Lectures Références

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Numerical Methods for PDE: Finite Differences and Finites Volumes

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2009

1. Finite Difference(FD) and Finite volume(FV) : Overview
2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions

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Finite volume approximation : principle

- Space decomposition in contrôl volumes : $\Omega = \bigcup_{i=1}^{N} \mathcal{C}_i$. where $\mathcal{C}_i$ are non interserting volumes (or Cells).
- Projection of the initial solution on the space of functions constants on cells :

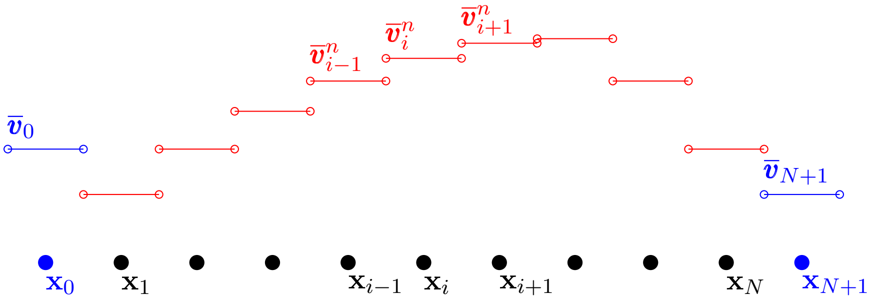
$$\overline{\boldsymbol{v}}_i^0 = \frac{1}{a_i} \int_{\mathcal{C}_i} \boldsymbol{u}_0(x) dx \quad \text{where} \quad a_i = \int_{\mathcal{C}_i} dx$$

- Integrate the conservative law on $(t^n, t^{n+1}) \times \mathcal{C}_i$:

$$\int_{t^n}^{t^{n+1}} \int_{\mathcal{C}_i} \left(\partial_t \boldsymbol{u} + \partial_x \left[f(\boldsymbol{u}) \right] \right) dx\ dt = 0$$

- Define $\overline{\boldsymbol{v}}_i^{n+1} = \dfrac{1}{a_i} \displaystyle\int_{\mathcal{C}_i} \boldsymbol{u}(t^{n+1}, x) dx$ using $\overline{\boldsymbol{v}}_i^n$ and approximated flux on cells interfaces.

1D mesh for the discretization (approximated solution)



Finite volume approximation 1D : $\mathcal{C}_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$

$$\int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\partial_t \boldsymbol{u} + \partial_x \left[f(\boldsymbol{u})\right]\right) dx\ dt = 0$$

$$\Longrightarrow a_i \overline{\boldsymbol{u}}_i^{n+1} - a_i \overline{\boldsymbol{u}}_i^{n} + \int_{t^n}^{t^{n+1}} f\left[\boldsymbol{u}(t, x_{i+\frac{1}{2}})\right] dt - \int_{t^n}^{t^{n+1}} f\left[\boldsymbol{u}(t, x_{i-\frac{1}{2}})\right] dt = 0$$

Finite volume : Conservative scheme

$$\frac{\overline{\boldsymbol{v}}_i^{n+1} - \overline{\boldsymbol{v}}_i^{n}}{t^{n+1} - t^n} + \frac{\boldsymbol{\phi}_{i+\frac{1}{2}} - \boldsymbol{\phi}_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = 0$$

where $\boldsymbol{\phi}_{i\pm\frac{1}{2}} \simeq \dfrac{1}{t^{n+1} - t^n} \displaystyle\int_{t^n}^{t^{n+1}} f\left[\boldsymbol{u}(t, x_{i\pm\frac{1}{2}})\right] dt$

Conservative schemes : General framework

$$\overline{\boldsymbol{v}}_i^{n+1} - \overline{\boldsymbol{v}}_i^n + \frac{\delta t}{\delta x_i}\left(\boldsymbol{\phi}_{i+\frac{1}{2}} - \boldsymbol{\phi}_{i-\frac{1}{2}}\right) = 0$$

$$\boldsymbol{\phi}_{i\pm\frac{1}{2}} \equiv \boldsymbol{\phi}_{i\pm\frac{1}{2}}\left(\overline{\boldsymbol{v}}\right) \simeq \frac{1}{\delta t}\int_{t^n}^{t^n+\delta t} f\left[\boldsymbol{u}(t, x_{i\pm\frac{1}{2}})\right] dt$$

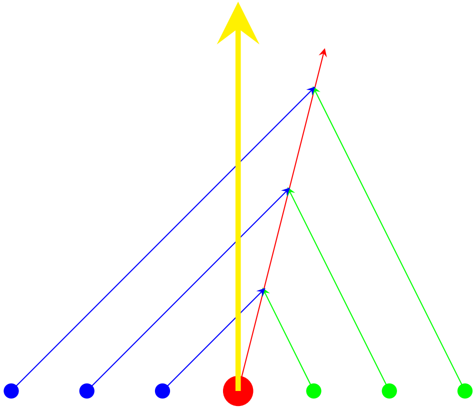
Examples :

- Explicit centered scheme :

$$\boldsymbol{\phi}_{i\pm\frac{1}{2}} \equiv \boldsymbol{\phi}\left(\overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}-\frac{1}{2}}, \overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}+\frac{1}{2}}\right) = \frac{f\left(\overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}-\frac{1}{2}}\right) + f\left(\overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}+\frac{1}{2}}\right)}{2}$$

- Explicit Godunov schemes : $\boldsymbol{\phi}_{i\pm\frac{1}{2}} \equiv \boldsymbol{\phi}\left(\overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}-\frac{1}{2}}, \overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}+\frac{1}{2}}\right) =$ $f\left[\mathcal{R}\left(\overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}-\frac{1}{2}}, \overline{\boldsymbol{v}}^n_{i\pm\frac{1}{2}+\frac{1}{2}}, \sigma_{i\pm\frac{1}{2}} = 0\right)\right]$

Godunov Numerical flux : $\overline{\boldsymbol{v}}_{i+\frac{1}{2}} = \mathcal{R}(\overline{\boldsymbol{v}}_i^n, \overline{\boldsymbol{v}}_{i+1}^n, \sigma = 0)$



$$\sigma = \frac{x - x_{i+\frac{1}{2}}}{t - t^n}$$

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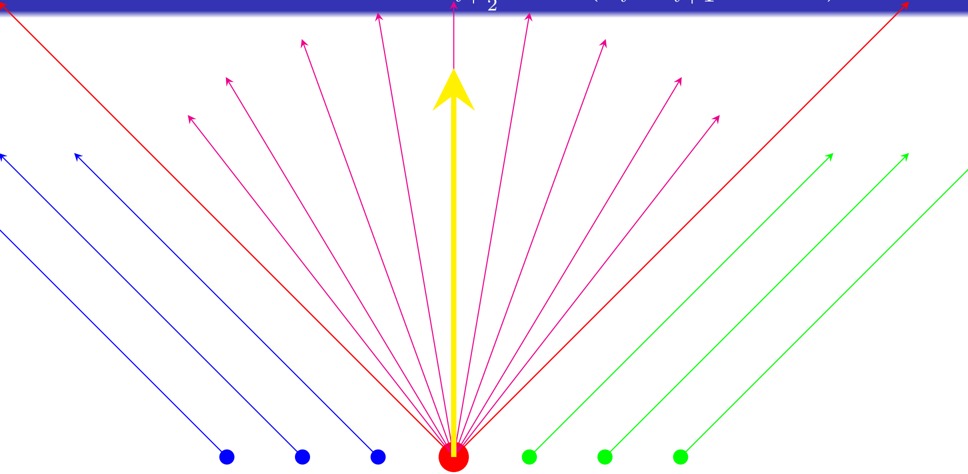

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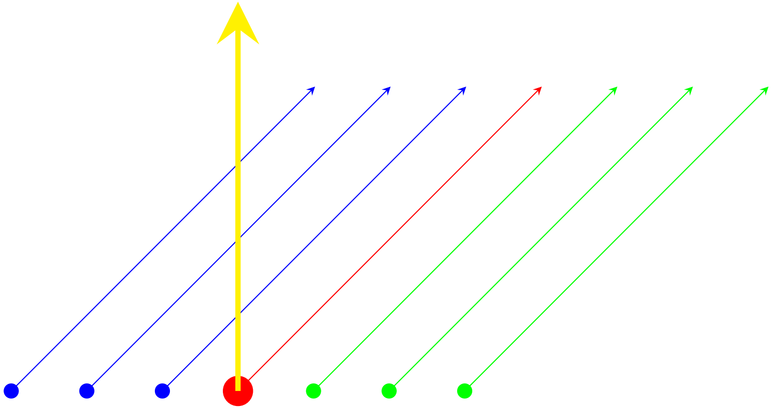

$$\sigma = \frac{x - x_{i+\frac{1}{2}}}{t - t^n}$$

Numerical schemes of 3 points : $\phi_{i\pm\frac{1}{2}} = \phi\left(\overline{v}_{i\pm\frac{1}{2}-\frac{1}{2}}, \overline{v}_{i\pm\frac{1}{2}+\frac{1}{2}}\right)$

- Lax-Friedrish scheme :

$$\phi^{LF}(\overline{v}_i, \overline{v}_{i+1}) = \frac{1}{2}\left[f(\overline{v}^n_{i+1}) + f(\overline{v}^n_i) - \frac{\delta x}{\delta t}(\overline{v}_{i+1} - \overline{v}_i)\right]$$

- Murman-Roe scheme

$$\phi^{MR}(\overline{v}_i, \overline{v}_{i+1}) = \frac{1}{2}\left(f(\overline{v}^n_{i+1}) + f(\overline{v}^n_i) - |\beta^n_{i+\frac{1}{2}}|\,(\overline{v}^n_{i+1} - \overline{v}^n_i)\right)$$

- Lax-Wendrof scheme

$$\phi^{LW}_{i+\frac{1}{2}} = \frac{1}{2}\left[f(\overline{v}^n_{i+1}) + f(\overline{v}^n_i) - \frac{\delta t}{\delta x}\beta^n_{i+\frac{1}{2}}\left(f(\overline{v}^n_{i+1}) - f(\overline{v}^n_i)\right)\right]$$

where $\beta_{i+\frac{1}{2}} = \beta(\overline{v}_i, \overline{v}_{i+1}) = \dfrac{f(\overline{v}^n_{i+1}) - f(\overline{v}^n_i)}{\overline{v}^n_{i+1} - \overline{v}^n_i}$

General case : Numerical schemes of $p+q+1$ points

$$\overline{\boldsymbol{v}}_i^{n+1} = \overline{\boldsymbol{v}}_i^n - \frac{\delta t}{\delta x}\left(\boldsymbol{\phi}_{i+\frac{1}{2}} - \boldsymbol{\phi}_{i-\frac{1}{2}}\right) = 0$$

where

$$\boldsymbol{\phi}_{i\pm\frac{1}{2}} = \boldsymbol{\phi}\left(\overline{\boldsymbol{v}}_{i\pm\frac{1}{2}-\frac{1}{2}-p}, \cdots, \overline{\boldsymbol{v}}_{i\pm\frac{1}{2}-\frac{1}{2}}, \overline{\boldsymbol{v}}_{i\pm\frac{1}{2}+\frac{1}{2}}, \cdots, \overline{\boldsymbol{v}}_{i\pm\frac{1}{2}+\frac{1}{2}+q}\right) \equiv \boldsymbol{\phi}_{i\pm\frac{1}{2}}\left(\overline{\boldsymbol{v}}\right)$$

More generally we defined

$$\overline{\boldsymbol{v}}^{n+1} = \sum_{m=0}^{s} \mathcal{G}_m\left(\delta t, \delta \mathbf{x}, \overline{\boldsymbol{v}}^{n+1-m}\right)$$

Definitions

$$\overline{\boldsymbol{v}}^{n+1} = \sum_{m=0}^{s} \mathcal{G}_m \left(\delta t, \delta \mathbf{x}, \overline{\boldsymbol{v}}^{n+1-m}\right)$$

- Truncation error : it is a vector $\boldsymbol{\mathcal{E}}$ of size N defined by

$$\boldsymbol{\mathcal{E}}^{n+1} = \frac{\overline{\boldsymbol{u}}^{n+1} - \displaystyle\sum_{m=0}^{s} \mathcal{G}_m \left(\delta t, \delta \mathbf{x}, \overline{\boldsymbol{u}}^{n+1-m}\right)}{\delta t}$$

- Approximation error : it is a vector $\boldsymbol{e}$ of size N defined by

$$\boldsymbol{e}^{n+1} = \overline{\boldsymbol{u}}(t^n+\delta t)-\overline{\boldsymbol{v}}^{n+1} = \overline{\boldsymbol{u}}(t^n+\delta t)-\sum_{m=0}^{s} \mathcal{G}_m \left(\delta t, \delta \mathbf{x}, \overline{\boldsymbol{v}}^{n+1-m}\right)$$

Definitions

- Consistency and Accurecy : A numerical scheme is consistent of p'th order accurate in time and k'th order accurate in space if

$$\|\boldsymbol{\mathcal{E}}^{n+1}\| = O(\delta t^p) + O(\delta \mathbf{x}^k) \quad \text{with } p > 0 \text{ and } k > 0$$

- Stability : the scheme is stable for a given norm $\|\|$ if, for any time $T > 0$, $\exists M \in \mathbb{N}^*$, $\exists \delta t_0 \in \mathbb{R}^+$, $\exists \delta \mathbf{x}_0 \in \mathbb{R}^+$ and a constant C_T such that $\|\overline{\boldsymbol{v}}^{n+1}\| \leq C_T \sum_{m=0}^{M} \|\overline{\boldsymbol{v}}^m\|$

- Linear Stability :

$$\|\overline{\boldsymbol{v}}^{n+1}\| \leq (1 + \alpha \delta h)\|\overline{\boldsymbol{v}}^n\|$$

- Convergence : for any $N_* = \frac{T}{\delta t}$ we have

$$\lim_{\delta h \to 0} ||\boldsymbol{e}^{N_*}|| = 0.$$

Consistency for conservative schemes

The consistency of conservative schemes is achived when,

$$\text{if } \boldsymbol{u}_j \equiv \alpha \Longrightarrow \boldsymbol{\phi}_{i\pm\frac{1}{2}} \equiv \boldsymbol{\phi}_{i\pm\frac{1}{2}}(\boldsymbol{u}) = \frac{1}{\delta t}\int_{t^n}^{t^n+\delta t} f(\alpha)\, dt = f(\alpha)$$

Lipschitz Continuity of the numerical flux is also expected :

$$|\boldsymbol{\phi}_{i\pm\frac{1}{2}}(\overline{\boldsymbol{v}}) - f(\alpha)| \leq L\|\overline{\boldsymbol{v}} - \boldsymbol{u}\|_\infty$$

where $\boldsymbol{u}_j \equiv \alpha$ and L is the Lipschitz constant.

CFL condition : necessary condition for stability

A numerical scheme can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as δt and δx go to zero.

For a 3 points schemes it gives :

$$\frac{\delta t}{\delta x} \max_{\boldsymbol{u}} \left(|f'(\boldsymbol{u})|\right) \leq 1$$

Numerical flux and schemes : Linear case : $f(\boldsymbol{u}) = c\boldsymbol{u}$

- Explicit centered scheme :

$$\overline{\boldsymbol{v}}_i^{n+1} - \overline{\boldsymbol{v}}_i^n + \frac{c\delta t}{2\delta x}\left(\overline{\boldsymbol{v}}_{i+1}^n - \overline{\boldsymbol{v}}_{i-1}^n\right) = 0$$

- Explicit Godunov/Murman-Roe/Upwind scheme :

$$\overline{\boldsymbol{v}}_i^{n+1} - \overline{\boldsymbol{v}}_i^n + \frac{c\delta t}{2\delta x}\left(\overline{\boldsymbol{v}}_{i+1}^n - \overline{\boldsymbol{v}}_{i-1}^n\right) = \frac{|c|\delta t}{2\delta x}\left(\overline{\boldsymbol{v}}_{i+1}^n - 2\overline{\boldsymbol{v}}_i^n + \overline{\boldsymbol{v}}_{i-1}^n\right)$$

- Explicit Lax-Wendrof scheme :

$$\overline{\boldsymbol{v}}_i^{n+1} - \overline{\boldsymbol{v}}_i^n + \frac{c\delta t}{2\delta x}\left(\overline{\boldsymbol{v}}_{i+1}^n - \overline{\boldsymbol{v}}_{i-1}^n\right) = \frac{c^2\delta t^2}{2\delta x^2}\left(\overline{\boldsymbol{v}}_{i+1}^n - 2\overline{\boldsymbol{v}}_i^n + \overline{\boldsymbol{v}}_{i-1}^n\right)$$

Truncation error : Linear case : $f(\boldsymbol{u}) = c\boldsymbol{u}$

Use $\partial_t \boldsymbol{u} = -c\partial_x \boldsymbol{u}$

- Explicit centered scheme :

$$\mathcal{E}_i^{n+1} = \frac{c^2\delta t}{2}\partial_x^2\boldsymbol{u} - \frac{c^3\delta t^2}{6}\partial_x^3\boldsymbol{u} + \frac{c\delta x^2}{6}\partial_x^3\boldsymbol{u} + 0(\delta t^3) + 0(\delta x^4)$$

- Explicit Godunov/Murman-Roe/Upwind scheme :

$$\mathcal{E}_i^{n+1} = \frac{|c|\delta x}{2}\left(\frac{|c|\delta t}{\delta x} - 1\right)\partial_x^2\boldsymbol{u} + 0(\delta t^2) + 0(\delta x^3)$$

- Explicit Lax-Wendrof scheme :

$$\mathcal{E}_i^{n+1} = -\frac{|c|\delta x^2}{6}\left(\left(\frac{|c|\delta t}{\delta x}\right)^2 - 1\right)\partial_x^3\boldsymbol{u} + 0(\delta t^3) + 0(\delta x^3)$$

Consistancy + Stability $\Longrightarrow$ Convergence.

Case of a scheme defined by $\overline{\boldsymbol{v}}^{n+1} = \mathcal{N}(\overline{\boldsymbol{v}}^n)$ where

$$\mathcal{N}_i(\overline{\boldsymbol{v}}^n) = \overline{\boldsymbol{v}}_i^n - \frac{\delta t}{\delta x}\left(\boldsymbol{\phi}_{i+\frac{1}{2}}(\overline{\boldsymbol{v}}^n) - \boldsymbol{\phi}_{i-\frac{1}{2}}(\overline{\boldsymbol{v}}^n)\right)$$

$$\boldsymbol{e}^{n+1} = \overline{\boldsymbol{u}}^{n+1} - \mathcal{N}(\overline{\boldsymbol{v}}^n) = \mathcal{N}(\overline{\boldsymbol{u}}^n) - \mathcal{N}(\overline{\boldsymbol{v}}^n) + \overline{\boldsymbol{u}}^{n+1} - \mathcal{N}(\overline{\boldsymbol{u}}^n)$$

$$\boldsymbol{e}^{n+1} = \mathcal{N}(\overline{\boldsymbol{u}}^n) - \mathcal{N}(\overline{\boldsymbol{v}}^n) + \delta t \mathcal{E}^{n+1}$$

- Stability (contractive schemes) $||\mathcal{N}(\overline{\boldsymbol{v}}) - \mathcal{N}(\boldsymbol{u})|| \leq ||\overline{\boldsymbol{v}} - \boldsymbol{u}||$
- More general need : $||\mathcal{N}(\overline{\boldsymbol{v}}) - \mathcal{N}(\boldsymbol{u})|| \leq (1 + \alpha\delta t)||\overline{\boldsymbol{v}} - \boldsymbol{u}||$

Therefore

$$||\boldsymbol{e}^{n+1}|| \leq (1 + \alpha\delta t)||\boldsymbol{e}^n|| + \delta t||\mathcal{E}^{n+1}||$$

Assuming that $||\mathcal{E}^{n+1}|| \leq \mathcal{E}$ for all n,

$$||\boldsymbol{e}^{n+1}|| \leq \exp(\alpha T)\left(||\boldsymbol{e}^0|| + T||\mathcal{E}||\right)$$

Definitions

Case of conservatif scheme defined by $\overline{\boldsymbol{v}}^{n+1} = \mathcal{N}(\overline{\boldsymbol{v}}^n)$ where

$$\mathcal{N}_i(\overline{\boldsymbol{v}}^n) = \overline{\boldsymbol{v}}_i^n - \frac{\delta t}{\delta x}\left(\boldsymbol{\phi}_{i+\frac{1}{2}}(\overline{\boldsymbol{v}}^n) - \boldsymbol{\phi}_{i-\frac{1}{2}}(\overline{\boldsymbol{v}}^n)\right)$$

- Consistent schemes

$$\boldsymbol{\phi}\left(\boldsymbol{u}, ..., \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}, ..., \boldsymbol{u}\right) = f(\boldsymbol{u}).$$

- Monotone schemes

$$\frac{\partial \mathcal{N}_i(\overline{\boldsymbol{v}})}{\partial \overline{\boldsymbol{v}}_j} \geq 0 \qquad \forall i, j$$

- TVD (Total Variation Disminishing) schemes

$$||\overline{\boldsymbol{v}}^{n+1}||_{BV} \leq ||\overline{\boldsymbol{v}}^n||_{BV} \quad \text{where} \quad ||\overline{\boldsymbol{v}}||_{BV} = \sum_j |\overline{\boldsymbol{v}}_{j+1} - \overline{\boldsymbol{v}}_j|$$

Entropy consistent schemes

A scheme is entropy consistent if, for any convex entropy $\boldsymbol{\eta}$ and the associated flux $\boldsymbol{\Psi}(\boldsymbol{\eta})$, there exist a numerical entropy flux $\Theta_{i+\frac{1}{2}}$ locally lipchitz,

$$\Theta_{i+\frac{1}{2}} = \Theta\left(\overline{\boldsymbol{v}}_{i-p}, ..., \overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_{i+1}, ..., \overline{\boldsymbol{v}}_{i+q}\right),$$

such that

$$\begin{cases} \Theta\left(\boldsymbol{\eta}(\boldsymbol{u}), ..., \boldsymbol{\eta}(\boldsymbol{u}), \boldsymbol{\eta}(\boldsymbol{u}), ..., \boldsymbol{\eta}(\boldsymbol{u})\right) = \boldsymbol{\Psi}(\boldsymbol{\eta}) \\ \boldsymbol{\eta}\left(\overline{\boldsymbol{v}}_i^{n+1}\right) - \boldsymbol{\eta}\left(\overline{\boldsymbol{v}}_i^{n}\right) + \dfrac{\delta t}{\delta x}\left(\Theta_{i+\frac{1}{2}} - \Theta_{i-\frac{1}{2}}\right) \leq 0 \end{cases}$$

Fondamental Theorem

$$\overline{\boldsymbol{v}}_h(x,t) = \overline{\boldsymbol{v}}_i^n \text{ for } (x,t) \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [t^n, t^n + \delta t)$$

Lax-Wendrof Theorem

Let us consider a conservative and consistent scheme and assume that

- $h = \delta x \rightarrow 0$ with $\frac{\delta t}{\delta x}$ constant.
- $\|\overline{\boldsymbol{v}}_h(x,t)\|_{L^\infty(\mathbb{R}\times(0,+\infty))} \leq C$
- $\overline{\boldsymbol{v}}_h(x,t) \longrightarrow \overline{\boldsymbol{v}}_*(x,t)$ in $L^1_{loc}(\mathbb{R} \times (0,+\infty))$ when $h \longrightarrow 0$.

Then $\overline{\boldsymbol{v}}_*(x,t)$ is a weak solution of the conservation law.

Moreover, if the numerical scheme is entropy consistent, $\overline{\boldsymbol{v}}_*(x,t)$ is a weak entropy solution of the conservation law.

proof

Let φ be a C_1^0 test function with compact support and $\varphi_i^n = \varphi(x_j, t^n)$ multiply the conservative numerical scheme with φ_i^n and sum overall i and n

$$\delta t \delta x \sum_{n=0}^{\infty} \sum_{i=-\infty}^{+\infty} \frac{\overline{\boldsymbol{v}}_i^{n+1} - \overline{\boldsymbol{v}}_i^n}{\delta t} \varphi_i^n = -\delta t \delta x \sum_{n=0}^{\infty} \sum_{i=-\infty}^{+\infty} \frac{\boldsymbol{\phi}_{i+\frac{1}{2}} - \boldsymbol{\phi}_{i-\frac{1}{2}}}{\delta x} \varphi_i^n$$

$$-\delta x \sum_{i=-\infty}^{+\infty} \overline{\boldsymbol{v}}_i^0 \varphi_i^0 - \delta t \delta x \sum_{n=0}^{\infty} \sum_{i=-\infty}^{+\infty} \frac{\varphi_i^{n+1} - \varphi_i^n}{\delta t} \overline{\boldsymbol{v}}_i^{n+1}$$
$$= \delta t \delta x \sum_{n=0}^{\infty} \sum_{i=-\infty}^{+\infty} \frac{\varphi_{i+1}^n - \varphi_i^n}{\delta x} \boldsymbol{\phi}_{i-\frac{1}{2}}$$

$$\int_0^{\infty} dt \int_{-\infty}^{\infty} dx \ \left(\overline{\boldsymbol{v}}_* \partial_t \phi + f(\overline{\boldsymbol{v}}_*) \partial_x \phi \right) = - \int_{-\infty}^{\infty} dx \ u_0(x) \phi(x, 0)$$

Usefull form of explicit conservatif schemes

Define

$$\Delta \boldsymbol{u}_m^n = \boldsymbol{u}_{m+\frac{1}{2}}^n - \boldsymbol{u}_{m-\frac{1}{2}}^n$$

- The incremental form of a conservatif scheme is

$$\overline{\boldsymbol{v}}_i^{n+1} = \overline{\boldsymbol{v}}_i^n + \mathcal{C}_{i+\frac{1}{2}} \Delta \overline{\boldsymbol{v}}_{i+\frac{1}{2}}^n - \mathcal{D}_{i-\frac{1}{2}} \Delta \overline{\boldsymbol{v}}_{i-\frac{1}{2}}^n$$

  where $\mathcal{C}_{i-\frac{1}{2}}$ and $\mathcal{D}_{i+\frac{1}{2}}$ are incremental coefficients.

- The viscosity form of a conservatif scheme is

$$\overline{\boldsymbol{v}}_i^{n+1} = \overline{\boldsymbol{v}}_i^n - \frac{\delta t}{2\delta x} \left(f(\overline{\boldsymbol{v}}_{i+1}^n) - f(\overline{\boldsymbol{v}}_{i-1}^n) \right) + \frac{1}{2} \Delta \left(\mathcal{Q}_i \Delta \overline{\boldsymbol{v}}_i^n \right)$$

  where $\Delta \left(\mathcal{Q}_i \Delta \overline{\boldsymbol{v}}_i^n \right) = \mathcal{Q}_{i+\frac{1}{2}} \Delta \overline{\boldsymbol{v}}_{i+\frac{1}{2}}^n - \mathcal{Q}_{i-\frac{1}{2}} \Delta \overline{\boldsymbol{v}}_{i-\frac{1}{2}}^n$

Explicit conservatif schemes that can be writen in viscous form

- An explicit consistent conservatif scheme is TVD when

$$\frac{\delta t}{\delta x}\left|\frac{\Delta f^n_{i+\frac{1}{2}}}{\Delta \overline{\boldsymbol{v}}^n_{i+\frac{1}{2}}}\right| \leq \mathcal{Q}_{i+\frac{1}{2}} \leq 1$$

- An explicit consistent conservatif scheme is TVD and L^∞ stable when

$$\frac{\delta t}{\delta x}\left|\frac{\Delta f^n_{i+\frac{1}{2}}}{\Delta \overline{\boldsymbol{v}}^n_{i+\frac{1}{2}}}\right| \leq \mathcal{Q}_{i+\frac{1}{2}} \leq \frac{1}{2}$$

Conservatif Consistent Monotone (CCM)

$$\frac{\partial \mathcal{N}_i(\overline{\boldsymbol{v}})}{\partial \overline{\boldsymbol{v}}_j} \geq 0 \qquad \forall i, j$$

These schemes are :

- Entropy consistent.
- L1-contractant : $||\overline{\boldsymbol{v}}_i^{n+1} - \tilde{\overline{\boldsymbol{v}}}_i^{n+1}||_1 < ||\overline{\boldsymbol{v}}_i^{n} - \tilde{\overline{\boldsymbol{v}}}_i^{n}||_1$
- TVD : $||\overline{\boldsymbol{v}}^{n+1}||_{BV} \leq ||\overline{\boldsymbol{v}}^{n}||_{BV}$.
- Monotony preserving
- At most first order accurate. ordre.

3-points schemes $\overline{\boldsymbol{v}}_i^{n+1} = \mathcal{N}_i(\overline{\boldsymbol{v}}^n)$: Lax-Friedrichs

$$\mathcal{N}_i(\overline{\boldsymbol{v}}) = \frac{\overline{\boldsymbol{v}}_{i-1} + \overline{\boldsymbol{v}}_{i+1}}{2} - \frac{\delta t}{2\delta \mathbf{x}} \left(f(\overline{\boldsymbol{v}}_{i+1}) - f(\overline{\boldsymbol{v}}_{i-1}) \right)$$

- Numerical flux $\boldsymbol{\phi}^{LF}(\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_{i+1}) = \frac{f(\overline{\boldsymbol{v}}_i)+f(\overline{\boldsymbol{v}}_{i+1})}{2} + \frac{\delta \mathbf{x}}{2\delta t}(\overline{\boldsymbol{v}}_i - \overline{\boldsymbol{v}}_{i+1})$
- Therefore, the scheme is conservative, consistent and the diffusion coefficient is : $\mathcal{Q}^{LF}_{i+\frac{1}{2}} = 1$.
- Under the CFL condition $\frac{\delta t}{\delta x} \max_{\boldsymbol{u} \in \mathbb{R}} \left(|f'(\boldsymbol{u})| \right) \leq 1$ the scheme is
  - Monotone : $\frac{\partial \mathcal{N}_i}{\partial \overline{\boldsymbol{v}}_j} \geq 0$ as $\frac{\partial \mathcal{N}_i}{\partial \overline{\boldsymbol{v}}_{i\pm1}} = \frac{1}{2}\left[1 \mp \frac{\delta t}{\delta \mathbf{x}} f'(\overline{\boldsymbol{v}}_{i\pm1}) \right]$
  - L^1-contractant : $||\mathcal{N}(\overline{\boldsymbol{v}}) - \mathcal{N}(\overline{\boldsymbol{u}})||_1 < ||\overline{\boldsymbol{v}} - \overline{\boldsymbol{u}}||_1$.
    It use the property $f(\alpha) - f(\beta) = f'(\xi)(\alpha - \beta)$ when f is C1.
- This scheme is conservatif consistent monotone
  Then it is TVD, positif, Entropy consistent
  But of order One.

3-points schemes $\overline{\boldsymbol{v}}_i^{n+1} = \mathcal{N}_i(\overline{\boldsymbol{v}}^n)$: Murman-Roe

$$\mathcal{N}_i(\overline{\boldsymbol{v}}) = \overline{\boldsymbol{v}}_i - \frac{\delta t}{2\delta \mathbf{x}}\left[f(\overline{\boldsymbol{v}}_{i+1}) - f(\overline{\boldsymbol{v}}_{i-1})\right] + \frac{\delta t}{2\delta \mathbf{x}}\left(|\beta^n_{i+\frac{1}{2}}|\,(\overline{\boldsymbol{v}}_{i+1} - \overline{\boldsymbol{v}}_i) - |\beta^n_{i-\frac{1}{2}}|\,(\overline{\boldsymbol{v}}_i - \overline{\boldsymbol{v}}_{i-1})\right)$$

where $\mathcal{A}_{i+\frac{1}{2}} = \mathcal{A}\left(\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_{i+1}\right) = |\Delta f_{i+\frac{1}{2}}|/|\Delta \overline{\boldsymbol{v}}_{i+\frac{1}{2}}|$

- Flux :

$$\boldsymbol{\phi}^{MR}(\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_{i+1}) = \frac{1}{2}\left(f^n_{i+1} + f^n_i + |\mathcal{A}^n_{i+\frac{1}{2}}|\,(\overline{\boldsymbol{v}}^n_{i+1} - \overline{\boldsymbol{v}}^n_i)\right)$$

- Viscosity : $\mathcal{Q}^{MR}_{i+\frac{1}{2}} = \dfrac{\delta t}{\delta x}|\mathcal{A}^n_{i+\frac{1}{2}}| = \dfrac{\delta t}{\delta x}\left|\dfrac{\Delta f^n_{i+\frac{1}{2}}}{\Delta \overline{\boldsymbol{v}}^n_{i+\frac{1}{2}}}\right|$
- This scheme is conservatif, consistent, TDV, non monotone, first order accurate.

Lax-Wendroff Scheme

$$\overline{\boldsymbol{v}}_i^{n+1} = \overline{\boldsymbol{v}}_i^n - \frac{\frac{\delta t}{\delta x}}{2}\left(f_{i+1}^n - f_{i-1}^n\right)$$
$$+\frac{\left(\frac{\delta t}{\delta x}\right)^2}{2}\left(\mathcal{A}_{i+\frac{1}{2}}^n\left(f_{i+1}^n - f_i^n\right) - \mathcal{A}_{i-\frac{1}{2}}^n\left(f_i^n - f_{i-1}^n\right)\right)$$

- Flux :

$$\boldsymbol{\phi}^{LW}\left(\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_{i+1}\right) = \frac{1}{2}\left(f_{i+1}^n + f_i^n - \frac{\delta t}{\delta x}\mathcal{A}_{i+\frac{1}{2}}^n\left(f_{i+1}^n - f_i^n\right)\right)$$

- Viscosity : $\mathcal{Q}_{i+\frac{1}{2}}^{LW} = \left(\frac{\delta t}{\delta x}\right)^2 |\mathcal{A}_{i+\frac{1}{2}}^n|^2 = \left(\frac{\delta t}{\delta x}\right)^2 \left|\frac{\Delta f_{i+\frac{1}{2}}^n}{\Delta \overline{\boldsymbol{v}}_{i+\frac{1}{2}}^n}\right|^2$
- This scheme is conservatif, consistent, non TDV, non monotone, second order accurate.

Godunov Scheme

$$\begin{aligned}\overline{\boldsymbol{v}}_i^{n+1} &= \frac{1}{\delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathcal{U}^h(x,t^{n+1})dx \\ &= \frac{1}{\delta x}\int_{x_{i-\frac{1}{2}}}^{x_i} \mathcal{U}\left(\overline{\boldsymbol{v}}_{i-1}^n, \overline{\boldsymbol{v}}_i^n, \xi_{i-\frac{1}{2}}\right)dx \\ &\quad + \frac{1}{\delta x}\int_{x_i}^{x_{i+\frac{1}{2}}} \mathcal{U}\left(\overline{\boldsymbol{v}}_i^n, \overline{\boldsymbol{v}}_{i+1}^n, \xi_{i+\frac{1}{2}}\right)dx\end{aligned}$$

$\mathcal{U}(\overline{\boldsymbol{v}}_l, \overline{\boldsymbol{v}}_r, \xi)$ is the entropy solution of the Riemann problem and $\xi_m = \xi_m(x) = \frac{x-x_m}{t^{n+1}-t^n}$.

$$\mathcal{U}^h(x,t) = \mathcal{U}\left(\overline{\boldsymbol{v}}_{i-1}^n, \overline{\boldsymbol{v}}_i^n, \xi_{i-\frac{1}{2}}\right) \quad \text{for} \quad \frac{\delta t}{\delta x}\max_{\boldsymbol{u}}\left(|f'(\boldsymbol{u})|\right) \le 1,$$

$t^n \le t < t^{n+1}$ and $\xi_{i-\frac{1}{2}}(x_{i-1}) \le \xi_{i-\frac{1}{2}}(x) \le \xi_{i-\frac{1}{2}}(x_i)$

Godunov Scheme

$$\partial_t \mathcal{U}^h + \partial_x f(\mathcal{U}^h) = 0, \quad \text{and} \quad \partial_t \boldsymbol{\eta}(\mathcal{U}^h) + \partial_x \boldsymbol{\Psi}(\mathcal{U}^h) \leq 0.$$

$\mathcal{U}^h$ is monotone.
Flux : $\boldsymbol{\phi}^G(\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_{i+1}) = f\left(\mathcal{U}_{i+\frac{1}{2}}\right)$ where

$$\mathcal{U}_{i+\frac{1}{2}} = \mathcal{U}\left(\overline{\boldsymbol{v}}_i^n, \overline{\boldsymbol{v}}_{i+1}^n, \xi_{i+\frac{1}{2}}(x_{i+\frac{1}{2}})\right)$$

Entropy Flux : $\boldsymbol{\Psi}\left(\mathcal{U}_{i+\frac{1}{2}}\right)$.
This scheme is conservatif consistent monotone
Then it is TVD, positif, Entropy consistent
But of order One.

Plan

1. Finite Difference(FD) and Finite volume(FV) : Overview
2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions