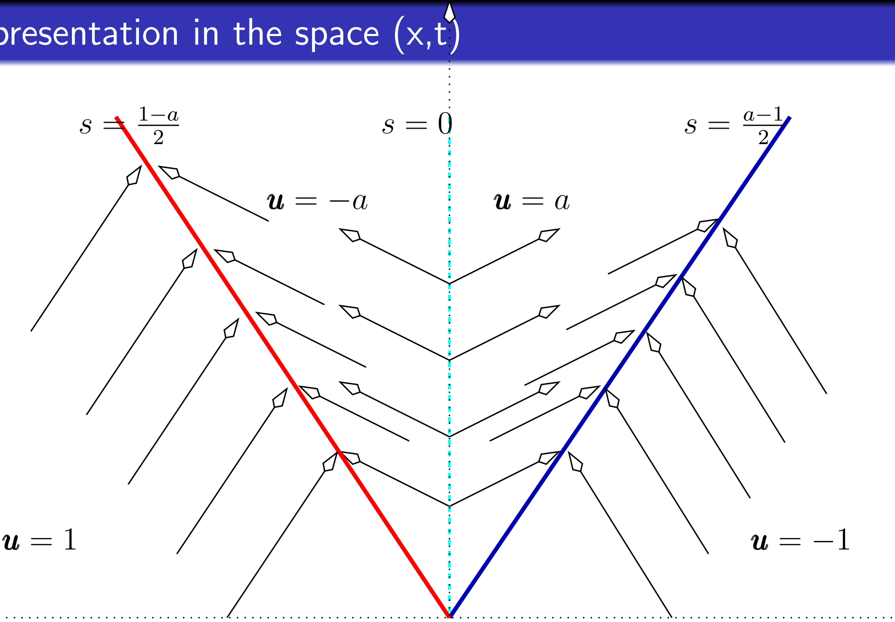
$$\boldsymbol{u}_0(x) = \begin{cases} \boldsymbol{u}_l & \text{when } x < 0 \\ \boldsymbol{u}_r & \text{when } x \geq 0 \end{cases}$$

where $\boldsymbol{u}_l$ and $\boldsymbol{u}_r$ are given constants.

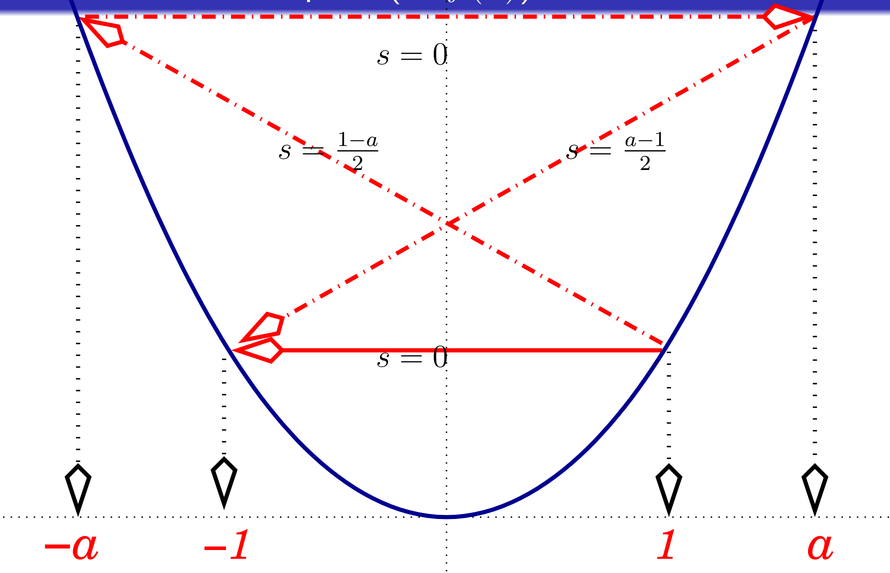
Let us consider a Riemann problem for the Burgers equation with $\boldsymbol{u}_l = 1$ and $\boldsymbol{u}_r = -1$. Therefore, for any $a \geq 1$, solutions of the following form are weak solutions of the Riemann problem.

$$\boldsymbol{u}(x,t) = \begin{cases} 1 & \text{when } \frac{x}{t} < \frac{1-a}{2} \\ -a & \text{when } \frac{1-a}{2} < \frac{x}{t} < 0 \\ a & \text{when } 0 < \frac{x}{t} < \frac{a-1}{2} \\ -1 & \text{when } \frac{x}{t} > \frac{a-1}{2} \end{cases}$$

Representation in the space (x,t)



Representation in the space $(u, f(u))$



What is Wrong ! How to cure it !

Some of the characteritics have their initial values on the discontinuity where the function is not defined in a strong sence.

Condition for uniqueness of the weak solution

Caracteristics will not have their initial values localized on a discontinuity.

For scalar equation this means that, if s is the local velocity of a discontinuity between $\boldsymbol{u}_l$ (on the left) and $\boldsymbol{u}_r$ (on the right), This gives the **Entropy Condition**

$$f'(\boldsymbol{u}_l) \geq s \geq f'(\boldsymbol{u}_r)$$

A simplify version is the **Olenik Condition**

$$\frac{f(\boldsymbol{u}) - f(\boldsymbol{u}_l)}{\boldsymbol{u} - \boldsymbol{u}_l} \geq s \geq \frac{f(\boldsymbol{u}) - f(\boldsymbol{u}_r)}{\boldsymbol{u} - \boldsymbol{u}_r}, \quad \forall \boldsymbol{u} \in (\boldsymbol{u}_l, \ \boldsymbol{u}_r)$$

Mathematical notion of entropy : for uniqueness

A convex function $\eta(\boldsymbol{u})$ is an entropy of the conservation law if there exist a flux function $\Psi(\boldsymbol{u})$ such as

$$\eta'(\boldsymbol{u})f'(\boldsymbol{u}) = \Psi'(\boldsymbol{u}) \quad pp. \tag{2}$$

Krushkov Familly “entropy/flux” :

$$\eta(\boldsymbol{u}) = |\boldsymbol{u} - k|, \quad \Psi(\boldsymbol{u}) = sgn(\boldsymbol{u} - k)\,(f(\boldsymbol{u}) - f(k)) \quad \forall k \in \mathbb{R}$$

Weak Entropy solution

A weak solution is called “Weak Entropy solution” if for any entropy/flux (η, Ψ) we have :

$$\partial_t \eta(\boldsymbol{u}) + \partial_x \Psi(\boldsymbol{u}) \leq 0 \tag{3}$$

jump conditions : $\Psi(\boldsymbol{u}_r) - \Psi(\boldsymbol{u}_l) \leq s\left[\eta(\boldsymbol{u}_r) - \eta(\boldsymbol{u}_l)\right]$

Uniqueness of entropy solution.

Theorem

If the piecewise continue functions $\boldsymbol{u}$ and v are weak entropy solutions of scalar conservation. Then

$$||\boldsymbol{u}(\cdot,t) - v(\cdot,t)||_{L^1(\mathbb{R})} \leq ||\boldsymbol{u}_0(\cdot) - v_0(\cdot)||_{L^1(\mathbb{R})} \quad \forall t \geq 0$$

As consequence, there is a unique piecewise continue weak entropy solutions of a scalar conservation law with the initial condition $\boldsymbol{u}_0(\cdot)$.

Uniqueness of entropy solution.

Total variation : $TV(\boldsymbol{u}(\cdot,t)) = \limsup_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_{\mathbb{R}}|\boldsymbol{u}(x+\varepsilon,t)-\boldsymbol{u}(x,t)|dx\right)$

Theorem

If $\boldsymbol{u}_0 \in L^1(\mathbb{R})\bigcap L^\infty(\mathbb{R})\bigcap BV(\mathbb{R})$ *and* $f \in C^1(\mathbb{R})$. *Then there is a unique entropy solution* $\boldsymbol{u}(x,t) \in L^\infty(\mathbb{R}\times\mathbb{R}^{+,*})\bigcap C^1(0,T;L^1(\mathbb{R}))$ *of the Cauchy problem, and*

$$||\boldsymbol{u}(\cdot,t)||_{L^\infty(\mathbb{R})} \leq ||\boldsymbol{u}_0(\cdot)||_{L^\infty(\mathbb{R})} \tag{4}$$

Moreover, $\boldsymbol{u}(\cdot,t) \in BV(\mathbb{R}) \quad \forall t > 0$ *and*

$$TV(\boldsymbol{u}(\cdot,t)) \leq TV(\boldsymbol{u}_0(\cdot)) \tag{5}$$

Riemann Problem

The Riemann problem consists of the "Hyperbolic" equation with the initial data defined as

$$\boldsymbol{u}_0(x) = \begin{cases} \boldsymbol{u}_l & \text{when } x < 0 \\ \boldsymbol{u}_r & \text{when } x > 0 \end{cases}$$

where $\boldsymbol{u}_l$ and $\boldsymbol{u}_r$ constants and the solution non trivial : $\boldsymbol{u}_l \neq \boldsymbol{u}_r$.

The Riemann problem is fundamental

To understanding the mathematical theory of hyperbolic problems
To derive Godunov type finite volume schemes.

If $\boldsymbol{u}(x,t)$ is a solution of the Riemann problem, then $\forall \alpha > 0$, $\boldsymbol{u}(x\alpha, t\alpha)$ is also a solution of the Riemann problem. Therefore $\boldsymbol{u}(x,t) = \boldsymbol{u}(\xi)$ with $\xi = \frac{x}{t}$.

Riemann Problem : Rarefaction wave

The aim here is to define a smooth solution of the Riemann problem. For any $t > 0$ this solution must satisfy the equation

$$\partial_t \boldsymbol{u} + f'(\boldsymbol{u})\partial_x \boldsymbol{u} = 0 \Longleftrightarrow \frac{1}{t}\Big(-\xi + f'(\boldsymbol{u})\Big)\,\partial_\xi \boldsymbol{u} = 0$$

Therefore

- either $\partial_\xi \boldsymbol{u} = 0$ and $\boldsymbol{u}(\xi)$ is constant.
- or $\partial_\xi \boldsymbol{u} \neq 0$ and

$$\mathcal{F}(\xi, \boldsymbol{u}) = f'(\boldsymbol{u}(\xi)) - \xi = 0$$

Moreover, if $f \in C^2(\mathbb{R})$

$$f"(\boldsymbol{u}(\xi))\partial_\xi \boldsymbol{u} = 1 \Longrightarrow \; f \text{ is nonlinear}$$

Riemann Problem : Rarefaction wave

$$\partial_\xi \boldsymbol{u} = 0 \quad \text{or} \quad f'(\boldsymbol{u}(\xi)) = \xi$$

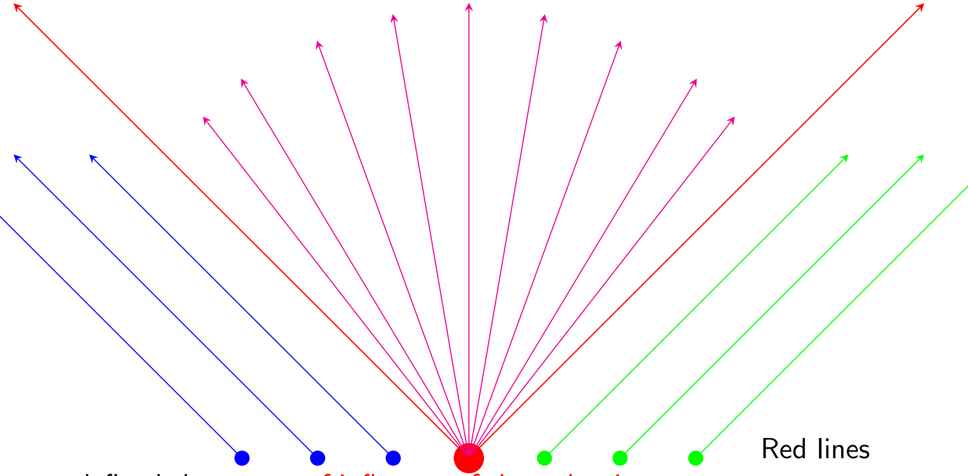
Caractérization of a single Rarefaction wave

$$\forall \xi \in \mathbb{R},\ \forall \zeta \in \mathbb{R} \quad : \quad \xi < \zeta \Longrightarrow f'(\boldsymbol{u}(\xi)) \leq f'(\boldsymbol{u}(\zeta))$$

The necessary condition to have a single rarefaction wave as solution of the Riemann problem is that

- f is nonlinear
- $f'(\boldsymbol{u}_l) \leq f'(\boldsymbol{u}_r)$
- $f'(\boldsymbol{u})$ is a monoton function, at least in the interval $(\boldsymbol{u}_l, \boldsymbol{u}_r)$

Riemann Problem : Single Rarefaction wave



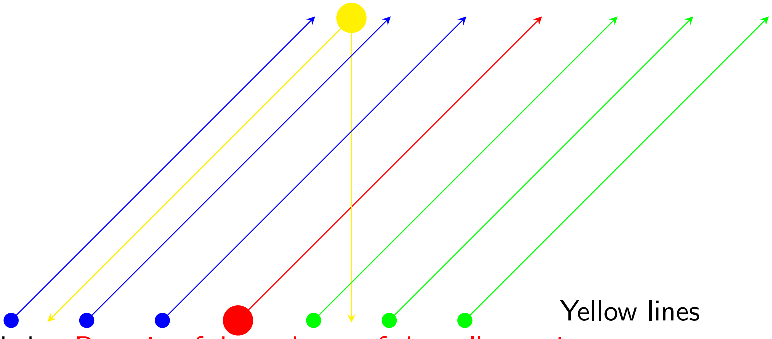
Red lines

defined the range of influence of the red point .

Riemann Problem : Single Contact Discontinuity ($m = 0$)

$$m = f(\boldsymbol{u}_r) - s\boldsymbol{u}_r = f(\boldsymbol{u}_l) - s\boldsymbol{u}_l = 0 \Longrightarrow f(\boldsymbol{u}) \equiv s\boldsymbol{u}$$

f is linear type

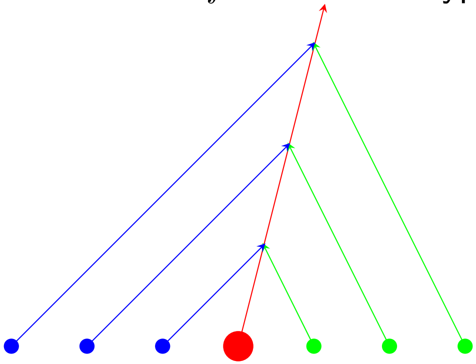


Yellow lines defined the Domain of dependence of the yellow point .

Riemann Problem : Shock wave ($m \neq 0$)

$$m = f(\boldsymbol{u}_r) - s\boldsymbol{u}_r = f(\boldsymbol{u}_l) - s\boldsymbol{u}_l \neq 0$$

$\Longrightarrow f$ is nonlinear type



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