Numerical Methods for PDE: Finite Differences and Finites Volumes

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Lectures Références:
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Plan

1. Finite Difference(FD) and Finite volume(FV) : Overview
2. Modelization and Simplified models of PDE.
3. Scalar Advection-Diffusion Eqation.
4. Approximation of a Scalar 1D ODE.
5. FD for 1D scalar poisson equation (elliptic).
6. FD for 1D scalar difusion equation (parabolic).
7. FD for 1D scalar advection-diffusion equation.
8. Scalar Nonlinear Conservation law : 1D (hyperbolic).
9. FV for scalar nonlinear Conservation law : 1D
10. Multi-Dimensional extensions

Plan

Stokes Fluides homogenous and isotropic

$\underline{\sigma} = -p\underline{I} + \mu\underline{\tau}$, no volumic forces (will be considered latter).

$$\left\{ \begin{array}{rcl} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) &=& 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p) &=& \mu \nabla \cdot \underline{\tau} \\ \partial_t (\rho e) + \nabla \cdot (\rho h \boldsymbol{u}) &=& \mu \nabla \cdot (\underline{\tau} \boldsymbol{u}) + \lambda \Delta T \end{array} \right. \quad (1)$$

where h is the specific enthalpy and define as $h = e + \frac{p}{\rho}$. The equation of states (perfect gaz one)

$$p = (\gamma - 1)\rho\varepsilon, \quad \varepsilon = e - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} = C_v T.$$

We assume a linear viscous stress tensor

$$\underline{\tau} = \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3}(\nabla \cdot \boldsymbol{u})\,\underline{I}$$

entropy inequality $\quad \partial_t(\rho S) + \nabla \cdot \left(\rho S \boldsymbol{u} - \lambda \frac{\nabla T}{T} \right) \geq 0 \quad (2)$

Non conservative form

Let us define the material derivative as $D_t(*) = \partial_t * + \boldsymbol{u} \cdot \nabla *$.

$$\begin{cases} D_t(\rho) &= -\rho \nabla \cdot \boldsymbol{u} \\ \rho D_t(\boldsymbol{u}) &= -\nabla p + \mu \nabla \cdot \underline{\tau} \\ \dfrac{\rho}{2} D_t(\boldsymbol{u} \cdot \boldsymbol{u}) &= -\boldsymbol{u} \cdot \nabla p + \mu \boldsymbol{u} \cdot \nabla \cdot \underline{\tau} \\ \rho D_t(e) &= -\nabla \cdot (p\boldsymbol{u}) + \mu \nabla \cdot (\underline{\tau}\boldsymbol{u}) + \nabla \cdot (\lambda \nabla T) \end{cases}$$

$\rho\varepsilon = \rho e - \rho \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2} \longrightarrow \rho D_t(\varepsilon) = \rho D_t(e) - \rho D_t\left(\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)$ Therefore,

$$\rho D_t(\varepsilon) = -p \nabla \cdot \boldsymbol{u} + \mu \underline{\tau} : \nabla \boldsymbol{u} + \lambda \Delta T$$

where we assume that λ is constant and it is used the fact that

$$\underline{\tau} : \nabla \boldsymbol{u} = \nabla \cdot (\underline{\tau}\boldsymbol{u}) - \boldsymbol{u} \cdot \nabla \cdot \underline{\tau} = \sum_{i,j} \underline{\tau}_{i,j} \partial_j \boldsymbol{u}_i$$

Primitive non conservative form.

$$\begin{cases} D_t(\rho) &= -\rho\nabla\cdot\boldsymbol{u} \\ \rho D_t(\boldsymbol{u}) &= -\nabla p + \mu\nabla\cdot\underline{\tau} \\ \rho D_t(\varepsilon) &= -p\nabla\cdot\boldsymbol{u} + \mu\underline{\tau} : \nabla\boldsymbol{u} + \lambda\Delta T \end{cases}$$

Using the relation $\varepsilon = \dfrac{p}{(\gamma-1)\rho}$ and

$$D_t(\varepsilon) = \frac{D_t(p)}{(\gamma-1)\rho} - \frac{pD_t(\rho)}{(\gamma-1)\rho^2} = \frac{D_t(p)}{(\gamma-1)\rho} + \frac{p\nabla\cdot\boldsymbol{u}}{(\gamma-1)\rho}$$

we obtain, with $c^2 = \frac{\gamma p}{\rho}$ (c is the sound speed),

$$D_t(p) = -\rho c^2\nabla\cdot\boldsymbol{u} + (\gamma-1)\left(\mu\underline{\tau} : \nabla\boldsymbol{u} + \lambda\Delta T\right)$$

Dimensionless quantities

$$[x], \quad [t], \quad [\rho], \quad [u], \quad [p] = [\rho][c]^2, \quad [h] = [e] = [c]^2,$$
$$[T], \quad [\mu], \quad [\lambda], \quad [\tau] = \frac{[\mu][u]}{[x]}$$

$\mathcal{S}t = \frac{[x]}{[u]\ [t]} = \frac{[x][\omega]}{[u]}$ Strouhal number.

$\mathcal{M}a = \frac{[u]}{[c]}$, Mach number

$\mathcal{R}e = \frac{[x]\ [\rho]\ [u]}{[\mu]}$, Reynolds number

$\mathcal{P}r = \frac{[\mu]\ [c]^2}{[\lambda]\ [T]}$ Prandtl number

Dimensionless Equations

Conservative

$$
\left\{
\begin{array}{lcl}
\mathcal{S}t\dfrac{\partial \hat{\rho}}{\partial \hat{t}} + \nabla_{\hat{x}} \cdot (\hat{\rho}\hat{\boldsymbol{u}}) & = & 0 \\
\mathcal{S}t\dfrac{\partial \hat{\rho}\hat{\boldsymbol{u}}}{\partial \hat{t}} + \nabla_{\hat{x}} \cdot (\hat{\rho}\hat{\boldsymbol{u}} \otimes \hat{\boldsymbol{u}}) + \dfrac{1}{\mathcal{M}a^2}\nabla_{\hat{x}}\hat{p} & = & \dfrac{1}{\mathcal{R}e}\nabla_{\hat{x}} \cdot \underline{\hat{\tau}} \\
\mathcal{S}t\dfrac{\partial \hat{\rho}\hat{e}}{\partial \hat{t}} + \nabla_{\hat{x}} \cdot ((\hat{\rho}\hat{e} + \hat{p})\hat{\boldsymbol{u}}) = \dfrac{\mathcal{M}a^2}{\mathcal{R}e}\nabla_{\hat{x}} \cdot \underline{\hat{\tau}}\hat{\boldsymbol{u}} & + & \dfrac{1}{\mathcal{P}r\mathcal{R}e}\nabla_{\hat{x}} \cdot (\nabla_{\hat{x}}\hat{T})
\end{array}
\right.
$$

Non conservative

$$
\left\{
\begin{array}{lcl}
\mathcal{S}t\dfrac{\partial \hat{\rho}}{\partial \hat{t}} + \hat{\boldsymbol{u}} \cdot \nabla_{\hat{x}}\hat{\rho} & = & -\hat{\rho}\nabla_{\hat{x}} \cdot \hat{\boldsymbol{u}} \\
\mathcal{S}t\hat{\rho}\dfrac{\partial \hat{\boldsymbol{u}}}{\partial \hat{t}} + \hat{\rho}\hat{\boldsymbol{u}} \cdot \nabla_{\hat{x}}\hat{\boldsymbol{u}} & = & -\dfrac{1}{\mathcal{M}a^2}\nabla_{\hat{x}}\hat{p} + \dfrac{1}{\mathcal{R}e}\nabla_{\hat{x}} \cdot \underline{\hat{\tau}} \\
\mathcal{S}t\hat{\rho}\dfrac{\partial \hat{\varepsilon}}{\partial \hat{t}} + \hat{\rho}\hat{\boldsymbol{u}} \cdot \nabla_{\hat{x}}\hat{\varepsilon} & = & -\hat{p}\nabla_{\hat{x}} \cdot \hat{\boldsymbol{u}} + \dfrac{\mathcal{M}a^2}{\mathcal{R}e}\underline{\hat{\tau}} : \nabla_{\hat{x}}\hat{\boldsymbol{u}} + \dfrac{\nabla_{\hat{x}} \cdot (\hat{\lambda}\nabla_{\hat{x}}\hat{T})}{\mathcal{P}r\mathcal{R}e}
\end{array}
\right.
$$

conservation laws : asymptotic

$$\left\{\begin{array}{rcl} \mathcal{S}t\dfrac{\partial\hat{\rho}}{\partial\hat{t}} + \hat{\boldsymbol{u}}\cdot\nabla_{\hat{x}}\hat{\rho} & = & -\hat{\rho}\nabla_{\hat{x}}\cdot\hat{\boldsymbol{u}} \\ \mathcal{S}t\hat{\rho}\dfrac{\partial\hat{\boldsymbol{u}}}{\partial\hat{t}} + \hat{\rho}\hat{\boldsymbol{u}}\cdot\nabla_{\hat{x}}\hat{\boldsymbol{u}} & = & -\dfrac{1}{\mathcal{M}a^2}\nabla_{\hat{x}}\hat{p} + \dfrac{1}{\mathcal{R}e}\nabla_{\hat{x}}\cdot\underline{\hat{\tau}} \\ \mathcal{S}t\hat{\rho}\dfrac{\partial\hat{\varepsilon}}{\partial\hat{t}} + \hat{\rho}\hat{\boldsymbol{u}}\cdot\nabla_{\hat{x}}\hat{\varepsilon} & = & -\hat{p}\nabla_{\hat{x}}\cdot\hat{\boldsymbol{u}} + \dfrac{\mathcal{M}a^2}{\mathcal{R}e}\underline{\hat{\tau}}:\nabla_{\hat{x}}\hat{\boldsymbol{u}} + \dfrac{\nabla_{\hat{x}}\cdot(\hat{\lambda}\nabla_{\hat{x}}\hat{T})}{\mathcal{P}r\mathcal{R}e} \end{array}\right.$$

$\mathcal{M}a \longrightarrow 0$ incompressible limit $\quad \nabla_x\cdot\boldsymbol{u} = 0, \quad \rho = \rho_0$

Adding a source term, the last equation gives (dimensioned form)

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot(\boldsymbol{u}\varepsilon) = \nabla\cdot(\lambda\nabla T) + S_\varepsilon$$

where $\lambda \equiv \frac{\lambda}{\rho_0}$ and we assume from now that $\boldsymbol{u}$ is given.

Necessary information to solve the PDE.

The PDE

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\boldsymbol{u}\varepsilon) = \nabla \cdot (\lambda \nabla T) + S_\varepsilon$$

requires some additional information in order to to be solvable.

Initial Condition + Boundary conditions.

- What is the solution at start of the process : $\varepsilon(t = 0, \mathbf{x}) = \varepsilon_0(\mathbf{x})$
- What are the boundary conditions :
  - Dirichlet : $\varepsilon\,(t, \mathbf{x} \in \partial\Omega_d) = \alpha(t, \mathbf{x})$
  - Neumann : $\boldsymbol{n} \cdot \boldsymbol{f}_\varepsilon\,(t, \mathbf{x} \in \partial\Omega_n) = \boldsymbol{\beta}(t, \mathbf{x})$

This defined an Initial, Boundary, Value Problem (IBVP).

Initial, Boundary, Value Problem (IBVP).

We assume that parameters C_v S_ε, $\varepsilon_0(\mathbf{x})$, $\alpha(t,\mathbf{x})$ and $\boldsymbol{\beta}(t,\mathbf{x})$ are given. Then the Initial, Boundary, Value Problem is formulated as :

Cauchy Problem ≡ Initial, Boundary, Value Problem (IBVP).

Find $\varepsilon(t,\mathbf{x})$ solution of the following problem :

$$\left\{\begin{array}{rll} \dfrac{\partial \varepsilon}{\partial t} + \nabla\cdot(\boldsymbol{u}\varepsilon) = \nabla\cdot(\lambda\nabla T) + S & \forall \mathbf{x}\in\Omega, & t>0, \\ \varepsilon(t,\mathbf{x}) = \varepsilon_0(\mathbf{x}) & \forall \mathbf{x}\in\Omega, & t=0, \\ \varepsilon(t,\mathbf{x}) = \alpha(t,\mathbf{x}) & \forall \mathbf{x}\in\partial\Omega_d, & t>0, \\ \varepsilon\boldsymbol{u}\cdot\boldsymbol{n} - \lambda\nabla T\cdot\boldsymbol{n} = \boldsymbol{\beta}(t,\mathbf{x}) & \forall \mathbf{x}\in\partial\Omega_n, & t>0, \end{array}\right.$$

Initial, Boundary, Value Problem (IBVP)

If C_v is constant, we can use a scaling such as to set $C_v \equiv 1$.

Linear Convection diffusion Equation .

$$
\left\{
\begin{array}{ll}
\dfrac{\partial T}{\partial t} + \nabla \cdot (\boldsymbol{u} T) = \nabla \cdot (\lambda \nabla T) + S_\theta, & \forall \mathbf{x} \in \Omega, \quad t > 0, \\
\text{Initial Conditions} & \\
T(t, \mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in \Omega, \quad t = 0, \\
\text{Boundary conditions} & \\
T(t, \mathbf{x}) = \alpha(t, \mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_d, \quad t > 0, \\
T \boldsymbol{u} \cdot \boldsymbol{n} - \lambda \nabla T \cdot \boldsymbol{n} = \boldsymbol{\beta}(t, \mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_n, \quad t > 0,
\end{array}
\right.
$$

Dimensional analysis and asymptotic limits.

Let us consider the flux expression

$$\boldsymbol{f}_\varepsilon = \varepsilon \boldsymbol{u} - \lambda \nabla T.$$

The magnitude of the Flux is estimates as

$$[\boldsymbol{f}_\varepsilon] = [C_v][T][\boldsymbol{u}] - [\lambda]\frac{[T]}{[\mathbf{x}]} = -[\lambda]\frac{[T]}{[\mathbf{x}]}\left(1 - Pe\right)$$

Where the Peclet Pe is defined as :

$$Pe = \frac{[C_v] * [\boldsymbol{u}] * [\mathbf{x}]}{[\lambda]} \equiv \frac{[\boldsymbol{u}] * [\mathbf{x}]}{[\lambda]}$$

- When $Pe \longrightarrow 0$ the flux is of order $\boldsymbol{f}_\varepsilon \simeq -\lambda \nabla T$.
- When $Pe \longrightarrow +\infty$ the flux is of order $\boldsymbol{f}_\varepsilon \simeq \varepsilon \boldsymbol{u}$.

Reduced Models $Pe \longrightarrow 0$: parabolic

Diffusion-reaction equation

$$\left\{\begin{array}{rll} \dfrac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T) + S, & \forall \mathbf{x} \in \Omega, & t > 0, \\ T(t,\mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in \Omega, & t = 0, \\ T(t,\mathbf{x}) = \alpha(t,\mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_d, & t > 0, \\ -\lambda \nabla T \cdot \boldsymbol{n} = \boldsymbol{\beta}(t,\mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_n, & t > 0, \end{array}\right.$$

1D Diffusion equation : $\Omega = (a,b), \quad S \equiv 0.$

$$\left\{\begin{array}{rll} \dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial \mathbf{x}}\left(\lambda \dfrac{\partial T}{\partial \mathbf{x}}\right), & \forall \mathbf{x} \in (a,b), & t > 0, \\ T(t,\mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in (a,b), & t = 0, \\ T(t,\mathbf{x}) = \alpha(t,\mathbf{x}) & \text{if } \mathbf{x} = a \text{ or/and } \mathbf{x} = b, & t > 0, \\ -\lambda \nabla T \cdot \boldsymbol{n} = \boldsymbol{\beta}(t,\mathbf{x}) & \text{if } \mathbf{x} = a \text{ or/and } \mathbf{x} = b, & t > 0, \end{array}\right.$$

Reduced Models $Pe \longrightarrow \infty$: hyperbolic

Advection-reaction equation

$$\begin{cases} \dfrac{\partial T}{\partial t} + \nabla \cdot (\boldsymbol{u} T) = S, & \forall \mathbf{x} \in \Omega, \quad t > 0, \\ T(t, \mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in \Omega, \quad t = 0, \\ T(t, \mathbf{x}) = \alpha(t, \mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_d, \quad t > 0, \\ T \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{\beta}(t, \mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_n, \quad t > 0, \end{cases}$$

1D Advection equation : $\Omega = (a, b), \quad \boldsymbol{u} \equiv c, \quad S \equiv 0.$

$$\begin{cases} \dfrac{\partial T}{\partial t} + c \dfrac{\partial T}{\partial \mathbf{x}} = 0, & \forall \mathbf{x} \in (a, b), \quad t > 0, \\ T(t, \mathbf{x}) = T_0(\mathbf{x}) & \forall \mathbf{x} \in (a, b), \quad t = 0, \\ T(t, \mathbf{x}) = \alpha(t, \mathbf{x}) & \text{if } \mathbf{x} = a \text{ or/and } \mathbf{x} = b, \quad t > 0, \end{cases}$$

Reduced Models : $\frac{\partial T}{\partial t} \simeq 0$ with $Pe = 0$ elliptic

Poisson equation

$$\begin{cases} -\nabla \cdot (\lambda \nabla T) = S & \forall \mathbf{x} \in \Omega, \\ \\ T(\mathbf{x}) = \alpha(\mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_d, \\ -\lambda \nabla T \cdot \boldsymbol{n} = \boldsymbol{\beta}(\mathbf{x}) & \forall \mathbf{x} \in \partial\Omega_n, \end{cases}$$

1D Poisson equation : $\Omega = (a, b)$

$$\begin{cases} \frac{\partial}{\partial \mathbf{x}} \left(\lambda \frac{\partial T}{\partial \mathbf{x}} \right) + S = 0, & \forall \mathbf{x} \in (a, b), \\ T(\mathbf{x}) = \alpha(\mathbf{x}) & \text{if } \mathbf{x} = a \text{ or/and } \mathbf{x} = b, \\ -\lambda \nabla T \cdot \boldsymbol{n} = \boldsymbol{\beta}(\mathbf{x}) & \text{if } \mathbf{x} = a \text{ or/and } \mathbf{x} = b, \end{cases}$$

Properties of a first order PDE

Let us consider the following first order PDE :

$$\frac{\partial T}{\partial t} = \mathcal{L}(\partial)T, \quad T(\mathbf{x}, t = 0) \text{ given}$$

- This PDE is well posed, for some choise of norm, if $\|T(.,t)\| \leq C(t)\|T(.,0)\|$.
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