Linear algebra and scientific programming.

Let us consider a matrix $\underline{\mathcal{C}}$ define as :

$$\underline{\mathcal{C}} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

The aim here is to solve analytically and numericaly the ordinary differential equation (ODE)

$$\frac{d\boldsymbol{X}}{dt} = \underline{\mathcal{C}}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{X}(t=0) = \boldsymbol{X}^0$$

At the time $T = 3$ the solution of this equation is $\boldsymbol{X}(3) = \exp\left(3\underline{\mathcal{C}}\right)\boldsymbol{X}^0$. For $\boldsymbol{X}^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- Compute $\underline{\mathcal{C}}^2$, $\underline{\mathcal{C}}^3$ and $\underline{\mathcal{C}}^4$.
- Compute $\tilde{\boldsymbol{X}}^{(k)}(3) = \sum_{m=0}^{k} \frac{3^k}{k!}\underline{\mathcal{C}}^k$ for k = 1, 2, 3 and 4.
- Compute $\boldsymbol{X}(3) = \exp\left(3\underline{\mathcal{C}}\right)\boldsymbol{X}^0$ and errors $e^{(k)}(3) = \boldsymbol{X}(3) - \tilde{\boldsymbol{X}}^{(k)}(3)$ and is norm $\|e^{(k)}(3)\|_2$.

Let us now approximate the solution at the time $T = 3$ as follows :

$$\tilde{\boldsymbol{X}}(3)|_{\delta t} = \tilde{\boldsymbol{X}}^K \quad where \quad \delta t = \frac{3}{K}, \quad \tilde{\boldsymbol{X}}^{n+1} = \tilde{\boldsymbol{X}}^n + \delta t\underline{\mathcal{C}}\tilde{\boldsymbol{X}}^n, \quad 0 \leq n \leq K-1$$

- Compute $\tilde{\boldsymbol{X}}^K$ for $K = 1, 2, 3, 4, ...$ (take care of the case $N = 3$)
- What is the limit $\lim_{K \longrightarrow \infty} \tilde{\boldsymbol{X}}^K$, compare it to $\boldsymbol{X}(3)$.

For T=100, 1000, 10000, ...

- Compute, numerically with a Fortran 90 language and scilab : $\tilde{\boldsymbol{X}}^{(k)}(T)$ for differents k and compare it to the exact solution.
- Compute, numerically with a Fortran 90 language and scilab : $\tilde{\boldsymbol{X}}^K$ for differents K (with $\delta t = \frac{T}{K}$) and compare it to the exact solution.
- plot the evolution of $\|\tilde{\boldsymbol{X}}^n\|_2$ as a function of n,
  - for $T = 100$ and $K = 100$.
  - for $T = 100$ and $K = 1000$.
  - for $T = 100$ and $K = 10000$.
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