Exercise (20pts) The purpose in this exercise is to analyse numerical approximations of the advection problem :

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + c\dfrac{\partial \omega}{\partial x} = 0, & \forall x \in (0,1), \quad \forall t > 0, \\ \omega(t, x = 0) = 0 & \forall t \\ \omega(t, x = 1) = 0 & \forall t \end{cases} \tag{0.1}$$

where $\omega(t,x)$ is a scalar function. We consider a regular mesh defined by $t^n = n\delta t$ and $x_i = i\delta x$ where $\delta t > 0$ and $\delta x > 0$ are given. It is always assume that $\frac{\delta t}{\delta x}$ is bounded when δt and δx goes to zero. The numerical scheme define, for any n and i, an approximation ω_i^n of $\omega(t^n, x_i)$

Consider the following Finite difference scheme at (t^n, x_i) :

$$\frac{\omega_i^{n+1} - \omega_i^n}{\delta t} + \frac{c}{\delta x}\left[\boldsymbol{\phi}_\beta(\omega_i^n, \omega_{i+1}^n) - \boldsymbol{\phi}_\beta(\omega_{i-1}^n, \omega_i^n)\right] = 0$$

where β is a parameter of the scheme and

$$\boldsymbol{\phi}_\beta(\omega_i, \omega_{i+1}) = \frac{1}{2}\left[\omega_i + \omega_{i+1} - \frac{\beta \delta x}{c\delta t}(\omega_{i+1} - \omega_i)\right]$$

1. 4pts) Derive the condition to be satisfied by β in order to achieve consistency. Is $\beta = \frac{1}{\delta x}$ in this range ?
2. We consider the case where $\beta = \frac{|c|\delta t}{\delta x}$,
   - 3pts) Show that the numerical scheme is first order accurate in time and space.
   - 2pts) Derive a CFL condition to achive L^∞ stability.
3. We now consider the case where $\beta = \frac{c^2\delta t^2}{\delta x^2}$
   - 7pts) Show that the numerical scheme is second order accurate in space and time
   - 4pts) Derive a CFL condition to achive Von Neumann stability.

Exercise (5pts) The purpose in this exercise is to derive and analyse a numerical approximation of the following problem :

$$\begin{cases} \dfrac{\partial T}{\partial t} = \lambda \dfrac{\partial^2 T}{\partial x^2}, & \forall x \in (0,1), \quad t > 0, \\ T(t,x) = T_0(x) & \forall x \in (0,1), \quad t = 0, \\ T(t, x = 0) = 0 & \forall t \geq 0 \\ T(t, x = 1) = 0 & \forall t \geq 0 \end{cases} \tag{0.2}$$

We consider a regular mesh define by N points $x_j = j\delta x$ for $j = 1, \cdots, N$ and $\delta x = \frac{1}{N+1}$. The size of the mesh (N) is a given parameter.

For a given parameter $\theta > 0$ and $\theta \neq \frac{1}{2}$, we consider a Finite difference strategy where the approximated solution of (0.2) is defined by a vector $\boldsymbol{\omega}(t)$ (components are $\omega_i(t)$ for $i = 1, ..., N$) satisfying the following recurrence (from step n to step $n+1$)

$$\begin{cases} \boldsymbol{\omega}^{n+\theta} - \theta\delta t \underline{A}\boldsymbol{\omega}^{n+\theta} = \boldsymbol{\omega}^n \\ \boldsymbol{\omega}^{n+1-\theta} = \boldsymbol{\omega}^{n+\theta} + (1-2\theta)\delta t \underline{A}\boldsymbol{\omega}^{n+\theta} \\ \boldsymbol{\omega}^{n+1} - \theta\delta t \underline{A}\boldsymbol{\omega}^{n+1} = \boldsymbol{\omega}^{n+1-\theta} \end{cases} \tag{0.3}$$

where $\boldsymbol{\omega}^m = \boldsymbol{\omega}(m\delta t)$ and $\boldsymbol{\omega}^0$ is given. The matrix $\underline{A}$ is given by

$$\underline{\mathcal{A}} = \frac{\lambda}{\delta x^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

1. Define a matrix $\underline{\mathcal{B}}$ such that the numerical scheme (0.3) can be put under the form $\boldsymbol{\omega}^{n+1} = \underline{\mathcal{B}}\boldsymbol{\omega}^n$
2. Give an approximation of $\underline{\mathcal{B}}$ in the space spanned by $\{I, \underline{\mathcal{A}}, \underline{\mathcal{A}}^2\}$, where I is the identity matrix. Hint : Use the Taylor expansion $(I - \epsilon\underline{\mathcal{A}})^{-1} = I - \epsilon\underline{\mathcal{A}} + \epsilon^2\underline{\mathcal{A}}^2 + O(\epsilon^3), \quad \forall\epsilon \in \mathbb{R}, \quad |\epsilon| << 1$.
3. Choose the parameter $\theta = \theta_0$ such as $\underline{\mathcal{B}} = \exp(\delta t\underline{\mathcal{A}}) + O(\delta t^3)$.
4. Formulate the eigenvalues ν_j of the matrix as functions of μ_j the eigenvalues of $\underline{\mathcal{A}}$.
5. The L^2 stability is in this case equivalent to $|\nu_j| \leq 1$. Give a range of values of θ for which the numerical scheme (0.3) is L^2 stable $\forall\delta t$ and $\forall\delta x$. Verify that θ_0 is in this range.