Conservations laws : Derivation

Let us consider a subset depending on time $\mathcal{D}(t) \subset \mathbb{R}^3$. Initially, for $t = 0$, any material particle in $\mathcal{D}(0)$ is identified by its coordinate $\boldsymbol{\xi}$. We define by $\boldsymbol{x}(\boldsymbol{\xi}, t)$ the position at the time t of the particle that was initially at $\boldsymbol{\xi}$. The transformation $(\boldsymbol{\xi}, t) \mapsto \boldsymbol{x}(\boldsymbol{\xi}, t)$ is invertible and sufficiently regular. The material velocity $\boldsymbol{u}$ and jacobian of the transformation are :

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{x}}{\partial t} \quad \text{and} \quad \underline{J}(\boldsymbol{\xi}, t) = \nabla_{\boldsymbol{\xi}} \boldsymbol{x}(\boldsymbol{\xi}, t) = \left(\frac{\partial x_i}{\partial \xi_j} \right)_{1 \le i \le 3, 1 \le j \le 3}$$

For any function $f(\boldsymbol{x}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \mapsto \mathbb{R}$ continuously differentiable (that could represent a physical property), we define its particular derivative and sum over a moving volume :

$$\frac{df}{dt} = \frac{df(\boldsymbol{x}(\boldsymbol{\xi}, t), t)}{dt} \quad \text{and} \quad \mathcal{I}_f(t) = \int_{\mathcal{D}(t)} f(\boldsymbol{x}, t) d\boldsymbol{x}$$

The aim here is to estimate the integral over the volume $\mathcal{D}(t)$ as a function of the initial position, and its variation in time in order to establish some conservation properties :

1. Verify that
$$\frac{df}{dt} = \partial_t f + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} f \quad \text{and} \quad \mathcal{I}_f(t) = \int_{\mathcal{D}(0)} f(\boldsymbol{x}(\boldsymbol{\xi}, t), t) \det(\underline{J}) d\boldsymbol{\xi}$$

   Answer. By using the standard derivation formulas of composed functions we get :

$$\frac{df}{dt} = \partial_t f + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t} = \partial_t f + \sum_{i=1}^{3} u_i \frac{\partial f}{\partial x_i} = \partial_t f + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} f$$

   As for the integral, we perform a variable change in the first integral (by writing $\boldsymbol{x}$ as a function of $\boldsymbol{\xi}$), $\boldsymbol{x} : \mathcal{D}(0) \times \mathbb{R}^+ \mapsto \mathcal{D}(t)$. First we have $d\boldsymbol{x} = \det(\underline{J}) d\boldsymbol{\xi}$, then the integration domain becomes $\mathcal{D}(0)$ and the conclusion follows.

2. Show that
$$\frac{\partial}{\partial t} \left(\frac{\partial x_i(\boldsymbol{\xi}, t)}{\partial \xi_j} \right) = \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial \xi_j}$$

   Answer. Again by inverting the derivatives w.r.t time and space and then by applying the derivatives to the composed functions, we get :

$$\frac{\partial}{\partial t} \left(\frac{\partial x_i(\boldsymbol{\xi}, t)}{\partial \xi_j} \right) = \frac{\partial}{\partial \xi_j} \left(\frac{\partial x_i(\boldsymbol{\xi}, t)}{\partial t} \right) = \frac{\partial u_i}{\partial \xi_j} = \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial \xi_j}$$

3. Show that

$$\frac{\partial \det(\underline{J})}{\partial t} = \Big[\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}\Big] \det(\underline{J}).$$

Answer. One can easily verify that given a matrix M, its determinant can be written as $\det(M) = L_1 \cdot (L_2 \times L_3)$ where L_j are its lines ($\cdot$ is the canonical scalar product and $\times$ the cross product). Therefore the derivation of $\det(\underline{J})$ follows the rule of the derivation of a product of functions :

$$\frac{\partial \det(\underline{J})}{\partial t} = \frac{\partial L_1}{\partial t} \cdot (L_2 \times L_3) + L_1 \cdot \left(\frac{\partial L_2}{\partial t} \times L_3\right) + L_1 \cdot \left(L_2 \times \frac{\partial L_3}{\partial t}\right) \tag{0.1}$$

According to the previous question and the definition of $\underline{J}$, we have that

$$\frac{\partial L_i}{\partial t} = \left(\frac{\partial}{\partial t}\left(\frac{\partial x_i(\boldsymbol{\xi}, t)}{\partial \xi_j}\right)\right)_{1\leq j\leq 3} = \left(\sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial \xi_j}\right)_{1\leq j\leq 3} = \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} L_k$$

By replacing this relation into (0.1) and using some properties of the cross product and then by rearranging the different terms, we get :

$$\begin{aligned}
\frac{\partial \det(\underline{J})}{\partial t} &= \left(\sum_{k=1}^{3} \frac{\partial u_1}{\partial x_k} L_k\right) \cdot (L_2 \times L_3) + \left(\sum_{k=1}^{3} \frac{\partial u_2}{\partial x_k} L_k\right) \cdot (L_3 \times L_1) + \left(\sum_{k=1}^{3} \frac{\partial u_3}{\partial x_k} L_k\right) \cdot (L_1 \times L_2) \\
&= \frac{\partial u_1}{\partial x_1} L_1 \cdot (L_2 \times L_3) + \frac{\partial u_2}{\partial x_2} L_2 \cdot (L_3 \times L_1) + \frac{\partial u_3}{\partial x_3} L_3 \cdot (L_1 \times L_2) \\
&= \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) L_1 \cdot (L_2 \times L_3) = \Big[\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}\Big] \det(\underline{J}).
\end{aligned} \tag{0.2}$$

4. Show that the time derivative of $\mathcal{I}_f(t)$ can be written as

$$\frac{d\mathcal{I}_f(t)}{dt} = \int_{\mathcal{D}(t)} \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}(\boldsymbol{x}, t))\, f(\boldsymbol{x}, t)\right) d\boldsymbol{x}$$

Answer. By using the previous results and by performing a change of variables, we get :

$$\begin{aligned}
\frac{d\mathcal{I}_f(t)}{dt} &= \int_{\mathcal{D}(0)} \left(\frac{df(\boldsymbol{x}(\boldsymbol{\xi}, t), t)}{dt} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}, t), t))\, f(\boldsymbol{x}(\boldsymbol{\xi}, t), t)\right) \det(\underline{J}) d\boldsymbol{\xi} \\
&= \int_{\mathcal{D}(0)} (\partial_t f(\boldsymbol{x}(\boldsymbol{\xi}, t), t) + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}(\boldsymbol{\xi}, t), t) + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}, t), t))\, f(\boldsymbol{x}(\boldsymbol{\xi}, t), t)) \det(\underline{J}) d\boldsymbol{\xi} \\
&= \int_{\mathcal{D}(0)} (\partial_t f(\boldsymbol{x}(\boldsymbol{\xi}, t), t) + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}, t), t) f(\boldsymbol{x}(\boldsymbol{\xi}, t), t))\,) \det(\underline{J}) d\boldsymbol{\xi} \\
&= \int_{\mathcal{D}(t)} \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u}(\boldsymbol{x}, t) f(\boldsymbol{x}, t))\right) d\boldsymbol{x}
\end{aligned}$$

5. The conservation of mass, momentum and total energy can be formulated as :

$$\frac{d\mathcal{I}_\rho(t)}{dt} = 0, \quad \frac{d\mathcal{I}_{\rho\boldsymbol{u}}(t)}{dt} = \int_{\partial\mathcal{D}(t)} \underline{\sigma} \boldsymbol{n} dS + \int_{\mathcal{D}(t)} \boldsymbol{F} d\boldsymbol{x}, \tag{0.3}$$

$$\frac{d\mathcal{I}_{\rho e}(t)}{dt} = \int_{\partial\mathcal{D}(t)} \boldsymbol{u}\cdot(\underline{\sigma}\boldsymbol{n})\,dS + \int_{\mathcal{D}(t)} \boldsymbol{F}\cdot\boldsymbol{u}d\boldsymbol{x} - \int_{\partial\mathcal{D}(t)} \boldsymbol{q}\cdot\boldsymbol{n}dS \tag{0.4}$$

where $\underline{\sigma}$ is the tensor of external forces, $\boldsymbol{F}$ is the internal force and $\boldsymbol{q}$ is the heat flux.

Derive the associated system of partial differential equations (Euler equations) when $\underline{\sigma} = -pId$, $\boldsymbol{F} = \rho\boldsymbol{g}$ and $\boldsymbol{q} = -\lambda\nabla_{\boldsymbol{x}}T$. The pressure p and the temperature T are defined by the equation of state, for perfect gaz $p = (\gamma-1)\left(\rho e - \frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)$ and $T \equiv e - \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}$. The heat conduction λ can be a constant.

Answer. For the conservation of mass, the transformation of the first integral relation of (0.3) into a differential one is quite obvious (the integral being equal to zero foar each domain $\mathcal{D}(t)$ then ....) :

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{u}) = 0.$$

For the momentum equation we first use the Green formula in order to transform boundary integrals into volume integrals, secondly since we deal with a vector quantity the product inside the operator $\nabla_{\boldsymbol{x}}\cdot$ transforms into a Kronecker product :

$$\int_{\mathcal{D}(t)} \left(\frac{\partial\rho\boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{u}\otimes(\rho\boldsymbol{u})) - \mathrm{div}\underline{\sigma} - \boldsymbol{F}\right)d\boldsymbol{x} = 0$$

which is equivalent to

$$\frac{\partial\rho\boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{u}\otimes(\rho\boldsymbol{u})) + \nabla p - \rho\boldsymbol{g} = 0.$$

In order to derive the energy conservation equation 0.4, we apply the same technique as before :

$$\int_{\mathcal{D}(t)} \left(\frac{\partial\rho e}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\rho e\boldsymbol{u}) - \mathrm{div}(\underline{\sigma}\boldsymbol{u}) + \mathrm{div}\boldsymbol{q} - \boldsymbol{F}\cdot\boldsymbol{u}\right)d\boldsymbol{x} = 0$$

which is equivalent to :

$$\frac{\partial\rho e}{\partial t} + \nabla_{\boldsymbol{x}}\cdot((\rho e + p)\boldsymbol{u}) - \lambda\Delta T - \rho\boldsymbol{g}\cdot\boldsymbol{u} = 0.$$

Bonus question/ Homework

Formulate these equations under the form (called primitive variables or non conservative form) :

$$\frac{\partial\boldsymbol{V}}{\partial t} + \sum_{j=1}^{3}\underline{\boldsymbol{A}}_j(\boldsymbol{V})\frac{\partial\boldsymbol{V}}{\partial\boldsymbol{x}_j} = S(V)$$

where $\boldsymbol{V} = (\rho,\boldsymbol{u},T)^T$. Show that, for any vector $\boldsymbol{n}$ with $\|\boldsymbol{n}\| \neq 0$, the matrix $\boldsymbol{A} = \sum_{j=1}^{3}\boldsymbol{n}_j\underline{\boldsymbol{A}}_j$ is diagonalizable.