Maximum principle (Discrete) and stability of the Finite difference scheme.

1D Poisson Equation

$$-\frac{\partial^2 \omega}{\partial \boldsymbol{x}^2} = f(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in (0,1), \text{ with } \omega(0) = \omega(1) = 0$$

Then we consider the numerical scheme defined as

$$\begin{array}{llll} & +\frac{2+c_1}{\delta \boldsymbol{x}^2} w_1 & -\frac{1}{\delta \boldsymbol{x}^2} w_2 & = \beta_1 f_1 \\ -\frac{1}{\delta \boldsymbol{x}^2} w_{i-1} & +\frac{2}{\delta \boldsymbol{x}^2} w_i & -\frac{1}{\delta \boldsymbol{x}^2} w_{i+1} & = f_i \qquad \text{for } i = 2, \cdots, N-1 \\ -\frac{1}{\delta \boldsymbol{x}^2} w_{N-1} & +\frac{2+c_N}{\delta \boldsymbol{x}^2} w_N & & = \beta_N f_N \end{array} \tag{0.1}$$

where $\delta \boldsymbol{x} > 0$, $c_1 \geq 0$, $c_N \geq 0$, β_1 and β_N are given parameters of the scheme and $f_i = f(\boldsymbol{x}_i)$ with $0 < \boldsymbol{x}_i < 1$ are given for $i = 1, \cdot, N$. Let use

$$\boldsymbol{w}_h = \begin{pmatrix} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_N \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}_h = \begin{pmatrix} \beta_1 f_1 \\ \vdots \\ f_i \\ \vdots \\ \beta_N f_N \end{pmatrix}$$

1. Reforlumate the numerical scheme under the form $\underline{\boldsymbol{A}}_h \boldsymbol{w}_h = \boldsymbol{b}_h$ where $\underline{\boldsymbol{A}}_h = \underline{\boldsymbol{A}}_h^\star + Diag\left(\frac{c_1}{\delta \boldsymbol{x}^2}, 0, \cdots, 0, \frac{c_N}{\delta \boldsymbol{x}^2}\right)$ is a matrix (to be defined) with a banded structure.
2. Verify that $\underline{\boldsymbol{A}}_h$ is symetric and positive defined (SPD) :
$$\boldsymbol{v} \cdot \underline{\boldsymbol{A}}_h \boldsymbol{v} \geq 0 \quad \forall \boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{v} \cdot \underline{\boldsymbol{A}}_h \boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v} = 0$$
3. For any positive vector $\boldsymbol{b} \geq 0$ and $\boldsymbol{b} \neq 0$, verify that the vector $\boldsymbol{v}_b$ defined by $\underline{\boldsymbol{A}}_h \boldsymbol{v}_b = \boldsymbol{b}$ is such as $\boldsymbol{v}_b \neq 0$.
4. Define by p the minimun index where components of $\boldsymbol{v}_b$ are minimal. Verify that either $p = 1$ or $p = N$ and in any case $\boldsymbol{v}_b \geq 0$.
5. Appling the previous result for canonical basis $\boldsymbol{v}_b = \boldsymbol{e}_i : i = 1, \cdots, N$ find that $\underline{\boldsymbol{A}}_h$ is a monotone matrix (invertible and $\underline{\boldsymbol{A}}_h^{-1}$ is positive : $\left(\underline{\boldsymbol{A}}_h^{-1}\right)_{ij} \geq 0$).
6. Verify that $\left(\underline{\boldsymbol{A}}_h^\star\right)^{-1} - \left(\underline{\boldsymbol{A}}_h\right)^{-1} = \left(\underline{\boldsymbol{A}}_h\right)^{-1} \left(\underline{\boldsymbol{A}}_h - \underline{\boldsymbol{A}}_h^\star\right) \left(\underline{\boldsymbol{A}}_h^\star\right)^{-1}$ so that $\left(\underline{\boldsymbol{A}}_h^\star\right)^{-1} - \left(\underline{\boldsymbol{A}}_h\right)^{-1}$ is a positive matrix and therefore $\left(\underline{\boldsymbol{A}}_h\right)^{-1} \leq \left(\underline{\boldsymbol{A}}_h^\star\right)^{-1}$ and
$$\left\|\underline{\boldsymbol{A}}_h^{-1}\right\|_\infty = \max_{i=1}^{N} \left(\sum_{j=1}^{N} \left|\left(\underline{\boldsymbol{A}}_h^{-1}\right)_{ij}\right|\right) = \max_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\underline{\boldsymbol{A}}_h^{-1}\right)_{ij}\right) \leq \left\|\left(\underline{\boldsymbol{A}}_h^\star\right)^{-1}\right\|_\infty$$
7. Verify that $\underline{\boldsymbol{A}}_h^\star \boldsymbol{v} = \mathbf{1}$ for $\boldsymbol{v}_i = T(i\delta \boldsymbol{x})$ with $T(\boldsymbol{x}) = \frac{\boldsymbol{x}(1-\boldsymbol{x})}{2}$ and conclude that
$$\left\|\left(\underline{\boldsymbol{A}}_h^\star\right)^{-1}\right\|_\infty = \max_{i=1}^{N} \boldsymbol{v}_i \leq \frac{1}{8}, \qquad \text{therefore} \qquad \left\|\left(\underline{\boldsymbol{A}}_h\right)^{-1}\right\|_\infty \leq \frac{1}{8}$$