Numerical Methods for PDE: Finite Differences and Finites Volumes

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Lectures Références:
Roger Peyret (NICE ESSI : 89),
Tim Warburton (Boston MIT : 03-05),
Pierre Charrier (Bordeaux Matmeca 96-08)

Plan

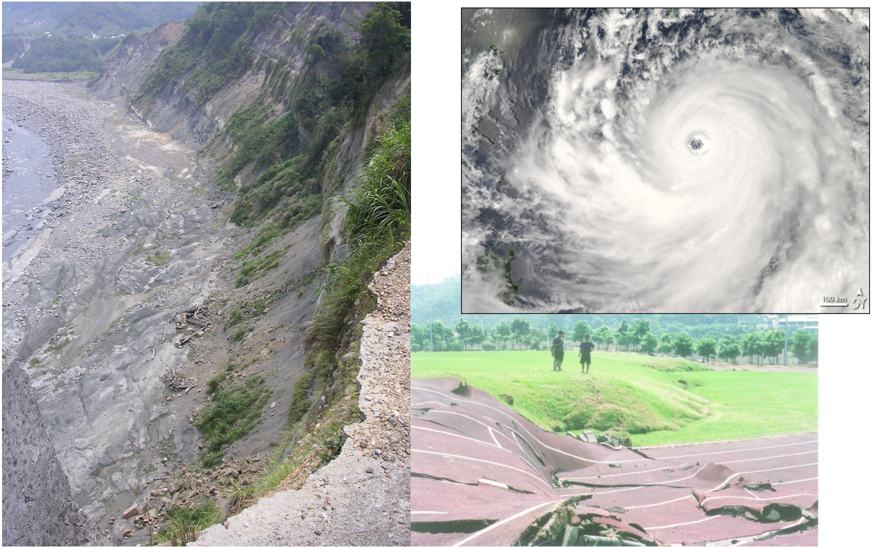
Introduction : Why we need computation ?

To enjoy and understand !

Introduction : Why we need computation ?

To prevent some humanity disaster, at least we hope so !

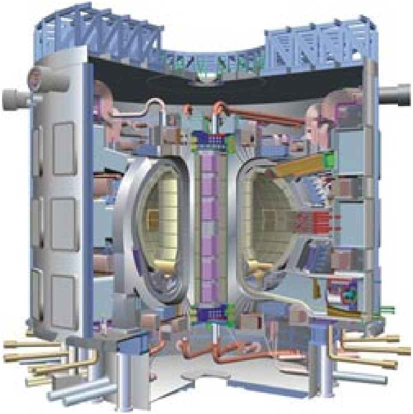
Introduction : Why we need computation ?



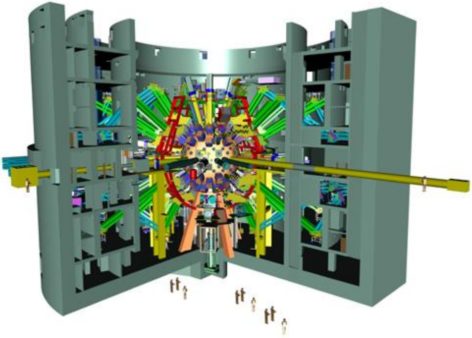
To prevent consequences of some natural disaster !

Introduction : Why we need computation ?

ITER Int. Project.



LMJ & NIF : Laser/plasmas



To help meet mankind's future energy needs : Fusion

Are computations possibles ?

Yes, Because we have

Fundamental Laws

Conservation of mass, Momentum, Energy.
Laws of the Thermodynamics : (1st, 2nd, 3th) Gauss, Ampère's, Faraday Laws

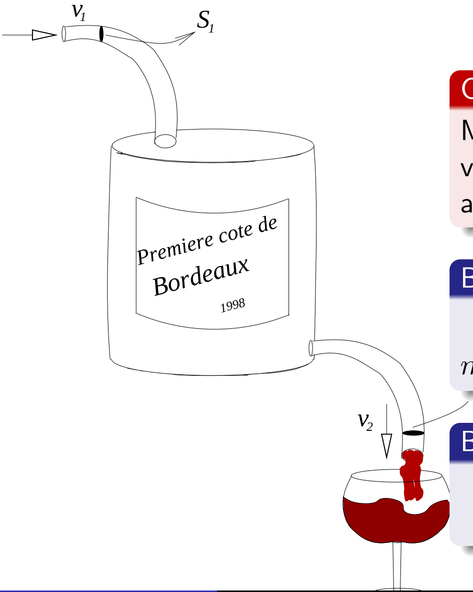
Computers are efficient

Tera-FLOPS computers availables 10^{12} FLoating point Operations Per Second.

Numerical approximation strategies

Finite volume, Finite element, Finite difference, Particles In Cells, ..., structured/unstructured mesh, Parallel programming (MPI),

Balance "relations" or equations !



Conservation Law : for Mass

Mass fluctuation, in a control volume, is the sum of outgoing and incoming mass :

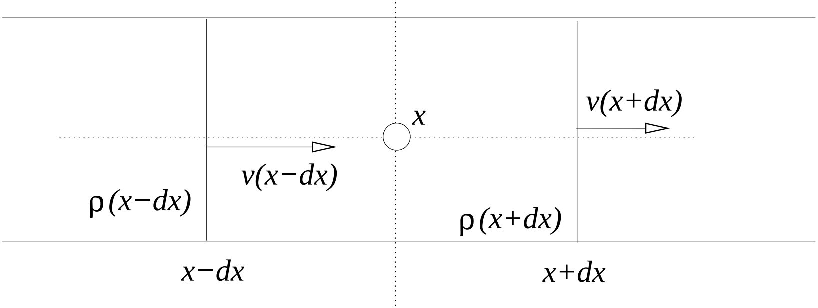
Balance Relation

$$m(t+dt) = m(t) + \rho_1 S_1 v_1 - \rho_2 S_2 v_2$$

Balance equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

Balance "relations" or equations !

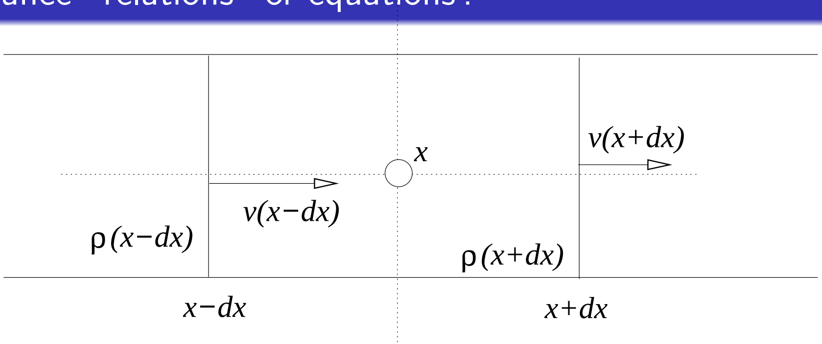


From relations to equations

$$m(x,t+dt) = m(x,t) \quad +S\int_t^{t+dt} f(x-dx,s)ds$$
$$-S\int_t^{t+dt} f(x+dx,s)ds$$

where the flux is defined by $f(x,t) = \rho(x,t)v(x,t)$

Balance “relations” or equations !



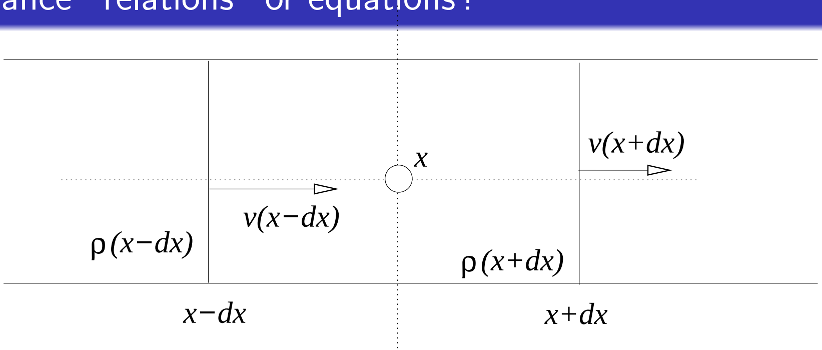
From relations to equations

$$m(x,t+dt) = m(x,t) + S dt f(x-dx,t) - S dt f(x+dx,t)$$

$$m(x,.) = 2 S dx \rho(x,.)$$

where the flux is defined by $f(x,t) = \rho(x,t) v(x,t)$

Balance "relations" or equations !



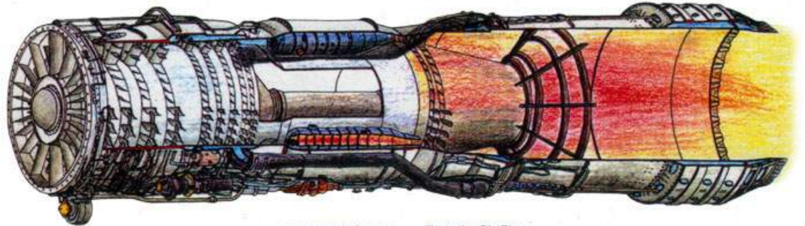
From relations to equations $dx \to 0$ and $dt \to 0$

$$\frac{\rho(x,t+dt)-\rho(x,t)}{dt} = -\frac{f(x+dx,t)-f(x-dx,t)}{2dx}$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0 \Longrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0$$

where the flux is defined by $f(x,t) = \rho(x,t)v(x,t)$

Balance equations : general 1D case



www.aerostories.org Illustration Ph. Ricco

$$\frac{\partial \omega}{\partial t} + \frac{\partial f(\omega)}{\partial x} = \mathcal{S}$$

where $\mathcal{S}$ can be defined, for example, by the chemistry process, geometrical topology ...

$$\omega(x,t) = \begin{pmatrix} \rho \\ \rho Y \\ \rho u \\ E \end{pmatrix} \quad \text{and} \quad f(\omega) = \begin{pmatrix} \rho u \\ \rho Y u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}$$

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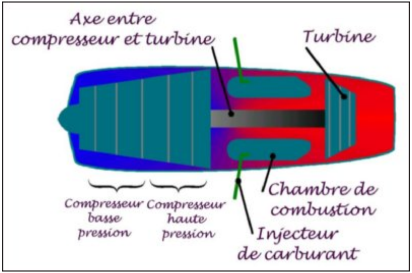

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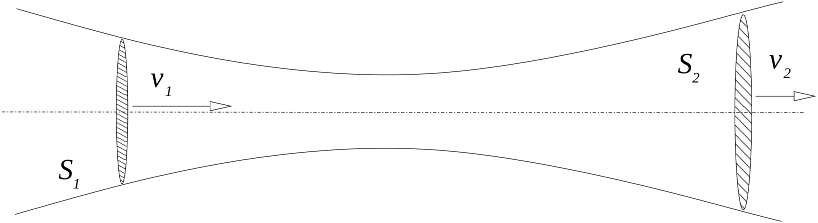

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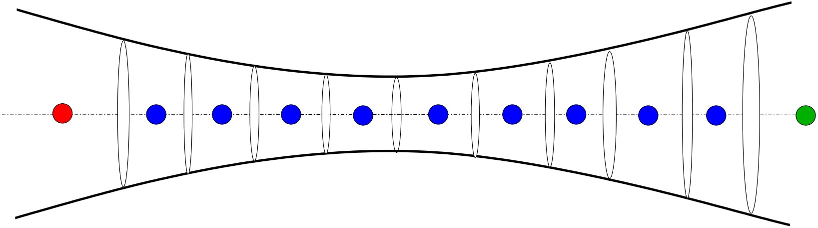
Finite volume Scheme : 1D case

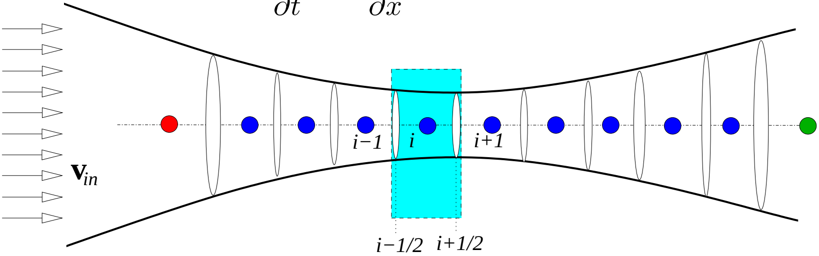
$$\frac{\partial \omega}{\partial t} + \frac{\partial f(\omega)}{\partial x} = \mathcal{S}$$



Mesh and Control volumes

Finite volume Scheme : 1D case

$$\frac{\partial \omega}{\partial t} + \frac{\partial f(\omega)}{\partial x} = \mathcal{S}$$

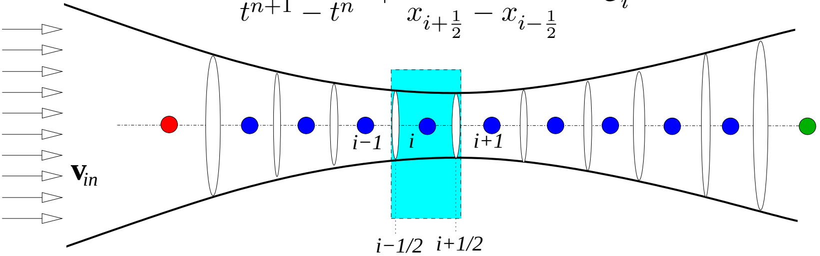


$$\frac{\omega_i^{n+1} - \omega_i^n}{t^{n+1} - t^n} + \frac{\phi^m_{i+\frac{1}{2}} - \phi^m_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \mathcal{S}_i^m$$

Balance relations on a Control volume

Finite volume Scheme : 1D case

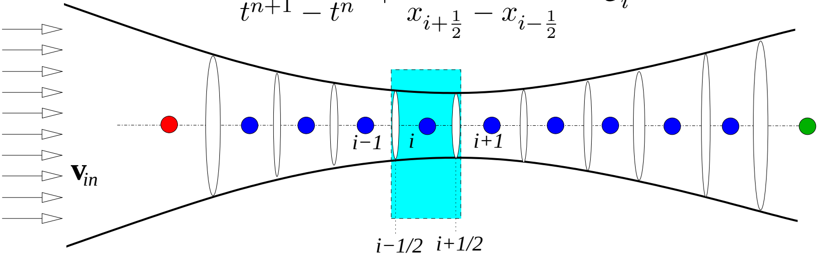
$$\frac{\omega_i^{n+1} - \omega_i^n}{t^{n+1} - t^n} + \frac{\phi^m_{i+\frac{1}{2}} - \phi^m_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \mathcal{S}_i^m$$



How to define the numerical flux $\phi \simeq f$

Finite volume Scheme : 1D case

$$\frac{\omega_i^{n+1} - \omega_i^n}{t^{n+1} - t^n} + \frac{\phi^m_{i+\frac{1}{2}} - \phi^m_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \mathcal{S}_i^m$$



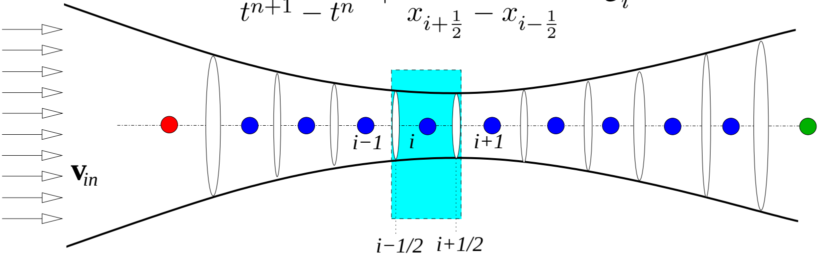
Centered scheme :

$$\phi_{i+\frac{1}{2}} = \frac{f_i + f_{i+1}}{2}$$

Accurate but Unstable !

Finite volume Scheme : 1D case

$$\frac{\omega_i^{n+1} - \omega_i^n}{t^{n+1} - t^n} + \frac{\phi^m_{i+\frac{1}{2}} - \phi^m_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \mathcal{S}_i^m$$



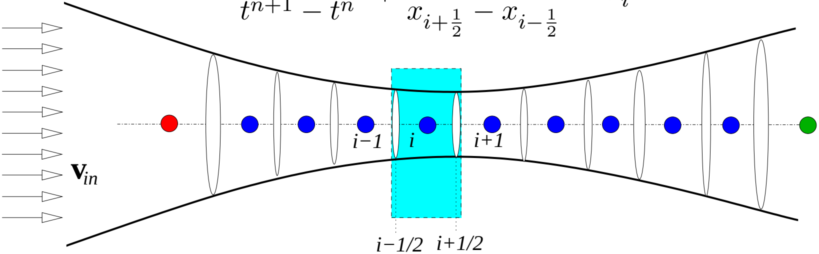
Upwind scheme (in this case) :

$$\phi_{i+\frac{1}{2}} = f_i \quad \text{and} \quad \phi_{i-\frac{1}{2}} = f_{i-1}$$

Less accurate but Stable $\surd$

Finite volume Scheme : 1D case

$$\frac{\omega_i^{n+1} - \omega_i^n}{t^{n+1} - t^n} + \frac{\phi^m_{i+\frac{1}{2}} - \phi^m_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} = \mathcal{S}_i^m$$

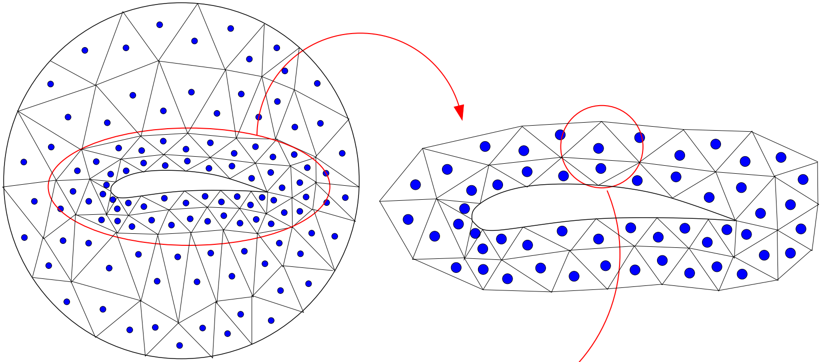


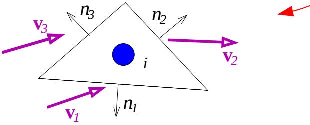
General rule :

The numerical flux should be consistent with the physics

Upwind : Follows information traveling in the correct direction.

Finite volume Scheme : 2D case





$$a_i \frac{\omega_i^{n+1} - \omega_i^n}{t^{n+1} - t^n} = - \sum_{j \in \vartheta(i)} \Phi_{i,j}$$

$\phi_{i,j}$ is now the flux crossing an interface from the cell i to j.

Need properties for Numerical Scheme

$$\phi_{i,j} = \phi(\omega_i, \omega_j)$$

Properties we want the numerical approximation to satisfy are :

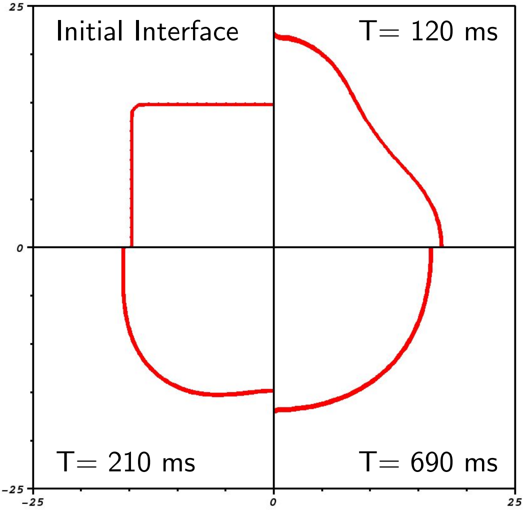
- Consistency, Stability, Convergence
- Accuracy : have a better result with a given mesh.
- Positivity and maximum principles $\rho \geq 0$, $||v|| \leq c$.
- Second thermodynamic law (entropy production) : Nonlinear cases.

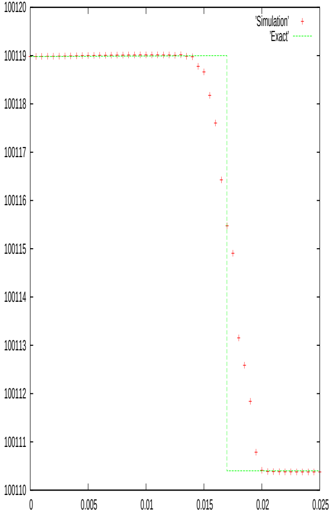
Lax Theorem for conservative systems

Consistency + Stability = Convergence

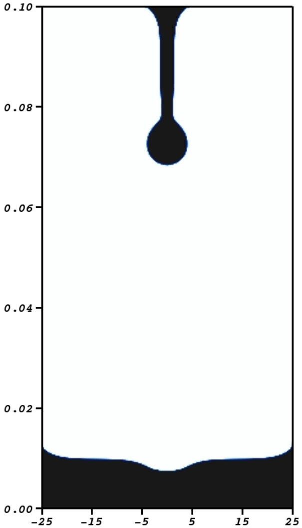
Low Mach interface flow with surface Tension

Laplace Law Recovered : $\delta p = \sigma \kappa$





Water drop in air, gravity and Surface tension



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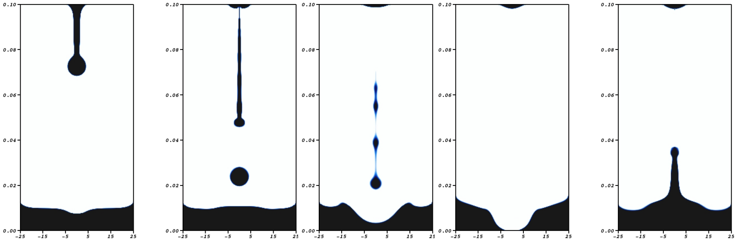
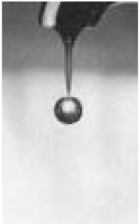
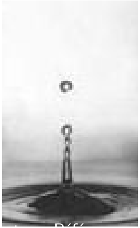
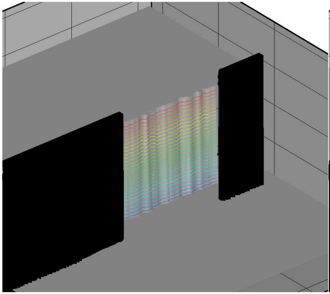
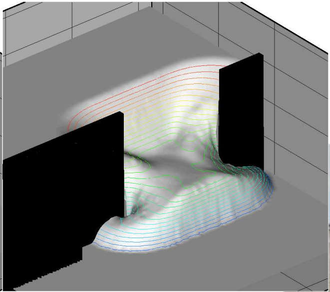


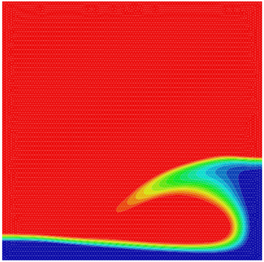


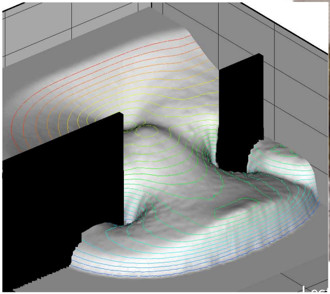


Dam Break : Shallow water and Multiphase flow

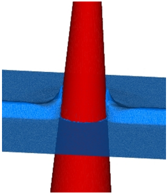


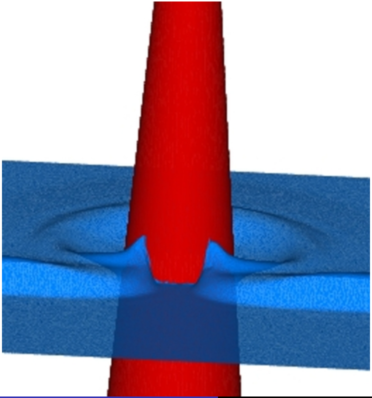


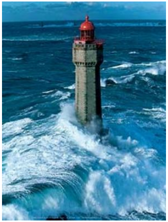




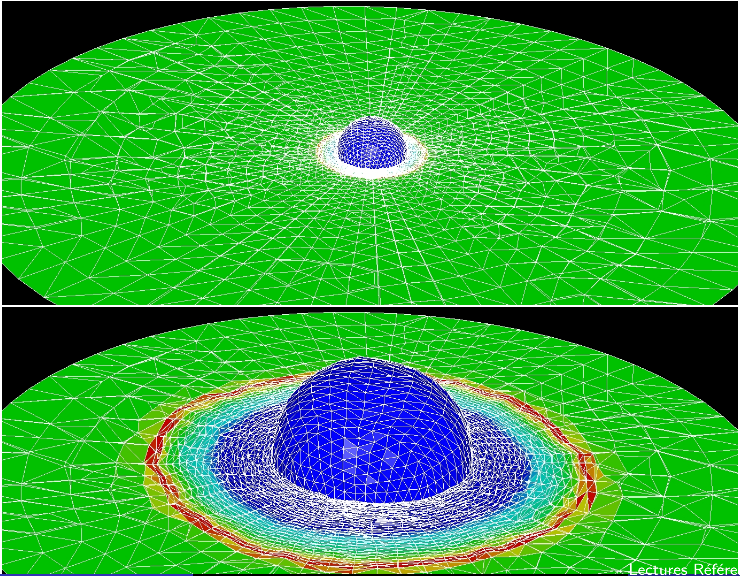
Shallow water and Multiphase flow



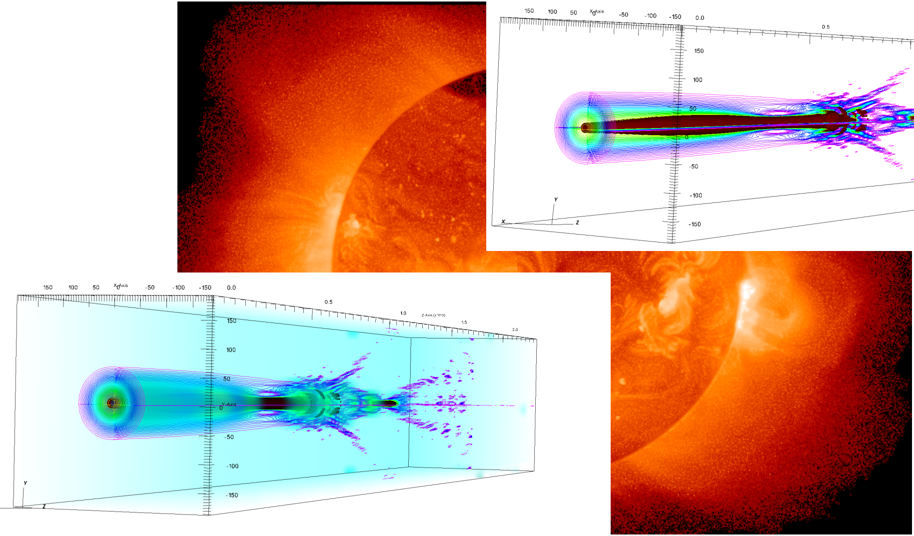




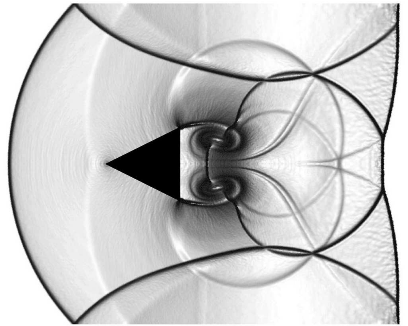
Explosion and propagation : Mesh follows the shock font.

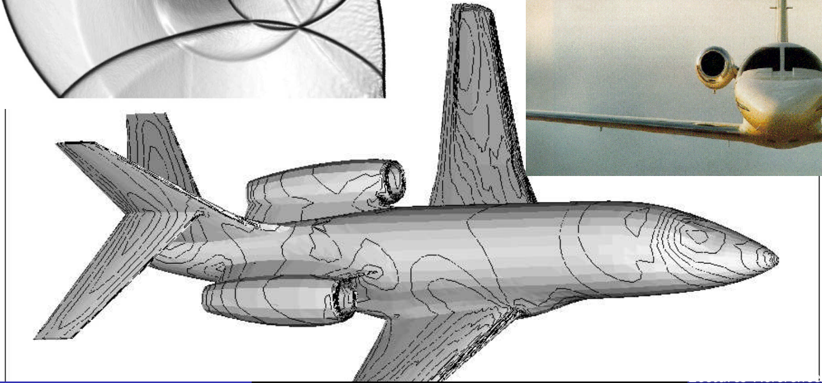


Laser / Plasmas Interactions for Nuclear Fusion

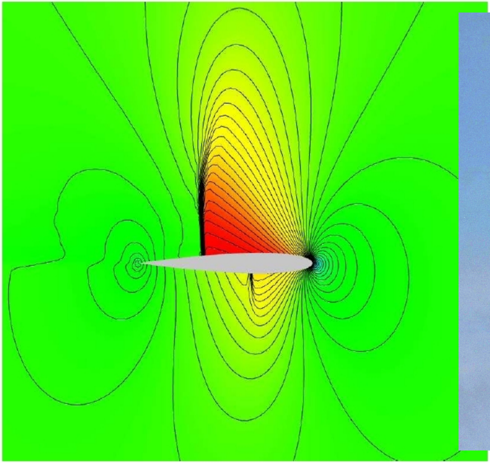


Computation and reality : Aerodynamic flow





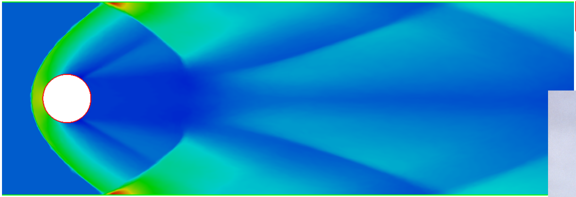
High speed aerodynamics : Shock and condensation

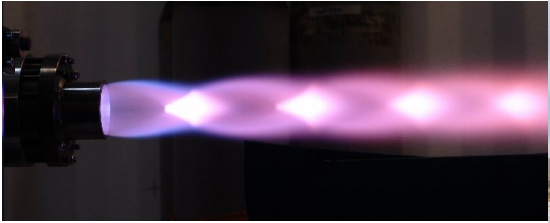


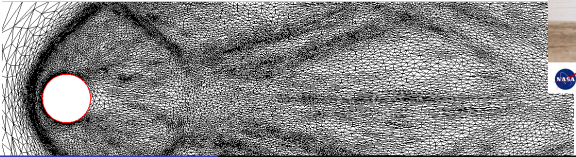
Condensation gives a view of the shock wave profile.

Shock waves interactions : Diamond structure

FluidBox







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2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions

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