Conservations laws : Hyperbolicity

Let us consider the Euler equations on the conservatibe form (it is assume that $\lambda \equiv 0$ and $\boldsymbol{g} \equiv 0$)

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u}) &= 0 \\ \frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{u} \otimes (\rho \boldsymbol{u})) + \nabla p &= 0 \\ \frac{\partial \rho e}{\partial t} + \nabla_{\boldsymbol{x}} \cdot ((\rho e + p)\boldsymbol{u}) &= 0 \end{aligned} \tag{0.1}$$

and the closure relation

$$p = (\gamma - 1)\rho\varepsilon \quad \text{where} \quad \varepsilon = e - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}$$

where γ is a constant ($\gamma = 1.4$ for air flow). The aim here is to prouve that the system (0.1) is Hyperbolic.

1. Using the vector $\mathbf{W} = \begin{pmatrix} \rho \\ \rho \boldsymbol{u} \\ \rho e \end{pmatrix}$ define the flux vectors $\boldsymbol{f}_1(\mathbf{W})$, $\boldsymbol{f}_2(\mathbf{W})$ and $\boldsymbol{f}_3(\mathbf{W})$ in order to put the system (0.1) under the following conservative form :

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{x}_1} + \frac{\partial \boldsymbol{f}_2}{\partial \boldsymbol{x}_2} + \frac{\partial \boldsymbol{f}_3}{\partial \boldsymbol{x}_3} = 0$$

2. Define the Matrics $\underline{\mathcal{A}}_1(\mathbf{W})$, $\underline{\mathcal{A}}_2(\mathbf{W})$ and $\underline{\mathcal{A}}_2(\mathbf{W})$ in order to put the system (0.1) under the following nonconservative form :

$$\frac{\partial \mathbf{W}}{\partial t} + \underline{\mathcal{A}}_1 \frac{\partial \mathbf{W}}{\partial \boldsymbol{x}_1} + \underline{\mathcal{A}}_2 \frac{\partial \mathbf{W}}{\partial \boldsymbol{x}_2} + \underline{\mathcal{A}}_3 \frac{\partial \mathbf{W}}{\partial \boldsymbol{x}_3} = 0$$

3. Let us consider now the set of variables $\mathbf{V} = \begin{pmatrix} \rho \\ \boldsymbol{u} \\ \varepsilon \end{pmatrix}$. Define the Matrics $\underline{\mathcal{B}}_1(\mathbf{V})$, $\underline{\mathcal{B}}_2(\mathbf{V})$ and $\underline{\mathcal{B}}_2(\mathbf{V})$ in order to put the system (0.1) under the following primitive nonconservative form :

$$\frac{\partial \mathbf{V}}{\partial t} + \underline{\mathcal{B}}_1 \frac{\partial \mathbf{V}}{\partial \boldsymbol{x}_1} + \underline{\mathcal{B}}_2 \frac{\partial \mathbf{V}}{\partial \boldsymbol{x}_2} + \underline{\mathcal{B}}_3 \frac{\partial \mathbf{V}}{\partial \boldsymbol{x}_3} = 0$$

4. Compute the eigenvalues λ_k and the associated right eigenvectors ($\boldsymbol{r}_k$) of the matrix $\mathcal{B}(\mathbf{V}, \boldsymbol{n}) = \sum_{k=1}^{3} \boldsymbol{n}_k \underline{\mathcal{B}}_k$ where $\|\boldsymbol{n}\| \neq 0$. (These computations are possible because the system is hyperbolic)
5. Compute $(\nabla_{\mathbf{V}} \lambda_k) \cdot \boldsymbol{r}_k$. (This is used to charaterize the waves associated to each eigenvalue)
6. Show that eigenvalues of the matrix $\mathcal{B}(\mathbf{V}, \boldsymbol{n})$ are also eigenvalues of the matrix $\mathcal{A}(\mathbf{V}, \boldsymbol{n}) = \sum_{k=1}^{3} \boldsymbol{n}_k \underline{\mathcal{A}}_k$

Hint : Use $\boldsymbol{n} = (1, 0, 0)^T$ frist before considering the more general case.