Conservations laws : Derivation

Let us consider a subset depending on time $\mathcal{D}(t) \subset \mathbb{R}^3$. Initially, for $t = 0$, any material particle in $\mathcal{D}(0)$ is identified by its coordinate $\boldsymbol{\xi}$. We define by $\boldsymbol{x}(\boldsymbol{\xi}, t)$ the position at the time t of the particle that was initially at $\boldsymbol{\xi}$. The transformation $(\boldsymbol{\xi}, t) \mapsto \boldsymbol{x}(\boldsymbol{\xi}, t)$ is invertible and sufficiently regular. The material velocity $\boldsymbol{u}$ and jacobian of the transformation are :

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{x}}{\partial t} \quad \text{and} \quad \underline{J}(\boldsymbol{\xi}, t) = \nabla_{\boldsymbol{\xi}} \boldsymbol{x}(\boldsymbol{\xi}, t) = \left(\frac{\partial x_i}{\partial \xi_j} \right)_{1 \le i \le 3, 1 \le j \le 3}$$

For any function $f(\boldsymbol{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \mapsto \mathbb{R}$ continuously differentiable (that could represent a physical property), we define its particular derivative and sum over a moving volume :

$$\frac{df}{dt} = \frac{df(\boldsymbol{x}(\boldsymbol{\xi}, t), t)}{dt} \quad \text{and} \quad \mathcal{I}_f(t) = \int_{\mathcal{D}(t)} f(\boldsymbol{x}, t) d\boldsymbol{x}$$

The aim here is to estimate the integral over the volume $\mathcal{D}(t)$ as a function of the intial position, and its variation in time in order to establish some conservation properties :

1. Verify that
$$\frac{df}{dt} = \partial_t f + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} f \quad \text{and} \quad \mathcal{I}_f(t) = \int_{\mathcal{D}(0)} f(\boldsymbol{x}(\boldsymbol{\xi}, t), t) \det(\underline{J}) d\boldsymbol{\xi}$$

2. Show that
$$\frac{\partial}{\partial t} \left(\frac{\partial x_i(\boldsymbol{\xi}, t)}{\partial \xi_j} \right) = \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial \xi_j}$$

3. Show that
$$\frac{\partial \det(\underline{J})}{\partial t} = \left[\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} \right] \det(\underline{J})$$

4. Therefore, show that the time derivative of $\mathcal{I}_f(t)$ can be written as
$$\frac{d\mathcal{I}_f(t)}{dt} = \int_{\mathcal{D}(t)} \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}(\boldsymbol{x}, t) f(\boldsymbol{x}, t)) \right) d\boldsymbol{x}$$

5. The conservation of mass, momentum and total energy can be formulated as :
$$\frac{d\mathcal{I}_\rho(t)}{dt} = 0, \quad \frac{d\mathcal{I}_{\rho \boldsymbol{u}}(t)}{dt} = \int_{\partial \mathcal{D}(t)} \underline{\sigma} \boldsymbol{n} dS + \int_{\mathcal{D}(t)} \boldsymbol{F} d\boldsymbol{x},$$
$$\frac{d\mathcal{I}_{\rho e}(t)}{dt} = \int_{\partial \mathcal{D}(t)} \boldsymbol{u} \cdot (\underline{\sigma} \boldsymbol{n}) \, dS + \int_{\mathcal{D}(t)} \boldsymbol{F} \cdot \boldsymbol{u} d\boldsymbol{x} - \int_{\partial \mathcal{D}(t)} \boldsymbol{q} \cdot \boldsymbol{n} dS$$
where $\underline{\sigma}$ is the tensor of external forces, $\boldsymbol{F}$ is the internal force and $\boldsymbol{q}$ is the heat flux.
Derive the associated system of partial differential equations (Euler equations) when $\underline{\sigma} = -pId$, $\boldsymbol{F} = \rho \boldsymbol{g}$ and $\boldsymbol{q} = -\lambda \nabla_{\boldsymbol{x}} T$. The pressure p and the temperature T are defined by the equation of state, for perfect gaz $p = (\gamma - 1) \left(\rho e - \frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right)$ and $T \equiv e - \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u}$. The heat conduction λ can be a constant.