Truncation error and order of accuracy : ODE

Numerical approximation is not an elegant subject. It is a collection of technical details and dirty work. However, is the more convenient way to solve real world problems.

The aim here is to analyse some properties of schemes for first order ODE :

$$\frac{dT}{dt} = \mathcal{F}(T) \quad \text{with} \quad T(t=0) = T^0, \quad T \in \mathbb{C}$$

Error analysis

Compute the truncation error of the following schemes for $\mathcal{F}(T) = -T$, $\mathcal{F}(T) = \exp(-T)$ and for general case :

1. Explicit Euler : $T^{n+1} = T^n + \delta t \mathcal{F}(T^n)$
2. Implicit Euler : $T^{n+1} = T^n + \delta t \mathcal{F}(T^{n+1})$
3. Midpoint Method : $T^{n+1} = T^n + \delta t \mathcal{F}\left(T^n + \frac{\delta t}{2}\mathcal{F}(T^n)\right)$
4. Crank-Nicholson scheme : $T^{n+1} = T^n + \delta t \frac{\mathcal{F}(T^n) + \mathcal{F}(T^{n+1})}{2}$
5. Heun's method $T^{n+1} = T^n + \delta t \left(\frac{\mathcal{F}(T^n) + \mathcal{F}(T^n + \delta t \mathcal{F}(T^n))}{2}\right)$
6. 2 levels leap-frog scheme : $T^{n+1} = T^{n-1} + 2\delta t \mathcal{F}(T^n)$
7. 2 levels Adams-Bashford scheme : $T^{n+1} = T^n + \frac{3\delta t}{2}\mathcal{F}(T^n) - \frac{\delta t}{2}\mathcal{F}(T^{n-1})$

Numerical application : Convergence and order of accuracy

For $\mathcal{F}(T) = -\alpha T$ and $T(0) = 1$, the exact solution is $T(t) = \exp(-\alpha t)$, for any $\alpha \in \mathbb{C}$. In numerical application fix the value of α and use $\delta t = \frac{2\pi}{Nm}$ successively for $Nm = 10$, $Nm = 20$, $Nm = 40$, $Nm = 80$ and $Nm = 160$. For any of the scheme defined in the previous section :

- Compute the numerical solution $\tilde{\boldsymbol{T}}_{Nm} = \left(T^1_{Nm}, T^2_{Nm}, \cdots, T^{Nm-1}_{Nm}, T^{Nm}_{Nm}\right)$.
- Compute and plot the error $\boldsymbol{e}_{Nm} = \boldsymbol{T}_{Nm} - \tilde{\boldsymbol{T}}_{Nm}$ where

$$\boldsymbol{T}_{Nm} = \left(\exp\left(-\frac{2\alpha\pi}{Nm}\right), \exp\left(-\frac{4\alpha\pi}{Nm}\right), \cdots, \exp\left(-\frac{2(Nm-1)\alpha\pi}{Nm}\right), \exp\left(-2\alpha\pi\right)\right)$$

- Compute $\frac{\log\left(\frac{\|\boldsymbol{e}_{10}\|_2}{\|\boldsymbol{e}_{20}\|_2}\right)}{\log(2)}$, $\frac{\log\left(\frac{\|\boldsymbol{e}_{20}\|_2}{\|\boldsymbol{e}_{40}\|_2}\right)}{\log(2)}$, $\frac{\log\left(\frac{\|\boldsymbol{e}_{40}\|_2}{\|\boldsymbol{e}_{80}\|_2}\right)}{\log(2)}$, $\frac{\log\left(\frac{\|\boldsymbol{e}_{80}\|_2}{\|\boldsymbol{e}_{160}\|_2}\right)}{\log(2)}$ and compare it to the order of accuracy of the scheme used and conclude.