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Adam Parusiński · Laurenţiu Păunescu

Lipschitz stratification of complex hypersurfaces in codimension 2

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Abstract. We show that the Zariski canonical stratification of complex hypersurfaces is locally bi-Lipschitz trivial along the strata of codimension two. More precisely, we study the Zariski equisingular families of surface, not necessarily isolated, singularities in $\mathbb{C}^3$. We show that a natural stratification of such a family, given by the singular set and the generic family of polar curves, provides a Lipschitz stratification in the sense of Mostowski. In particular such families are bi-Lipschitz trivial, with trivializations obtained by integrating Lipschitz vector fields.

Keywords. Stratifications, Zariski equisingularity, polar curves and surface singularities, Lipschitz stratifications

1. Introduction

In the geometric study of complex singular algebraic varieties or analytic spaces the notion of stratification plays an essential role. It is well known that there always exists a stratification that is topologically equisingular (i.e. trivial) along each stratum. This is usually achieved by means of a Whitney stratification. Another and entirely independent way of constructing such a stratification is Zariski equisingularity. A desirable important feature is the existence of a stratification that satisfies stronger equisingularity properties than the one given by Whitney conditions. This is known about Zariski (generic) equisingularity, though its precise geometric properties are still to be understood. For instance, it is well known that Zariski equisingular families of plane curve singularities are bi-Lipschitz trivial. The goal of this paper is to extend this observation to the next case, the families of surface singularities in $\mathbb{C}^3$.

In 1979 O. Zariski [29] presented a general theory of equisingularity for algebroid and algebraic hypersurfaces over an algebraically closed field of characteristic zero. Zariski's

Adam Parusiński: LJAD, Université Côte d'Azur, CNRS, UMR 7351, 06108 Nice, France; adam.parusinski@univ-cotedazur.fr

Laurenţiu Păunescu: School of Mathematics and Statistics, The University of Sydney, Sydney, NSW, 2006, Australia; laurentiu.paunescu@sydney.edu.au

Mathematics Subject Classification (2020): 32Sxx, 32B10, 14B05 theory is based on the notion of equisingularity along the strata defined by considering the discriminant loci of successive "generic" projections. This concept, now referred to as Zariski equisingularity or generic Zariski equisingularity, was called by Zariski himself algebro-geometric equisingularity, since it is defined by purely algebraic means but reflects several natural geometric properties. In [27] Zariski studied the case of strata of codimension 1. In this case the hypersurface is locally isomorphic to an equisingular (topologically trivial if the ground field is $\mathbb{C}$) family of plane curve singularities. Moreover, by [27, Theorem 8.1], Zariski's stratification satisfies Whitney's conditions along strata of codimension 1, and over $\mathbb{C}$, by [18], such an equisingular family of plane curves is bi-Lipschitz trivial, i.e. trivial via a local ambient bi-Lipschitz homeomorphism. In general, Zariski equisingularity implies Whitney's conditions, as shown by Speder [20]. For a survey on Zariski equisingularity and its recent applications see [16].

In 1985 T. Mostowski [9] introduced the notion of Lipschitz stratification of complex analytic spaces or algebraic varieties, by imposing local conditions on tangent spaces to the strata, stronger than Whitney's conditions. Mostowski's work was partly motivated by the question of Siebenmann and Sullivan [19] whether the number of local Lipschitz types on (real or complex) analytic spaces is countable. Mostowski's Lipschitz stratification satisfies the extension property of stratified vector fields from lower-dimensional to higher-dimensional strata, and therefore implies local bi-Lipschitz triviality. Its construction is similar to the one of Zariski, but involves considering many projections at each stage of the construction. It is related to the geometry of polar varieties, as shown by Mostowski in the case of hypersurface singularities in $\mathbb{C}^3$ (see [10]). In general, the construction of a Lipschitz stratifications is complicated and involves many stages. It was conjectured by J.-P. Henry and T. Mostowski that Zariski equisingular families of surface singularities in $\mathbb{C}^3$ admit natural Lipschitz stratifications by taking the singular locus and the family of "generic" polar curves as strata. We show this conjecture in this paper (see Theorem 2.1).

Recent works (see for instance [3,5,6,12,23]) show further development and progress on understanding the Lipschitz structure of singularities and its relation to other geometric phenomena appearing in the study of local properties of complex or real analytic or algebraic singular spaces. Among the major results and contributions we mention only the most important ones related to this paper: [1] where the case of the "inner" metric was considered and [11] where the equivalence of Zariski equisingularity and Lipschitz triviality for families of complex normal surface singularities was announced.

Our proof of Theorem 2.1 is based on local parameterizations of two geometric objects associated to such families: polar wedges and quasi-wings. Both originate from the classical wings introduced by Whitney [25]. Polar wedges are neighbourhoods of families of polar curves, the critical loci of corank 1 projections. Quasi-wings, originally introduced in [9], are neighbourhoods of curves on which this projection is regular (with control on the derivatives). Their local parameterizations, interesting in themselves, in the case of polar wedges originate from [2] and [22] and were recently considered in [11].

As we show, the quasi-wings and the polar wedges cover a neighbourhood of the singularity. The proof of this fact follows from the analytic wings construction of [17].

The definition of "generic projection" is crucial for Zariski's theory. Zariski's study of codimension 1 singularities (families of plane curve singularities) required just transverse projections. This is no longer the case for singularities in codimension 2. In [7] Luengo gave an example of a family of surface singularities in $\mathbb{C}^3$ that is Zariski equisingular for one transverse projection but not for a generic transverse projection. Therefore we make precise what we mean by "generic projection" in our context and we state it in our Transversality Assumptions. This is important since this condition can be computed and algorithmically verified.

2. Set-up and statement of results

Let $f(x, y, z, t) : (\mathbb{C}^{3+l}, 0) \to (\mathbb{C}, 0)$ be analytic. We suppose that $f(0, 0, 0, t) = 0$ for every $t \in (\mathbb{C}^l, 0)$, and regard f as an analytic family $f_t(x, y, z) = f(x, y, z, t)$ of analytic function germs parameterized by t. In what follows we suppress the germ notation for simplicity.

We let $\mathcal{X} = f^{-1}(0)$ and denote by Σ_f the singular set of $\mathcal{X}$. We always assume that the germs f_t are reduced, and that the system of coordinates is sufficiently generic (see the Transversality Assumptions below for a precise formulation). In particular, we assume that the restriction of the projection $\pi(x, y, z, t) = (x, y, t)$ to $\mathcal{X}$ is finite.

Denote by C_f the *polar set* of $\pi|_{\mathcal{X}}$, i.e. the closure of the critical locus of the projection π restricted to the regular part of $\mathcal{X}$. The set C_f can be understood as a family of space curves (polar curves) parameterized by t. Let

$$S = \{f(x, y, z, t) = f'_z(x, y, z, t) = 0\} = \Sigma_f \cup C_f. \tag{1}$$

The main goal of this paper is to show the following result (for the notion of Zariski equisingular families of hypersurface singularities in $(\mathbb{C}^3, 0)$ see Section 2.1, and for Mostowski's Lipschitz stratification see Section 2.2).

Theorem 2.1. *Suppose that the family $\mathcal{X}_t = f_t^{-1}(0)$ is generically linearly Zariski equisingular. Then it is bi-Lipschitz trivial. That is, there are neighbourhoods Ω of 0 in $\mathbb{C}^3 \times \mathbb{C}^l$, Ω_0 of 0 in $\mathbb{C}^3$, and U of 0 in $\mathbb{C}^l$, and a bi-Lipschitz homeomorphism*

$$\Phi : \Omega_0 \times U \to \Omega$$

satisfying $\Phi(x, y, z, t) = (\Psi(x, y, z, t), t)$, $\Phi(x, y, z, 0) = (x, y, z, 0)$, and

$$\Phi(\mathcal{X}_0 \times U) = \mathcal{X}.$$

Moreover, $\{\mathcal{X} \setminus S, S \setminus T, T\}$, where $T = \{0\} \times \mathbb{C}^l$, defines a Lipschitz stratification of $\mathcal{X}$ in the sense of Mostowski. In particular, the homeomorphism Φ can be obtained by integration of Lipschitz vector fields.

The nonparameterized version, i.e. if $l = 0$, of Theorem 2.1 was proven in [10], and the general version, as stated above, was conjectured by J.-P Henry and T. Mostowski more than twenty years ago. The bi-Lipschitz triviality for families of normal surface singularities in $\mathbb{C}^3$ was announced in [11]. Our proof uses some ideas of [11] and [1], in particular that of polar wedges. Nevertheless, the main idea of the proof is different from that of [11]. Moreover, we prove a much stronger bi-Lipschitz property, the existence of a Lipschitz stratification in the sense of Mostowski. This implies that the trivialization Φ can be obtained by integration of Lipschitz vector fields. There is a difference between arbitrary bi-Lipschitz trivializations, and the ones obtained by integration of Lipschitz vector fields (note that the bi-Lipschitz trivializations of [1,11,23] do not have this property). For instance, the latter implies the continuity of the Gaussian curvature (see [9, Section 10] and [15]).

The notion of Lipschitz stratification was defined by Mostowski in terms of regularity conditions on tangent spaces to strata, but to show that $\{\mathcal{X} \setminus S, S \setminus T, T\}$ is a Lipschitz stratification we do not use Mostowski's definition but an equivalent characterization based on the extension of stratified Lipschitz vector fields (see Section 2.2). For this we use two, in a way, complementary constructions, the polar wedges of [1, 11] (covering neighbourhoods of the critical loci of a generic linear projection) and the quasi-wings of [9] (covering their complements).

Both can be understood as a generalized version of the classical wings. Actually we need a strong analytic form of the latter given by [17], in order to construct, for an arbitrary real analytic arc not contained in polar wedges, first a complex analytic wing and then a quasi-wing containing it (see Proposition 7.7).

Many parts of the proof are fairly technical. In order to simplify the exposition we use the following strategy. For virtually all the geometric constructions of the proof, including the description of stratified Lipschitz vector fields on polar wedges in Proposition 5.5 or on quasi-wings in Proposition 8.4, the emphasis is on the nonparameterized case, i.e., with $l = 0$. The profound understanding of this case, properly stated, makes the parameterized case much easier.

2.1. Zariski equisingularity

Given a family of reduced analytic functions germs $f_t(x, y, z) : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ as above, we denote by $\Delta(x, y, t)$ the discriminant of the projection π restricted to $\mathcal{X}$. The zero set of $\Delta(x, y, t)$ is a family of plane curve singularities parameterized by t. We say that the family $\mathcal{X}_t$ is *Zariski equisingular* (*with respect to the projection* π) if $t \mapsto \{\Delta(x, y, t) = 0\}$ is an equisingular family of plane curves, that is, satisfies one of the standard equivalent definitions (see [26], [21, p. 623]). We shall often use the classical result saying that a family of equisingular plane curves admits a uniform Puiseux expansion with respect to parameters, in the sense of [17, Theorem 2.2]. We refer to it as the *Puiseux with parameter theorem*.

We say that the family $\mathcal{X}_t$ is *generically linearly Zariski equisingular* if it is Zariski equisingular after a generic linear change of coordinates x, y, z.

In the proof of Theorem 2.1 we use the following precise assumptions on f, called the Transversality Assumptions, implied by the generic linear Zariski equisingularity.

Let us denote by π_b the projection $\mathbb{C}^3 \times \mathbb{C}^l \to \mathbb{C}^2 \times \mathbb{C}^l$ parallel to $(0, b, 1, 0)$, that is, $\pi_b(x, y, z, t) = (x, y - bz, t)$. We denote by $\Delta_b(x, y, t)$ the discriminant of the projection π_b restricted to $\mathcal{X}$.

Transversality Assumptions. The tangent cone $C_0(\mathcal{X}_0)$ to $\mathcal{X}_0 = f_0^{-1}(0)$ does not contain the z-axis and, for b and t small, the family of the discriminant loci $\Delta_b = 0$ is an equisingular family of plane curve singularities with respect to b and t as parameters. Moreover, we suppose that $\Delta_0 = 0$ is transverse to the y-axis and that $x = 0$ is not a limit of tangent spaces to $\mathcal{X}_{\mathrm{reg}}$, the regular part of $\mathcal{X}$.

Remark 2.2. Since Zariski equisingular families are equimultiple (see [28] or [17, Proposition 1.13], the above assumptions imply the following. The tangent cone $C_0(\mathcal{X}_t)$ does not contain $(0, b, 1)$, for t and b small. The y-axis is transverse to every $\{(x, y);\ \Delta_b(x, y, t) = 0\}$, also for t and b small.

We now show that a generically linearly Zariski equisingular family satisfies the Transversality Assumptions after a linear change of coordinates x, y, z. First we need the following lemma.

Lemma 2.3. *The family $f_t(x, y, z) = 0$ is generically linearly Zariski equisingular if and only if, after a linear change of coordinates x, y, z, the family $f(x + az, y + bz, z, t) = 0$, for a, b, t small, is Zariski equisingular with respect to parameters a, b, t.*

Proof. The "if" part is obvious. We show the "only if" part. Let $\Delta(x, y, a, b, t)$ be the discriminant of $f(x + az, y + bz, z, t)$. By assumption there is an open subset $U \subset \mathbb{C}^2$ such that the family of plane curve germs $\Delta(x, y, a, b, t) = 0$ is equisingular with respect to t for every $(a, b) \in U$. Fix a small neighbourhood V of the origin in $\mathbb{C}^l$ so that the subset of parameters $(a, b, t) \in U \times V$ such that $\Delta(x, y, a, b, t) = 0$ changes its equisingularity type is a proper analytic subset $Y \subset U \times V$. The existence of such Y follows for instance from Zariski [26], where it is shown that a family of plane curve singularities is equisingular if and only if its discriminant by a transverse projection is equimultiple. (Equivalently, one may use semicontinuous invariants characterizing equisingularity such as the Milnor number for instance.) Then Y cannot contain $U \times \{0\}$ (this would contradict the Zariski equisingularity of $\Delta = 0$ for $(a, b) \in U$ arbitrary and fixed). Therefore, the family $f(x + az, y + bz, z, t) = 0$ is Zariski equisingular for the parameters a, b, t in a neighbourhood of any point of $(U \setminus Y) \times \{0\}$. This shows the claim. ∎

Suppose now that the family $f_t = 0$ is generically linearly Zariski equisingular and choose a generic line ℓ in the parameter space of $(a, b) \in U$ in the notation of the proof of the above lemma. The pencil of kernels of $\pi_{a,b}(x, y, z, t) = (x - at, y - bz, t)$, $(a, b) \in \ell$, corresponds to a hyperplane $H \subset \mathbb{C}^3$. Choose coordinates x, y, z so that $H = \{x = 0\}$ and ℓ corresponds to the pencil of projections parallel to $(0, b, 1) \in H$. In this system of coordinates, f satisfies the Transversality Assumptions.

2.2. Lipschitz stratification

In [9] T. Mostowski introduced a sequence of conditions on the tangent spaces to the strata of a stratified subset of $\mathbb{C}^n$ that imply the Lipschitz triviality of the stratification along each stratum. Mostowski showed the existence of such stratifications for germs of complex analytic subsets of $\mathbb{C}^n$. Note that there is no canonical Lipschitz stratification in the sense of Mostowski in general.

For more information about the Lipschitz stratification we refer the interested reader to [6, 9, 13, 14].

In [10, pp. 320–321, second example] Mostowski gave a criterion for a set to be a codimension 1 stratum of a Lipschitz stratification of a complex surface germ in $\mathbb{C}^3$. This criterion implies that a generic polar curve can be chosen as such a stratum. It is not difficult to complete Mostowski's argument and show Theorem 2.1 in the nonparameterized case ($l = 0$). In Section 6.1 we give a different proof which implies the parameterized case as well.

Mostowski's conditions imply the existence of extensions of Lipschitz stratified vector fields from lower-dimensional to higher-dimensional strata, a property which, as shown in [13], is equivalent to Mostowski's conditions. Let us recall this equivalent definition. For this it is convenient to express Mostowski's stratification in terms of its skeleton, that is, the union of the strata of dimensions $\leq k$. Let $X \subset \mathbb{C}^n$ be a complex analytic subset of dimension d and let

$$X = X^d \supset X^{d-1} \supset \cdots \supset X^l \neq \emptyset, \tag{2}$$

where $l \geq 0$, $X^{l-1} = \emptyset$, be its filtration by complex analytic sets such that every $X^k \setminus X^{k-1}$ is either empty or nonsingular of pure dimension k.

Our proof is based on the following characterization of a Lipschitz stratification.

Proposition 2.4 ([13, Proposition 1.5]). *The filtration (2) induces a Lipschitz stratification if and only if one of the following equivalent conditions holds:*

(i) *There exists $C > 0$ such that for every $W \subset X$ satisfying $X^{j-1} \subseteq W \subset X^j$, every Lipschitz stratified vector field on W with Lipschitz constant L, and bounded on $W \cap X^l$ by K, can be extended to a Lipschitz stratified vector field on X^j with Lipschitz constant $C(L + K)$.*

(ii) *There exists $C > 0$ such that for every $W = X^{j-1} \cup \{q\}$, $q \in X^j$, each Lipschitz stratified vector field on W with Lipschitz constant L, and bounded on $W \cap X^l$ by K, can be extended to a Lipschitz stratified vector field on $W \cup \{q'\}$, $q' \in X^j$, with Lipschitz constant $C(L + K)$.*

Here by a *stratified vector field* we mean a vector field tangent to strata. In our particular case, the stratification $\{\mathcal{X} \setminus S, S \setminus T, T\}$ is Lipschitz if and only if there is a constant $C > 0$ such that:

(L1) for every couple of points $q, q' \in S \setminus T$, every stratified Lipschitz vector field on $T \cup \{q\}$ with Lipschitz constant L, and bounded by K, can be extended to a Lipschitz stratified vector field on $T \cup \{q, q'\}$ with Lipschitz constant $C(L + K)$.

(L2) for every couple of points $q, q' \in \mathcal{X} \setminus S$, every stratified Lipschitz vector field on $S \cup \{q\}$ with Lipschitz constant L, and bounded by K, can be extended to a Lipschitz vector field on $S \cup \{q, q'\}$ with Lipschitz constant $C(L + K)$.

In order to prove conditions (L1) and (L2) we consider two geometric constructions, the quasi-wings of Mostowski [9] and the polar wedges of [1, 11], which, as sets, together cover the whole $\mathcal{X}$. We first prove (L1) in general and (L2) on polar wedges. This part of the proof is based on a complete description of stratified Lipschitz vector fields on polar wedges in terms of their parameterizations (see Section 5). Note that in order to compare points on polar wedges we work with fractional powers, using parameterizations over the same allowable sector (see Section 4.1 for more details).

In order to show (L2) on quasi-wings we employ the following strategy. If Mostowski's conditions fail then they fail along real analytic arcs $\gamma(s), \gamma'(s)$, $s \in [0, \varepsilon)$ (see [9, Lemma 6.2] or the valuative Mostowski conditions of [6]). For such arcs, however, if they are not in the union of polar wedges, we can construct quasi-wings containing them, say $\mathcal{QW}$ and $\mathcal{QW}'$ respectively, and then we show that the stratification $\{\mathcal{QW} \cup \mathcal{QW}' \setminus S, S \setminus T, T\}$ satisfies (L2) on the arcs $\gamma(s), \gamma'(s)$. For a precise statement and proof justifying this strategy the reader is referred to the rather technical Section 11.

2.3. *Notation and conventions*

For two complex function germs $f, g : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ we write:

- $|f(x)| \lesssim |g(x)|$ (or $f = O(g)$) if $|f(x)| \leq c|g(x)|, c > 0$ a given constant, in a neighbourhood of 0.
- $|f(x)| \sim |g(x)|$ if $|f(x)| \lesssim |g(x)| \lesssim |f(x)|$ in a neighbourhood of 0.
- $|f(x)| \ll |g(x)|$ (or $f = o(g)$) if $|f(x)|/|g(x)| \to 0$ as $\|x\| \to 0$.

While parameterizing analytic curve singularities or families of such singularities in $\mathbb{C}^2$ and $\mathbb{C}^3$ using the Puiseux theorem, we ramify in variable $x = u^n$. We often have to replace the exponent n by a multiple in order for such parameterizations to remain analytic, but we keep denoting it by n for simplicity. This makes no harm since we always work over an admissible sector as explained in Section 4.1. By an analytic unit we mean a nowhere vanishing analytic function or a germ of such a function.

3. Families of polar curves

In this section we discuss how the families of polar curves of $\mathcal{X}$, associated to the projections π_b, $b \in \mathbb{C}$, depend on b. The main result is Proposition 3.3 (nonparameterized case) and Proposition 3.4 (parameterized case). The proposition in the nonparameterized case appeared in the proof of the polar wedge lemma [1, Proposition 3.4]. The proofs of Propositions 3.3 and 3.4 are based on a key Lemma 3.1, coming from [2] and [22].

3.1. Nonparameterized case

For simplicity we first consider the case of $f(x, y, z)$ without parameter. We assume that the coordinate system satisfies the Transversality Assumptions and therefore the family

$$F(X, Y, Z, b) := f(X, Y + bZ, Z) \tag{3}$$

parameterized by $b \in \mathbb{C}$ is Zariski equisingular for b small. By this assumption the zero set of the discriminant $\Delta_F(X, Y, b)$ of F satisfies the Puiseux with parameter theorem. The set $F = F'_Z = 0$ is the union $S_F = \Sigma_F \cup C_F$ of the singular set Σ_F of F and the union C_F of a family of polar curves. The set S_F consists of finitely many irreducible components parameterized by

$$(u, b) \mapsto (u^n, Y_i(u, b), Z_i(u, b), b) \tag{4}$$

with Y_i, Z_i analytic. Then $(u^n, Y = Y_i(u, b), b)$ parameterizes a component of the discriminant locus $\Delta_F = 0$ of F.

The key lemma below is a version of [2, p. 278, first formula] or [22, a formula on p. 465].

Lemma 3.1.

$$Z_i = -\frac{\partial Y_i}{\partial b}. \tag{5}$$

Proof. We have

$$F(u^n, Y_i, Z_i, b) = 0 = F'_Z(u^n, Y_i, Z_i, b). \tag{6}$$

We differentiate the first identity with respect to b and use the second one to simplify the result:

$$0 = F'_Y \frac{\partial Y_i}{\partial b} + F'_Z \frac{\partial Z_i}{\partial b} + F'_b = f'_y(u^n, Y_i + bZ_i, Z_i)\left(\frac{\partial Y_i}{\partial b} + Z_i\right).$$

If $f'_y(u^n, Y_i + bZ_i, Z_i) \not\equiv 0$ then (5) holds. Note that in this case (4) parameterizes an irreducible component of C_F.

If $f'_y(u^n, Y_i + bZ_i, Z_i) \equiv 0$ then, in addition to (6), we have $F'_Y(u^n, Y_i, Z_i, b) = 0$. Thus in this case (4) parameterizes a component of Σ_F. By the formula

$$F'_Z(X, Y, Z, b) = bf'_y(X, Y + bZ, Z) + f'_z(X, Y + bZ, Z), \tag{7}$$

$(X, Y, Z, b) \in \Sigma_F$ if and only if $(x, y, z) = (X, Y + bZ, Z) \in \Sigma_f$, the singular set of f. Thus in this case the map

$$(u, b) \mapsto (u^n, y_i(u, b), z_i(u, b)), \quad y_i = Y_i + bZ_i,\ z_i = Z_i, \tag{8}$$

parameterizes a component of Σ_f. Moreover, by the Transversality Assumptions, the projection of Σ_f on the x-axis is finite. Consequently, both $y_i = Y_i + bZ_i$ and Z_i are independent of b and (5) trivially holds. ∎

We note that if $f'_y(u^n, Y_i + bZ_i, Z_i) \not\equiv 0$, i.e. if (4) parameterizes a component of C_F, then (8) parameterizes a family of polar curves in $f^{-1}(0)$ defined by the projection π_b. In both cases, the functions $y_i(u,b)$, $z_i(u,b) = Z_i(u,b)$, and $Y_i(u,b)$ are related by

$$z_i = -\partial Y_i/\partial b, \quad y_i = Y_i + bz_i, \quad \partial y_i/\partial b = b\partial z_i/\partial b. \tag{9}$$

In particular, the expansion of y_i cannot have a term linear in b.

By the Zariski equisingularity assumption, for any two distinct branches $Y_i(u,b)$, $Y_j(u,b)$ there is $k_{ij} \in \mathbb{N}_{\geq 0}$ such that $Y_i(u,b) - Y_j(u,b) = u^{k_{ij}} \operatorname{unit}(u,b)$. Note that, by the transversality to the y-axis, we have $k_{ij} \geq n$. By (9) this implies the following result.

Lemma 3.2. *For $i \neq j$ there is $k_{ij} \in \mathbb{N}_{\geq n}$ such that*

$$\begin{aligned} y_i(u,b) - y_j(u,b) &= u^{k_{ij}} \operatorname{unit}(u,b), \\ z_i(u,b) - z_j(u,b) &= O(u^{k_{ij}}). \end{aligned} \tag{10}$$

The next result, which we will prove later in the more general parameterized case, is crucial.

Proposition 3.3. *There are integers $m_i \in \mathbb{N}_{\geq n}$ such that*

$$\begin{aligned} y_i(u,b) &= y_i(u,0) + b^2 u^{m_i} \varphi_i(u,b), \\ z_i(u,b) &= z_i(u,0) + b u^{m_i} \psi_i(u,b), \end{aligned} \tag{11}$$

and either $\varphi_i(0,0), \psi_i(0,0) \neq 0$, or, if (8) parameterizes a component of Σ_f, then $\varphi_i \equiv \psi_i \equiv 0$.

3.2. Parameterized case

We extend the results of the previous subsection to the parameterized family

$$F(X,Y,Z,b,t) := f(X, Y + bZ, Z, t), \tag{12}$$

with f satisfying the Transversality Assumptions. Thus F is now Zariski equisingular with respect to the parameters b and t and therefore the discriminant $\Delta_f(X,Y,b,t)$ of F with respect to Z satisfies the Puiseux with parameter theorem. Similarly to the non-parameterized case, $S_F = \{F = F'_Z = 0\}$ is parameterized by

$$(u,b,t) \mapsto (u^n, Y_i(u,b,t), Z_i(u,b,t), b, t), \tag{13}$$

and consists of the singular locus Σ_F and the union C_F of a family of of polar curves, now parameterized by b and t.

Lemma 3.1 still holds (with the same proof) so we have $Z_i = -\partial Y_i/\partial b$. Then

$$(u,b,t) \mapsto p_i(u,b,t) = (u^n, y_i(u,b,t), z_i(u,b,t), t), \quad y_i = Y_i + bZ_i,\ z_i = Z_i, \tag{14}$$

parameterize in $\mathbb{C}^3 \times \mathbb{C}^l$ families of polar curves with respect to the projections π_b with t being a parameter, or branches of the singular locus Σ_f. The relations (9) are still satisfied.

Also the counterpart of Proposition 3.3 holds. We give its proof below.

Proposition 3.4. *There are integers $m_i \in \mathbb{N}_{\ge n}$ and functions $\varphi_i(u,b,t)$, $\psi_i(u,b,t)$ such that*

$$\begin{aligned} y_i(u,b,t) &= y_i(u,0,t) + b^2u^{m_i}\varphi_i(u,b,t), \\ z_i(u,b,t) &= z_i(u,0,t) + bu^{m_i}\psi_i(u,b,t). \end{aligned} \tag{15}$$

Moreover, either $\varphi_i \equiv \psi_i \equiv 0$ if (14) parameterizes a branch of Σ_f, or $\varphi_i(0,0,0) \neq 0$, $\psi_i(0,0,0) \neq 0$ if (14) parameterizes a family of polar curves.

Proof. If $y_i(u,b,t)$ and $z_i(u,b,t)$ are independent of b then (14) parameterizes a branch of the singular locus Σ_f. Therefore we suppose that one of them, and hence both by (9), depend notrivially on b. Expand $\frac{\partial z_i}{\partial b}(u,b,t) = \sum_{k\ge m} a_k(b,t)u^k$ with $a_m(b,t) \not\equiv 0$. To prove the result it suffices to show that $a_m(0,0) \neq 0$.

Suppose that $a_m(0,0) = 0$. Then there exists a solution $(b(u), t(u))$ of the equation $\frac{\partial z_i}{\partial b}(u,b,t) = 0$ with $(b(0), t(0)) = 0$.

By the last identity of (9), $(b(u), t(u))$ also solves $\frac{\partial y_i}{\partial b} = 0$. Recall that $f'_z + bf'_y$ vanishes identically on (8). Thus computing $\frac{\partial}{\partial b}(f'_z + bf'_y)$ on (14), and replacing (u,b,t) by $(u, b(u), t(u))$, we get

$$0 = \frac{\partial}{\partial b}(f'_z + bf'_y) = (f''_{zy} + bf''_{yy})\frac{\partial y}{\partial b} + (f''_{zz} + bf''_{yz})\frac{\partial z}{\partial b} + f'_y = f'_y. \tag{16}$$

Therefore, in this case, (14) parameterizes a component of Σ_f. ∎

Corollary 3.5.

$$\begin{aligned} Y_i(u,b,t) &= y_i(u,b,t) - bz_i(u,b,t) \\ &= y_i(u,0,t) - bz_i(u,0,t) + b^2u^{m_i}\,\mathrm{unit}(u,b,t). \end{aligned} \tag{17}$$

Proof. Using (15) we get

$$\begin{aligned} Y_i(u,b,t) &= y_i(u,b,t) - bz_i(u,b,t) \\ &= y_i(u,0,t) - bz_i(u,0,t) + b^2u^{m_i}(\varphi_i(u,b,t) - \psi_i(u,b,t)). \end{aligned}$$

Differentiating with respect to b and applying (9), we conclude that $\varphi_i(u,b,t) - \psi_i(u,b,t)$ is a unit (as ψ_i is a unit by (15)). ∎

The following lemma follows from the Zariski equisingularity assumption.

Lemma 3.6.

$$\begin{aligned} y_i(u,b,t) - y_j(u,b,t) &= u^{k_{ij}}\,\mathrm{unit}(u,b,t), \\ z_i(u,b,t) - z_j(u,b,t) &= O(u^{k_{ij}}), \\ Y_i(u,b,t) - Y_j(u,b,t) &= u^{k_{ij}}\,\mathrm{unit}(u,b,t), \end{aligned} \tag{18}$$

and $y_i(u,b,t) = O(u^n)$, $z_i(u,b,t) = O(u^n)$.

Remark 3.7. Note that by Proposition 3.4, $m_i \neq m_j$ implies $k_{ij} \le \min\{m_i, m_j\}$.

Lemma 3.8. *Let $p_i(u,0,t) = (u^n, y_i(u,0,t), z_i(u,0,t))$ parameterize a family of polar curves. Then* $\operatorname{dist}(p_i(u,0,t), \Sigma_f) \gtrsim |u|^{m_i}$.

Proof. Fix a component Σ_r of Σ_f parameterized by $(u^n, \tilde{y}_r(u,t), \tilde{z}_r(u,t), t)$. By Proposition 3.3 and Zariski equisingularity,

$$y_i(u,b,t) - \tilde{y}_r(u,t) = (y_i(u,0,t) - \tilde{y}_r(u,t)) + u^{m_i} b^2 \,\text{unit} = u^{k_{ir}} \text{unit},$$

which is possible only if $m_i \geq k_{ir} \geq n$. ∎

4. Polar wedges

In this section we consider polar wedges in the sense of [1] and [11]. These are neighbourhoods of polar curves that play a crucial role in our proof of Theorem 2.1. The formal definition is the following.

Definition 4.1 (Polar wedge). We define a *polar wedge*, denoted by $\mathcal{PW}_i$, the image of the map $p_i(u,b,t)$ defined by (14) (for $|b| < \varepsilon$ with $\varepsilon > 0$ small) that parameterizes a family of polar curves associated to the projection π_b.

Thus if $p_i(u,b,t)$ of (14) is independent of b, that is, parameterizes a branch of the singular set Σ_f, then it does not define a polar wedge. Two polar wedges (defined for the same ε) either coincide as sets or are disjoint for $u \neq 0$. Moreover, either $k_{ij} \leq \min\{m_i, m_j\}$ or $k_{ij} > m_i = m_j$.

4.1. Allowable sectors

Let $\mathcal{PW}_i$ be a polar wedge parameterized by p_i and let θ be an n-th root of unity. Then $p_i(\theta u, b, t)$ could be identical to $p_i(u,b,t)$ or not, but it always parameterizes the same polar wedge as a set. In order to avoid confusion and also to compare two different polar wedges we work over allowable sectors. We say that a sector $\Xi = \Xi_I = \{u \in \mathbb{C}; \arg u \in I\}$ is *allowable* if the interval $I \subset \mathbb{R}$ is of length strictly smaller than $2\pi/n$. If we consider only $u \in \Xi$ then $x = u^n \neq 0$ uniquely defines u. That means that over such an x, every point in the union of polar wedges is attained by a unique parameterization.

Therefore we may write such a parameterization (14) in terms of x, b, t assuming implicitly that we work over a sector Ξ,

$$p_i(x,b,t) = (x, y_i(x,b,t), z_i(x,b,t), t) \tag{19}$$

with

$$\begin{aligned} y_i(x,b,t) &= y_i(x,0,t) + b^2 x^{m_i/n} \varphi_i(x,b,t), \\ z_i(x,b,t) &= z_i(x,0,t) + b x^{m_i/n} \psi_i(x,b,t). \end{aligned} \tag{20}$$

Remark 4.2. We note that any two points in polar wedges $p_i(u_1,b_1,t_1)$ and $p_j(u_2,b_2,t_2)$ can be compared using parameterizations over the same allowable sector. Indeed, given nonzero u_1, u_2 there always exists an n-th root of unity θ and an allowable sector Ξ that contains u_1 and θu_2 and an index k such that $p_j(u_2,b_2,t_2) = p_k(\theta u_2, b_2, t_2)$.

4.2. Distance in polar wedges

For a fixed allowable sector, we will give formulas for the distance between points inside one polar wedge and the distance between points of different polar wedges. Note that these formulas imply, in particular, that different polar wedges do not intersect outside $T = \{x = y = z = 0\}$. In order to avoid heavy notation we do not use special symbols for the restriction of a polar wedge to an allowable sector.

Proposition 4.3. *For every polar wedge $\mathcal{PW}_i$ and for $x_1, x_2, b_1, b_2, t_1, t_2$ sufficiently small,*

$$\begin{aligned}\|p_i(x_1,b_1,t_1)-p_i(x_2,b_2,t_2)\| &\sim \max\{|t_1-t_2|, |x_1-x_2|, |b_1-b_2|\,|x_1|^{m_i/n}\}\\ &\sim \max\{|t_1-t_2|, |x_1-x_2|, |b_1-b_2|\,|x_2|^{m_i/n}\}.\end{aligned} \tag{21}$$

For every pair of polar wedges $\mathcal{PW}_i, \mathcal{PW}_j$, if $k_{ij} \le \min\{m_i, m_j\}$ (in particular if $m_i \neq m_j$) then

$$\begin{aligned}\|p_i(x_1,b_1,t_1) - p_j(x_2,b_2,t_2)\| &\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |x_1|^{k_{i,j}/n}\}\\ &\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |x_2|^{k_{i,j}/n}\},\end{aligned} \tag{22}$$

and if $m_i = m_j = m$ then

$$\begin{aligned}\|p_i(x_1,b_1,t_1) - p_j(x_2,b_2,t_2)\| &\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |x_1|^{k_{i,j}/n}, |b_1 - b_2|\,|x_1|^{m/n}\}\\ &\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |x_2|^{k_{i,j}/n}, |b_1 - b_2|\,|x_2|^{m/n}\}.\end{aligned} \tag{23}$$

Corollary 4.4.

$$\begin{aligned}&\|p_i(x_1,b_1,t_1) - p_j(x_2,b_2,t_2)\|\\ &\qquad\sim \|p_i(x_1,b_1,t_1) - p_j(x_1,b_1,t_1)\| + \|p_j(x_1,b_1,t_1) - p_j(x_2,b_2,t_2)\|.\end{aligned}$$

Corollary 4.5 (Lipschitz property). *There is $c > 0$ such that for all x_1, x_2, b_1, b_2, t sufficiently small,*

$$\begin{aligned}\|p_i(x_1,b_1,0) - p_j(x_2,b_2,0)\| &\le c\|p_i(x_1,b_1,t) - p_j(x_2,b_2,t)\|\\ &\le c^2\|p_i(x_1,b_1,0) - p_j(x_2,b_2,0)\|.\end{aligned}$$

Proof of Proposition 4.3. We divide the proof into four steps. In the first two steps we reduce the proofs of (21)–(23) to simpler cases. In particular, while considering the formula (21) we suppose below that $i = j$.

1. First reduction. We claim that it suffices to prove (21)–(23) for $t_1 = t_2$. This follows from

$$\begin{aligned}\|p_i(x_1,b_1,t_1) - p_j(x_2,b_2,t_2)\| &\sim |t_1 - t_2| + \|p_i(x_1,b_1,t_1) - p_j(x_2,b_2,t_2)\|\\ &\sim |t_1 - t_2| + \|p_i(x_1,b_1,t_2) - p_j(x_2,b_2,t_2)\|,\end{aligned}$$

which we show now. The first property is obvious, because $|t_1 - t_2|$ is a part of $\|p_i(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\|$.

Secondly, $p_i(x, b, t_1) - p_i(x, b, t_2) = O(t_1 - t_2)$ because $p_i(u^n, b, t)$ is analytic. This implies that

$$\begin{aligned}\|p_i(x_1, b_1, t_1) &- p_j(x_2, b_2, t_2)\| \\ &\leq \|p_i(x_1, b_1, t_1) - p_i(x_1, b_1, t_2)\| + \|p_i(x_1, b_1, t_2) - p_j(x_2, b_2, t_2)\| \\ &\lesssim |t_1 - t_2| + \|p_i(x_1, b_1, t_2) - p_j(x_2, b_2, t_2)\|.\end{aligned}$$

A similar computation gives

$$\|p_i(x_1, b_1, t_2) - p_j(x_2, b_2, t_2)\| \lesssim |t_1 - t_2| + \|p_i(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\|.$$

This completes the proof of the first reduction.

2. Second reduction. We claim that it suffices to show the formulas of the proposition for $t = t_1 = t_2$, $x_1 = x_2$. The argument is similar to the one above. The property $p_i(x_1, b, t) - p_i(x_2, b, t) = O(x_1 - x_2)$ follows from the following lemma.

Lemma 4.6. *For each i we have*

$$\begin{aligned}|y_i(u_1, b, t) - y_i(u_2, b, t)| &= O(|u_1^n - u_2^n|), \\ \left|u_1 \frac{\partial y_i}{\partial u}(u_1, b, t) - u_2 \frac{\partial y_i}{\partial u}(u_2, b, t)\right| &= O(|u_1^n - u_2^n|),\end{aligned}$$

and similar bounds hold for z_i in place of y_i.

Proof. If (u_1, b, t), (u_2, b, t) are in the same allowable sector then we have

$$|u_1^n - u_2^n| \sim |u_1 - u_2| \max\{|u_1|^{n-1}, |u_2|^{n-1}\},$$

that is, both sides are comparable up to a constant depending only on the sector. Denote $y_i(u, b, t) = u^n \hat{y}_i(u, b, t)$ and suppose $|u_2| \geq |u_1|$. Then

$$\begin{aligned}|y_i(u_1, b, t) - y_i(u_2, b, t)| &\lesssim |(u_1^n - u_2^n)\hat{y}_i(u_1, b, t)| + |u_2^n|\,|\hat{y}_i(u_1, b, t) - \hat{y}_i(u_2, b, t)| \\ &\lesssim |u_1^n - u_2^n| + |u_2^n|\,|u_1 - u_2| \sim |u_1^n - u_2^n|.\end{aligned}$$

This shows the first formula; the second one can be shown in a similar way. ∎

3. Proof of (21) and (22). We assume $t = t_1 = t_2$, $x = x_1 = x_2$. Then (21) follows from (15) and the fact that $b \mapsto b\psi(b)$ is bi-Lipschitz (ψ a unit), and (22) follows from

$$\begin{aligned}y_i&(x, b_1, t) - y_j(x, b_2, t) \\ &= (y_i(x, 0, t) - y_j(x, 0, t)) + (b_1^2 x^{m_1/n}\varphi_i(x, b_1, t) - b_2^2 x^{m_2/n}\varphi_j(x, b_2, t))\end{aligned}$$

and a similar formula for $z_i(x, b_1, t) - z_j(x, b_2, t)$.

4. Proof of (23). We assume $t = t_1 = t_2$, $x = x_1 = x_2$ and $m = m_1 = m_2$. Then

$$\begin{aligned} y_i(x,b_1,t) - y_j(x,b_2,t) &= (y_i(x,b_1,t) - y_j(x,b_1,t)) + (y_j(x,b_1,t) - y_j(x,b_2,t)) \\ &= x^{k_{ij}/n}\text{unit} + x^{m/n}(b_1^2\varphi_j(x,b_1,t) - b_2^2\varphi_j(x,b_2,t)) \\ &= x^{k_{ij}/n}\text{unit} + x^{m/n}(b_1-b_2)O(\|(b_1,b_2)\|). \end{aligned} \tag{24}$$

$$z_i(x,b_1,t) - z_j(x,b_2,t) = O(x^{k_{ij}/n}) + x^{m/n}(b_1-b_2)(\text{unit} + O(\|(b_1,b_2)\|)). \tag{25}$$

Now (23) follows from (24), (25). Indeed, we may consider separately the three cases: $|x|^{k_{i,j}/n} \sim |b_1-b_2|\,|x|^{m/n}$, $|x|^{k_{i,j}/n}$ dominant, and $|b_1-b_2|\,|x|^{m/n}$ dominant, and suppose that b_1, b_2 are small in comparison to the units. ■

5. Stratified Lipschitz vector fields on polar wedges

In this section we completely describe stratified Lipschitz vector fields on polar wedges in terms of their parameterizations. Note that these descriptions are valid only over allowable sectors (see Remark 4.2).

Let $\mathcal{PW}_i$ be a polar wedge parameterized by (14). We call the polar set C_i parameterized by $p_i(u,t) := p_i(u,0,t)$ the *spine* of $\mathcal{PW}_i$. A vector field on $\mathcal{PW}_i$ is *stratified* if it is tangent to the strata T, $C_i \setminus T$, and to $\mathcal{PW}_i \setminus C_i$.

5.1. *Stratified Lipschitz vector fields on a single polar wedge*

Let $p_{i*}(v)$ be a vector field defined on a subset of $\mathcal{PW}_i$, where

$$v(u,b,t) = \alpha(x,b,t)\frac{\partial}{\partial t} + \beta(x,b,t)\frac{\partial}{\partial x} + \delta(x,b,t)\frac{\partial}{\partial b}.$$

We always suppose the vector field $p_{i*}(v)$ is well defined on $\mathcal{PW}_i$, that is, independent of b if $x = 0$, and it is stratified, that is, tangent to T and $C_i \setminus T$:

$$p_{i*}(v) = \beta\frac{\partial}{\partial x} + \left(\beta\frac{\partial y_i}{\partial x} + \delta\frac{\partial y_i}{\partial b} + \alpha\frac{\partial y_i}{\partial t}\right)\frac{\partial}{\partial y} + \left(\beta\frac{\partial z_i}{\partial x} + \delta\frac{\partial z_i}{\partial b} + \alpha\frac{\partial z_i}{\partial t}\right)\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial t}.$$

The independence from b if $x = 0$ implies that both $\alpha(0,b,t)$ and $\beta(0,b,t)$ are independent of b, and the tangency to T ensures that in fact $\beta(0,b,t) = 0$. The tangency to $C_i \setminus T$ implies $\delta(x,0,t) = 0$. We also note that $p_{i*}(\frac{\partial}{\partial b})$ is always zero on $x = 0$.

Suppose that a function $h(u,b,t)$ defines a function $\tilde h = h \circ p_i^{-1}$ on $\mathcal{PW}_i$, that is, $h(0,b,t)$ does not depend on b. Then, by Proposition 4.3, $\tilde h$ is Lipschitz iff

$$|h(u_1,b_1,t_1) - h(u_2,b_2,t_2)| \lesssim |t_1-t_2| + |u_1^n - u_2^n| + |b_1-b_2|\,|u_2|^m. \tag{26}$$

Proposition 5.1. *The vector fields $p_{i*}(\frac{\partial}{\partial t})$, $p_{i*}(u\frac{\partial}{\partial u})$, $p_{i*}(b\frac{\partial}{\partial b})$ are stratified Lipschitz on $\mathcal{PW}_i$.*

Proof. We show that each coordinate of these vector fields is Lipschitz. For this computation it is more convenient to use the parameter u instead of x since these vector fields are analytic in u, b, t. For clarity we also drop the index i coming from the parameterization (14).

The t-coordinate of $p_*(\frac{\partial}{\partial t})$ equals $1 = \frac{\partial t}{\partial t}$ and is Lipschitz. The x-coordinate of $p_*(\frac{\partial}{\partial t})$ vanishes identically. Let us show, using Proposition 3.4 and Lemma 4.6, that the y-coordinate of $p_*(\frac{\partial}{\partial t})$ is Lipschitz (the argument for the z-coordinate is similar) :

$$\begin{aligned}
&\left|\frac{\partial y}{\partial t}(u_1, b_1, t_1) - \frac{\partial y}{\partial t}(u_2, b_2, t_2)\right| \\
&\qquad\qquad \leq \left|\frac{\partial y}{\partial t}(u_1, b_1, t_1) - \frac{\partial y}{\partial t}(u_1, b_1, t_2)\right| + \left|\frac{\partial y}{\partial t}(u_1, b_1, t_2) - \frac{\partial y}{\partial t}(u_2, b_1, t_2)\right| \\
&\qquad\qquad\quad + \left|\frac{\partial y}{\partial t}(u_2, b_1, t_2) - \frac{\partial y}{\partial t}(u_2, b_2, t_2)\right| \\
&\qquad\qquad \lesssim |t_1 - t_2| + |u_1^n - u_2^n| + |b_1 - b_2|\,|u_2|^m \\
&\qquad\qquad \sim \max\{|t_1 - t_2|, |u_1^n - u_2^n|, |b_1 - b_2|\,|u_2|^m\}.
\end{aligned}$$

A similar computation works for $p_*(x\frac{\partial}{\partial x}) = \frac{1}{n} p_*(u\frac{\partial}{\partial u})$:

$$\begin{aligned}
&\left|u_1\frac{\partial y}{\partial u}(u_1, b_1, t_1) - u_2\frac{\partial y}{\partial u}(u_2, b_2, t_2)\right| \\
&\quad \leq \left|u_1\frac{\partial y}{\partial u}(u_1, b_1, t_1) - u_1\frac{\partial y}{\partial u}(u_1, b_1, t_2)\right| + \left|u_1\frac{\partial y}{\partial u}(u_1, b_1, t_2) - u_2\frac{\partial y}{\partial u}(u_2, b_1, t_2)\right| \\
&\quad\quad + \left|u_2\frac{\partial y}{\partial u}(u_2, b_1, t_2) - u_2\frac{\partial y}{\partial u}(u_2, b_2, t_2)\right| \\
&\quad \lesssim |t_1 - t_2| + |u_1^n - u_2^n| + |b_1 - b_2|\,|u_2|^m \\
&\quad \sim \max\{|t_1 - t_2|, |u_1^n - u_2^n|, |b_1 - b_2|\,|u_2|^m\}.
\end{aligned}$$

All the other cases can be checked in a similar way. ■

Proposition 5.2. *The vector field of the form $p_{i*}(v)$, defined on a subset U of $\mathcal{PW}_i$ containing C_i, is stratified Lipschitz iff the following conditions are satisfied:*

(1) *α satisfies (26);*

(2) *$|\beta| \lesssim |x|$ and β satisfies (26);*

(3) *$|\delta| \lesssim |b|$ and $\delta x^{m/n}$ satisfies (26).*

Proof. If $p_{i*}(v)$ is Lipschitz then so is its t-coordinate, that is, α. We claim that if α satisfies (26) so do $\alpha\frac{\partial y_i}{\partial t}$ and $\alpha\frac{\partial z_i}{\partial t}$. This follows from Proposition 5.1 because the product of two Lipschitz functions is Lipschitz. This shows that $p_{i*}(\alpha\frac{\partial}{\partial t})$ is Lipschitz. By subtracting it from $p_{i*}(v)$ we may assume that $\alpha \equiv 0$.

If $p_{i*}(v)$ is Lipschitz then so is its x-coordinate, that is, β. Let $(x, b, t) \in p_i^{-1}(U)$. Then, by (21) in Proposition 4.3 and the Lipschitz property between $p_i(x, b, t)$ and $p_i(0, b, t)$, we have $|\beta| \lesssim |x|$ as claimed.

To use a similar argument to "the product of Lipschitz functions is Lipschitz", we need the following elementary generalization.

Lemma 5.3. *Suppose $h : X \to \mathbb{C}$ is a Lipschitz function on a metric space X and let $L_h := \{f : X \to \mathbb{C};\ f$ Lipschitz on $X, |f| \lesssim |h|\}$. If $f, g \in L_h$, then $\xi := fg/h \in L_h$ (here ξ is understood to be equal to 0 on the zero set of h).*

Proof. Suppose $|h(q_2)| \geq |h(q_1)|$. Then $|fg(q_2) - fg(q_1)| \lesssim |h(q_2)| \operatorname{dist}(q_1, q_2)$ and

$$\begin{aligned}
|\xi(q_2) - \xi(q_1)| &\leq \frac{|fg(q_2)h(q_1) - fg(q_1)h(q_2)|}{|h(q_1)h(q_2)|} \\
&\leq \frac{|fg(q_2)h(q_1) - fg(q_1)h(q_1)| + |fg(q_1)h(q_1) - fg(q_1)h(q_2)|}{|h(q_1)h(q_2)|} \\
&\lesssim \operatorname{dist}(q_1, q_2).
\end{aligned}$$ ∎

We apply the above lemma to $f = \beta$, $g = p_{i*}(x\frac{\partial}{\partial x})$, and $h = x$ to complete the proof of (2). Thus, by subtracting $p_{i*}(\beta\frac{\partial}{\partial x})$ from $p_{i*}(v)$ we may assume that $\beta \equiv 0$.

Consider now $p_{i*}(\delta\frac{\partial}{\partial b}) = (0, \delta\frac{\partial y_i}{\partial b}, \delta\frac{\partial z_i}{\partial b}, 0)$. By Proposition 5.1, $p_{i*}(b\frac{\partial}{\partial b})$ is Lipschitz and by (15) it satisfies $\|p_{i*}(b\frac{\partial}{\partial b})\| \lesssim |b|\,|x^{m/n}|$. Therefore if $\delta x^{m/n}$ satisfies (26) then $p_{i*}(\delta\frac{\partial}{\partial b})$ is Lipschitz if we apply Lemma 5.3 to $f = \delta x^{m/n}$, $g = p_{i*}(b\frac{\partial}{\partial b})$, and $h = bx^{m/n}$.

Conversely, if $p_{i*}(\delta\frac{\partial}{\partial b})$ is Lipschitz so is its z-coordinate $\delta\frac{\partial z_i}{\partial b}$. Moreover, because $p_{i*}(\delta\frac{\partial}{\partial b})$ is stratified (tangent to C_i), $\delta\frac{\partial z_i}{\partial b}$ is zero if $b = 0$. Therefore, since $\frac{\partial z_i}{\partial b} \sim x^{m/n}$ by (15) and (21) and the Lipschitz property between $p_i(x, 0, t)$ and $p_i(x, b, t)$, we have $|\delta| \lesssim |b|$. By Lemma 5.3 applied to $f = \delta\frac{\partial z_i}{\partial b}$, $g = bx^{m/n}$ and $h = b\frac{\partial z_i}{\partial b}$, we conclude that $\delta x^{m/n}$ satisfies (26). ∎

5.2. *Lipschitz vector fields on the union of two polar wedges*

Consider two polar wedges $\mathcal{PW}_i$ and $\mathcal{PW}_j$ parameterized by $p_i(x, b, t)$ and $p_j(x, b, t)$, over the same allowable sector (see Section 4.1 for more details).

Let $\tilde h$ be a function defined on a subset of $\mathcal{PW}_i \cup \mathcal{PW}_j$ by two functions $h_k(x, b, t)$, $k = i, j$. Then, by Proposition 4.3, $\tilde h$ is Lipschitz iff so are its restrictions $\tilde h_i$ and $\tilde h_j$ to $\mathcal{PW}_i$ and $\mathcal{PW}_j$ respectively, and

$$\begin{aligned}
&|h_i(x_1, b_1, t_1) - h_j(x_2, b_2, t_2)| \\
&\qquad \lesssim |t_1 - t_2| + |x_1 - x_2| + |x_2|^{k_{ij}/n} + |b_1 - b_2|\,|x_2|^{m/n}, \qquad (27)
\end{aligned}$$

where $m = \min\{m_i, m_j\}$.

Proposition 5.4. *The vector fields given by $p_{k*}(v)$, $k = i, j$, where v are $\frac{\partial}{\partial t}$, $x\frac{\partial}{\partial x}$, or $b\frac{\partial}{\partial b}$, are Lipschitz on $\mathcal{PW}_i \cup \mathcal{PW}_j$.*

Proof. By Corollary 4.4 and Proposition 5.1 it suffices to check only (27) for $t = t_1 = t_2$, $u = u_1 = u_2$, and $b = b_1 = b_2$. In this case the result follows from $\|p_i - p_j\| \lesssim u^{k_{ij}}$ and $(p_i - p_j)(u, b, t) = u^{k_{ij}} q(u, b, t)$, with q analytic (see Lemma 3.6). ∎

For $k = i, j$ let $p_{k*}(v_k)$ be a vector field on a subset of $W_{\Xi,k}$ given by

$$v_k(x, b; t) = \alpha_k \frac{\partial}{\partial t} + \beta_k \frac{\partial}{\partial x} + \delta_k \frac{\partial}{\partial b}.$$

Proposition 5.5. *The vector field given by $p_{k*}(v_k)$, $k = i, j$, defined on a subset U of $\mathcal{PW}_i \cup \mathcal{PW}_j$ containing $C_i \cup C_j$ is stratified Lipschitz iff the following conditions are satisfied:*

(0) *each $p_{k*}(v_k)$ is stratified Lipschitz on $U \cap \mathcal{PW}_k$;*

(1) *α_i, α_j satisfy (27);*

(2) *β_i, β_j satisfy (27);*

(3) *$\delta_i x^{m/n}, \delta_j x^{m/n}$ satisfy (27).*

Proof. The proof is similar to the proof of Proposition 5.2 and it is based on Lemma 5.3 and Proposition 5.4. ∎

Remark 5.6. If $\tilde{h}_i, \tilde{h}_j$ are stratified Lipschitz on $\mathcal{PW}_i$ and $\mathcal{PW}_j$ respectively, then, by Corollary 4.4, it suffices to check (27) for $t = t_1 = t_2$, $u = u_1 = u_2$, and $b = b_1 = b_2$. Therefore, in this case, (27) can be replaced by

$$|h_i(x, b, t) - h_j(x, b, t)| \lesssim |x|^{k_{ij}/n}. \tag{28}$$

6. Proof of Theorem 2.1: Part I

We prove the conclusion of Theorem 2.1 on $\mathcal{PW}$, that is, the union of the polar wedges and the singular set Σ_f.

6.1. *Extension of stratified Lipschitz vector fields on polar wedges in the nonparameterized case*

Let $X = \{f(x, y, z) = 0\}$, $S = \{f(x, y, z) = f'_z(x, y, z) = 0\}$, and suppose f satisfies the Transversality Assumptions. We show that $\{\mathcal{PW} \setminus S, S \setminus \{0\}, \{0\}\}$ is a Lipschitz stratification of $\mathcal{PW}$ in the sense of Mostowski.

Given $q_0 \in S \setminus \{0\}$ and a vector $v_0 = v(q_0)$ tangent to S, suppose q_0 belongs to a component S_i (a polar curve or a branch of the singular locus) of S parameterized by

$$p_i(x) = (x, y_i(x), z_i(x)), \quad q_0 = p_i(x_0)$$

and $v_0 = p_{i*}(\beta_0 \frac{\partial}{\partial x})$. Then the vector field on S defined on each S_j by $v_j = p_{j*}(\beta x \frac{\partial}{\partial x})$, with $\beta = \beta_0/x_0$, defines a Lipschitz extension of v_0. This shows (L1).

Consider a stratified Lipschitz vector field v on $S \cup \{q_0\}$ with $q_0 = p_i(x_0, b_0) \in \mathcal{PW}_i$ defined by $p_{j*}v_j$ on the component S_j of S, where

$$v_j(x, b) = \beta_j \frac{\partial}{\partial x} + \delta_j \frac{\partial}{\partial b}.$$

Thus, for $j \neq i$, the functions β_j and δ_j are defined only for $b = 0$ (and hence $\delta_j = 0$ since the vector field is stratified). The functions β_i and δ_i are defined on $\{(x,b); b = 0\} \cup \{(x_0,b_0)\}$. Denote $\beta_0 = \beta_i(x_0,b_0)$, $\delta_0 = \delta_i(x_0,b_0)$. By Propositions 5.2 and 5.5 it suffices to extend β_j and δ_j to two families of functions, still denoted by β_j, δ_j, that satisfy the conditions given in those propositions. For all j we define

$$\beta_j(x,b) = (\beta_0 - \beta_i(x_0,0))\frac{b}{b_0}\frac{x^{m_j/n}}{x_0^{m_i/n}} + \beta_j(x,0), \tag{29}$$

$$\delta_j(x,b) = (\delta_0 b)/b_0. \tag{30}$$

Then, because $|\beta_0 - \beta_i(x_0,0)| \le CL|b_0|\,|x_0|^{m_i/n}$, where L is the Lipschitz constant of the vector field v and C is a universal constant, the first summand of the right-hand side of (29) satisfies (2) of Propositions 5.2 and 5.5. The argument for (30) is similar because $|\delta_0| \le CL|b_0|$. This completes the proof of Theorem 2.1 for $\mathcal{PW}$ in the nonparameterized case.

6.2. *Parameterized case*

By Corollary 4.5 and Propositions 5.2, 5.5, the map $\mathcal{X}_0 \times T \to \mathcal{X}$, restricted to $\mathcal{PW} \cap \mathcal{X}_0$, defined in terms of the parameterizations of polar wedges by

$$(p_i(0,x,b),t) \mapsto p_i(x,b,t),$$

is not only Lipschitz but also establishes a bijection between the Lipschitz vector fields. Therefore, by Proposition 2.4, $\{\mathcal{PW} \setminus S, S \setminus T, T\}$ is a Lipschitz stratification if and only if so is its intersection with $\mathcal{X}_0$, and the latter is a Lipschitz stratification by the nonparameterized case. We use here the easy observation that the cartesian product of a Lipschitz stratification by a smooth space is also Lipschitz (actually the cartesian product of two Lipschitz stratifications is Lipschitz).

6.3. *Examples*

In [10] Mostowski gives a criterion for the codimension 1 stratum of a Lipschitz stratification. In particular he proposes the following example (we change the order of variables so it follows our notation): $f(x,y,z) = z^2 - (y^3 + y^2x^2)$. The singular set Σ_f of $X = \{f = 0\}$ is the x-axis but Mostowski shows that $\{X \setminus \Sigma_f, \Sigma_f \setminus \{0\}, \{0\}\}$ is not a Lipschitz stratification of X. By solving the system $f = \partial f/\partial z - b\partial f/\partial y = 0$ one can check that there is one polar wedge with $n = 1$ and $m = 4$ given by

$$y = -x^2 + b^2x^4\varphi(x,b), \quad z = 3bx^4\psi(x,b),$$

and one has to add a generic polar curve, or just a curve $y = -x^2 + b^2x^4 + \cdots$, $z = 3bx^4 + \cdots$, to Σ_f to get the one-dimensional stratum. In [10, Section 7] Mostowski studies the case of surface singularities in $\mathbb{C}^3$ and shows in particular the following result; we give an alternative proof.

Proposition 6.1. *If X has an isolated singularity but there is an $m_i > n$ then $\{X \setminus \{0\}, \{0\}\}$ is not a Lipschitz stratification of X.*

Proof. Let $q_0 = p(x_0, b_0) \in X \setminus \{0\}$ be on the polar wedge parameterized by $p(x, b) = (x, y(x, b), z(x, b))$, $x = u^n$, where y, z are as in (11). Let $v_0 = p_*(\frac{\partial}{\partial b})$ be the vector tangent to X at $q_0 = p(x_0, b_0)$. We extend it by 0 to $\{0\}$ and get a Lipschitz vector field on $\{0\} \cup \{q_0\}$ with Lipschitz constant $L = C x_0^{m/n-1}$, where $C > 0$ depends only on the polar wedge. Suppose we extend this vector field to $q_1 = p(x_1, b_1)$, $x_0 = x_1$ by $v_1 = p_*(\alpha_1 \frac{\partial}{\partial x} + \delta_1 \frac{\partial}{\partial b})$ so that the extended vector field has Lipschitz constant $L_1 = C_1 L$. By the Lipschitz property of the x-coordinate of this vector field, $|\alpha_1| \le C_1 L \|q_0 - q_1\| \sim C_1 L |b_0 - b_1|\, |x_0|^{m/n}$. Therefore, we can subtract from v_1 the vector $p_*(\alpha_1 \frac{\partial}{\partial x})$ without changing the Lipschitz constant significantly (just changing C_1). Thus we may assume that $\alpha_1 = 0$. By the Lipschitz property of the y- and z-coordinates of this vector field,

$$\begin{aligned} b_0 x_0^{m/n} \tilde\varphi(x_0, b_0) - \delta_1 b_1 x_0^{m/n} \tilde\varphi(x_0, b_1) &= O(|b_0 - b_1| x_0^{m/n}) L_1, \\ x_0^{m/n} \tilde\psi(x_0, b_0) - \delta_1 x_0^{m/n} \tilde\psi(x_0, b_1) &= O(|b_0 - b_1| x_0^{m/n}) L_1, \end{aligned} \tag{31}$$

where $\tilde\varphi, \tilde\psi$ are units. Considering (31) as a system of linear equations in 1 (in front of the first summands of both equations) and δ_1, by Cramer's rule,

$$1 \lesssim |L_1| \sim |x_0^{m/n-1}|, \quad |\delta_1| \lesssim |L_1| \sim |x_0^{m/n-1}|,$$

which is impossible if we let $x_0 \to 0$, because by our assumption $m > n$. ∎

7. Quasi-wings

Quasi-wings were introduced by Mostowski [9, Section 5] in order to show the existence of a Lipschitz stratification in the complex analytic case. In this construction Mostowski used several corank 1 projections, instead of a single one, to cover the whole complement of Σ_f in $\mathcal{X}$ by quasi-wings. We use quasi-wings to study Lipschitz vector fields on the complement of $\mathcal{PW}$.

The main idea of the construction is as follows (the details will follow later). Given a real analytic arc $p(s)$, $s \in [0, \varepsilon)$, of the form

$$p(s) = (s^n, y(s), z(s), t(s)), \quad y(s) = O(s^n), z(s) = O(s^n), \tag{32}$$

our goal is to embed $p(s)$ in a quasi-wing $\mathcal{QW}$ (kind of cuspidal neighbourhood of $p(s)$ in $\mathcal{X}$) that is the graph of a root of f over a set $\mathcal{W}_q$, the image of

$$q(u, v, t) = (u^n, y(u, t) + u^{\tilde l} v, t),$$

where $u, v \in \mathbb{C}$ are supposed small. Geometrically, $\mathcal{W}_q$ is a cuspidal neighbourhood of $\pi(p(s))$, which we call a wedge, and $\mathcal{QW}$ is its lift to $\mathcal{X}$. Thus $\mathcal{QW}$ admits a parameterization of the form $p(u, v, t) = (q(u, v, t), z(u, v, t))$ such that $p(s) = p(s, 0, t(s))$. We shall make the following assumptions on $p(s)$:

(1) $p(s)$ is not included in S and for every polar branch C_i there is an exponent l_i such that $s^{l_i} \sim \operatorname{dist}(p(s), C_i) \sim \operatorname{dist}(\pi(p(s)), \pi(C_i))$. A similar assumption is made on every branch of the singular locus Σ_f. In particular, we have $\operatorname{dist}(p(s), S) \sim \operatorname{dist}(\pi(p(s)), \pi(S))$.

(2) For every polar branch C_i we have $l_i \le m_i$ (for the definition of m_i see Proposition 3.4). This implies that $p(s)$ is not included in $\mathcal{PW}_i$.

We have the following requirement on $\mathcal{QW}$:

(3) $s^{\tilde{l}} \lesssim \operatorname{dist}(p(s), S) \sim \operatorname{dist}(\pi(p(s)), \pi(S))$,

that is, $\mathcal{QW}$ does not touch S (except along T), and this property is preserved by the projection to the t, x, y-space.

Then $\mathcal{PW} \cap \mathcal{QW}$ is just the T stratum and as we will show in Proposition 7.3,

(4) $\mathcal{QW}$ is the graph of a root of f whose first order partial derivatives are all bounded. In particular, the projection $\mathcal{QW} \to \mathcal{W}_q$ is bi-Lipschitz.

In the formal definition of quasi-wings we will require that $\tilde{l}$ is chosen minimal for (3), i.e. $s^{\tilde{l}} \sim \operatorname{dist}(p(s), S) \sim \operatorname{dist}(\pi(p(s)), \pi(S))$ (we seek the maximal possible set $\mathcal{W}_q$ satisfying the above properties). We show in Proposition 7.7 that each real analytic arc satisfying (1) and (2) can be embedded in a quasi-wing. In general, any real analytic arc that is not embedded in the singular locus satisfies (1) or (2) after a small linear change of coordinates and therefore can be embedded in a quasi-wing in the new coordinates (see Corollary 7.8). We note that our construction of quasi-wings differs significantly from the one of Mostowski. We use the Puiseux with parameter theorem and arcwise analytic trivializations of [17]. The latter reference also provides a crucial partial Lipschitz property; see Remark 7.6 that we use in the proof of Proposition 7.7. Consequently, our construction of quasi-wings can be extended to the real analytic set-up. Mostowski uses instead the bound on derivatives of holomorphic functions (Schwarz's Lemma).

7.1. Regular wedges and quasi-wings

Let $\Delta(x, y, t)$ denote the discriminant of $f(x, y, z, t)$. The discriminant locus $\Delta = 0$ is the finite union of families of analytic plane curves parameterized by

$$(u, t) \mapsto (u^n, y_i(u, t), t). \tag{33}$$

By the Zariski equisingularity assumption we have

$$y_i(u, t) - y_j(u, t) = u^{k_{ij}} \operatorname{unit}(u, t),$$

and by the Transversality Assumptions, $y_i(u, t) = O(u^n)$. Note that y_i of (33) either is the projection of a polar branch, denoted by $y_i(u, 0, t)$ in (15) and from now on indexed by $i \in I_C$, or parameterizes the projection of a branch of the singular locus Σ_f, and it will be indexed by $i \in I_\Sigma$.

Given an analytic family of analytic arcs

$$q(u,t) = (u^n, y(u,t), t), \tag{34}$$

we assume $y(u,t) = O(u^n)$ and that for each discriminant branch (33), $y(u,t)$ satisfies, for some integers $\tilde{l}_i$,

$$y(u,t) - y_i(u,t) = u^{\tilde{l}_i} \mathrm{unit}(u,t).$$

Remark 7.1. As both $y(u,t) = O(u^n)$ and $y_i(u,t) = O(u^n)$ it follows that $\tilde{l}_i \geq n$.

Consider the map

$$q(u,v,t) = (u^n, y(u,t) + u^{\tilde{l}} v, t), \tag{35}$$

defined for complex v, $|v| < \varepsilon$ with $\varepsilon > 0$ small, and denote its image by $\mathcal{W}_q$. We suppose $\tilde{l} \geq \max_i \tilde{l}_i$, that is, the image of q, for $u \neq 0$, is inside the complement of the discriminant locus $\Delta = 0$.

Lemma 7.2. *Let $g(u,v,z,t) = f(q(u,v,t),z)$. If $\tilde{l} \geq \max_i \tilde{l}_i$ then the discriminant of g satisfies*

$$\Delta_g = u^N \mathrm{unit}(u,v,t). \tag{36}$$

Proof. Write the discriminant of f

$$\Delta(u^n, y, t) = \mathrm{unit}(u,y,t) \prod_i (y - y_i(u,t))^{d_i}.$$

Then, by the assumption $\tilde{l} \geq \max_i \tilde{l}_i$,

$$\Delta_g(u,v,t) = \Delta(u^n, y(u,t) + vu^{\tilde{l}}, t) = u^{\sum \tilde{l}_i d_i} \mathrm{unit}(u,v,t). \qquad \blacksquare$$

Therefore, by the Puiseux with parameter theorem, after a ramification in u, we may assume that the roots of g are analytic functions of the form $z_\tau(u,v,t) = z_\tau(u^n, y(u,t) + vu^{\tilde{l}}, t)$ and that for every pair of such roots,

$$z_\tau(u,v,t) - z_\nu(u,v,t) \sim u^{r_{\tau\nu}}. \tag{37}$$

Moreover, by transversality of the projection π, $z_\tau(u,v,t) = O(u^n)$.

Proposition 7.3. *Suppose $\tilde{l}_i \leq m_i$ for every projection (33) of a polar branch. Then the (first order) partial derivatives of the roots $z_\tau(x,y,t)$ of f over $\mathcal{W}_q$ (the image of (35)) are bounded. Therefore, the roots of g are of the form*

$$z_\tau(u,v,t) = z_\tau(u,t) + vu^{\tilde{l}} \tilde{\psi}(u,v,t) \tag{38}$$

with $\tilde{\psi}(u,v,t)$ analytic.

Proof. The derivative $\frac{\partial}{\partial t}(z_\tau(x,y,t))$ is bounded on $\mathcal{W}_q$ because $z_\tau(u,v,t)$ is analytic in t. Similarly $x\frac{\partial}{\partial x}(z_\tau(x,y,t))$ is $O(x)$ because $z_\tau(u,v,t)$ is analytic in u and

$$x\frac{\partial z_\tau}{\partial x} \simeq u\frac{\partial z_\tau}{\partial u} \lesssim u^n.$$

Finally, $\frac{\partial}{\partial y}(z_\tau(x,y,t))$ is bounded on $\mathcal{W}_q$ by the conditions $\tilde{l}_i \le m_i$, $\tilde{l}_i \le \tilde{l}$, and (15). Indeed, since $f(x,y,z_\tau(x,y,t),t) \equiv 0$, on the graph of z_τ we have

$$0 = \frac{\partial}{\partial y} f(x,y,z_\tau(x,y,t),t) = f'_y + \frac{\partial z_\tau}{\partial y} f'_z.$$

If $|\frac{\partial z_\tau}{\partial y}| > N$, then, by (7), the graph of $z_\tau(x,y,t)$ on $\mathcal{W}_q$ would intersect a polar wedge $\mathcal{P}\mathcal{W}_i$ for $b = (\frac{\partial z_\tau}{\partial y})^{-1}$. This is only possible if $\tilde{l}_i \ge \min\{\tilde{l}, m_i\}$. If $\tilde{l}_i = \min\{\tilde{l}, m_i\}$ then this intersection is empty provided both b and v are sufficiently small (and hence N large). ∎

We now introduce a version of quasi-wings and nicely-situated quasi-wings of [9].

Definition 7.4 (Quasi-wings). We say that the image of $q(u,v,t)$ of (35) is a *regular wedge* $\mathcal{W}_q$ if $\tilde{l} = \max_{i\in I_C\cup I_\Sigma} \tilde{l}_i$ and $\tilde{l}_i \le m_i$ for every $i \in I_C$. Then by a *quasi-wing* $\mathcal{Q}\mathcal{W}_\tau$ over $\mathcal{W}_q$ we mean the image of an analytic map $p_\tau(u,v,t) = (q(u,v,t), z_\tau(u,v,t))$, where z_τ is a root of $f(q_t(u,v),z)$.

We say that two quasi-wings $\mathcal{Q}\mathcal{W}_\tau, \mathcal{Q}\mathcal{W}_\nu$ are *nicely-situated* if they lie over the same regular wedge $\mathcal{W}_q$.

7.2. *Construction of quasi-wings*

Consider a real analytic arc $p(s)$, $s \in [0,\varepsilon)$, of the form

$$\begin{aligned} &p(s) = (s^n, y(s), z(s), t(s)), \quad \pi(p(s)) = q(s) = (s^n, y(s), t(s)),\\ &y(s) = O(s^n), \quad z(s) = O(s^n). \end{aligned} \tag{39}$$

Under some additional assumptions we construct in Proposition 7.7 a quasi-wing containing the arc $p(s)$. For this we use (in the proof of Lemma 7.5) the arcwise analytic trivializations of [17] and construct, following [17, Proposition 7.3], a complex analytic wing containing $q(s)$.

Let

$$(u^n, y_i(u,t), z_i(u,t), t), \qquad i \in I_C,$$

be a parameterization of the polar branch C_i, and let

$$(u^n, y_k(u,t), z_k(u,t), t), \qquad k \in I_\Sigma,$$

be a parameterization of the branch Σ_k of the singular set Σ_f.

Lemma 7.5. *Let $q(s) = (s^n, y(s), t(s))$, $y(s) = O(s^n)$, be a real analytic arc at the origin. For each polar branch C_i, parameterized as above, denote $q_i(u,t) = (u^n, y_i(u,t), t)$ and let $\tilde{l}_i = \operatorname{ord}_s(y(s) - y_i(s, t(s)))$. Then there is a complex analytic wing parameterized by*

$$q(u,t) = (u^n, y(u,t), t), \quad y(u,t) = O(u^n),$$

containing $q(s)$, that is, satisfying $y(s) = y(s, t(s))$, such that $y(u,t) - y_i(u,t)$ equals $u^{\tilde{l}_i}$ times a unit. In particular, over the same allowable sector we have

$$\|(u_1^n, y(u_1,t_1), t_1) - (u_2^n, y_i(u_2,t_2), t_2)\| \sim \max\{|t_1 - t_2|, |u_1^n - u_2^n|, |u_2|^{\tilde{l}_i}\} \tag{40}$$

and $\operatorname{ord}_s \operatorname{dist}(q(s), \pi(C_i)) = \tilde{l}_i$.

Proof. By [17, Theorem 3.3] there is an arcwise analytic local trivialization $\Phi : \mathbb{C}^2 \times T \to \mathbb{C}^2 \times T$ preserving the discriminant locus $\Delta = 0$. In particular, Φ is of the form

$$\Phi(x,y,t) = (\Psi_1(x,t), \Psi_2(x,y,t), t), \tag{41}$$

it is complex analytic with respect to t, and both Φ and Φ^{-1} are real analytic on real analytic arcs. By [17, Proposition 3.7] we may require $\Psi_1(x,t) = x$, so the allowable sectors are preserved.

By the arc-analyticity of Φ^{-1}, there exists a real analytic arc $(s^n, \tilde{y}(s), t(s))$ such that $\Phi(s^n, \tilde{y}(s), t(s)) = (s^n, y(s), t(s))$. Then, by the arcwise analyticity of Φ, the map $q(s,t) = \Phi(s^n, \tilde{y}(s), t)$ is analytic in both s and t, and its complexification $q(u,t)$ is a complex analytic wing containing $q(s)$.

Remark 7.6. Arcwise analytic trivializations of [17] satisfy a *partial Lipschitz property*, namely they are bi-Lipschitz in the last variable, i.e., Ψ_1 with respect to x and Ψ_2 with respect to y, etc.; see [17, property (Z3) of Theorem 3.3].

By the partial Lipschitz property,

$$s^{\tilde{l}_i} \sim |y(s) - y_i(s,t(s))| = |\Psi_2(s^n, \tilde{y}(s), t(s)) - \Psi_2(s^n, y_i(s,0), t(s))| \sim |\tilde{y}(s) - y_i(s,0)|.$$

This implies, again by the partial Lipschitz property of Ψ_2, that $s^{\tilde{l}_i} \sim y(s,t) - y_i(s,t)$. Therefore $y(u,t) - y_i(u,t)$, being analytic, equals $u^{\tilde{l}_i}$ times a unit.

Since $y(u,t) = O(u^n)$, $y_i(u,t) = O(u^n)$, and $y(u,t) - y_i(u,t) \sim u^{\tilde{l}_i}$, the proof of (40) can be obtained in a similar, even simpler, way as the formula (22) of Proposition 4.3. ∎

We set

$$\begin{aligned} l_i &:= \operatorname{ord}_s \operatorname{dist}(p(s), C_i) \le \tilde{l}_i := \operatorname{ord}_s \operatorname{dist}(\pi(p(s)), \pi(C_i)), \quad i \in I_C, \\ l_k &:= \operatorname{ord}_s \operatorname{dist}(p(s), \Sigma_k) \le \tilde{l}_k := \operatorname{ord}_s \operatorname{dist}(\pi(p(s)), \pi(\Sigma_k)), \quad k \in I_\Sigma, \\ l &:= \max\{l_i, l_k\}, \quad \tilde{l} := \max\{\tilde{l}_i, \tilde{l}_k\}. \end{aligned}$$

Proposition 7.7 (Existence of quasi-wings I). *Assume that the arc $p(s)$ satisfies*

$$\forall i \in I_C, \quad m_i \geq \tilde{l}_i, \tag{42}$$

$$\forall j \in I := I_C \cup I_\Sigma, \quad l_j = \tilde{l}_j. \tag{43}$$

Then there is a regular wedge W_q containing the projection $q(s) = \pi(p(s))$ and parameterized by $q(u,v,t) = (u^n, y(u,t) + vu^{\tilde{l}}, t)$, $q(u,t) := q(u,0,t)$, satisfying $q(s,t(s)) = q(s)$ and such that $\pi^{-1}(W_q)$ is a finite union of nicely-situated quasi-wings. One of these quasi-wings contains $p(s)$.

Proof. If we apply Lemma 7.5 to $q(s) = \pi(p(s))$ then we get $\tilde{l}_i = l_i$, thus $l = \tilde{l}$ and therefore

$$s^{l_i} \sim \operatorname{dist}(\pi(p(s)), \pi(C_i)) \sim |y(s) - y_i(s, t(s))| \sim |\tilde{y}(s) - y_i(s, 0)|.$$

A similar property holds for each component Σ_k of the singular locus.

The map

$$q(u,v,t) = (u^n, y(u,t) + u^l v, t),$$

for v small, parameterizes a regular wedge $\mathcal{W}_q$. The inverse image $\pi^{-1}(\mathcal{W}_q) \cap \mathcal{X}$ is a finite union of nicely-situated quasi-wings, and one of them contains $p(s)$. ∎

Corollary 7.8 (Existence of quasi-wings II). *Suppose that $p(s) = (s^n, y(s), z(s), t(s))$ is a real analytic arc in $\mathcal{X}$ and is not contained in the singular locus Σ_f. Then, for b_0 small and generic, $p(s)$ belongs to a quasi-wing in the coordinates x, Y_{b_0}, z, t, where $Y_{b_0} := y - b_0 z$.*

(Here by generic we mean in $\{b \in \mathbb{C};\ |b| < \varepsilon\} \setminus A$, where A is finite. Moreover, we show that one may choose $\varepsilon > 0$ independent of $p(s)$.)

Proof. Recall that

$$\tilde{l}_i := \operatorname{ord}_s \operatorname{dist}(\pi(p(s)), \pi(C_i)), \quad \tilde{l}_k := \operatorname{ord}_s \operatorname{dist}(\pi(p(s)), \pi(\Sigma_k)).$$

If $\tilde{l}_i = l_i \leq m_i$ for all $i \in I_C$ and $\tilde{l}_k = l_k$ for all $k \in I_\Sigma$ then the result follows from Proposition 7.7. Nevertheless, whether this is satisfied or not, it follows from Lemma 7.5 that $\tilde{l}_i = \operatorname{ord}_s(y(s) - y_i(s, t(s)))$.

We denote $\pi_b(x, y, z, t) := (x, y - bz, t)$ and by $C_{i,b}$ the associated polar set. By the Transversality Assumptions, $\mathcal{X}$ is Zariski equisingular with respect to π_b for b sufficiently small (that defines ε). We claim that if $\tilde{l}_i > l_i$ and $l_i \leq m_i$ then $\operatorname{ord}_s \operatorname{dist}(\pi_b(p(s)), \pi_b(C_i)) = l_i$ for $b \neq 0$. Indeed, otherwise this order is strictly greater than l_i and then, again by Lemma 7.5,

$$|y(s) - y_i(s, t(s)) - b(z(s) - z_i(s, t(s)))| \ll s^{l_i}.$$

From $\tilde{l}_i > l_i$ we have $|y(s) - y_i(s, t(s))| \ll s^{l_i}$ and so $|z(s) - z_i(s, t(s))| \ll s^{l_i}$, which contradicts $\operatorname{ord}_s \operatorname{dist}(p(s), C_i) = l_i$. Moreover, we claim that $\operatorname{ord}_s \operatorname{dist}(\pi_b(p(s)), \pi_b(C_{i,b}))$ $= l_i$ for $b \neq 0$ and small. Indeed, by (17),

$$Y_b(s, b, t(s)) - (y(s) - bz(s))$$
$$= (y_i(s, t(s)) - y(s)) - b(z_i(s, t(s)) - z(s)) + b^2 s^{m_i} \operatorname{unit}(s, b, t(s)).$$

The first summand is of size $s^{\tilde{l}_i}$, the second of size bs^{l_i}, and the third of size $b^2 s^{m_i}$. Therefore the claim follows for small $b \neq 0$ because $l_i \leq m_i$.

If $l_i > m_i$ then $\operatorname{ord}_s \operatorname{dist}(p(s), C_{i,b}) = m_i$ for $b \neq 0$. Therefore, in general, only for finitely many b, one for each C_i, we do not have $\operatorname{ord}_s \operatorname{dist}(p(s), C_{i,b}) \leq m_i$.

Finally, by a similar argument, $\operatorname{ord}_s \operatorname{dist}(p(s), \Sigma_k) = \operatorname{ord}_s \operatorname{dist}(\pi_b(p(s)), \pi_b(\Sigma_k))$ for all b but one.

Thus the statement follows from Proposition 7.7. ■

7.3. Basic properties of quasi-wings

Let $p(s)$ be an arc as given in (39) satisfying the assumptions of Proposition 7.7 and let $\mathcal{QW}$ be the quasi-wing constructed in the proof of this proposition. Let $p(u, v, t) = (q(u, v, t), z(u, v, t))$ be its parameterization. Then, by Lemma 7.5, $\tilde{l}_i = \operatorname{ord}_s(y(s) - y_i(s, t(s)))$ and $\operatorname{dist}(p(s), C_i) \sim \operatorname{dist}(p(s), \mathcal{PW}_i) \sim s^{l_i}$ (and recall $\tilde{l}_i = l_i \geq m_i$).

We shall show that the distances from $\mathcal{QW}$ to $\mathcal{PW}_i$ and to Σ_k are constant, that is, they are of order u^{l_i} and u^{l_k} respectively. This follows from their construction that uses arcwise trivializations of [17] and the partial Lipschitz property of these trivializations (see Remark 7.6).

Recall that $\mathcal{QW}$ is constructed as follows. Let (41) be an arcwise trivialization preserving the discriminant locus $\Delta = 0$. Then there is an arc $q_0(s) = (s^n, \tilde{y}(s), 0)$ such that $\Phi(u^n, \tilde{y}(u), t)$ is a complex analytic wing containing $q(s) = \Phi(s^n, \tilde{y}(s), t(s))$. The lift of Φ is an arcwise analytic trivialization of $\mathcal{X}$ (see [17, proof of Theorem 3.3]. Let us denote this lift by

$$\tilde{\Phi}(x, y, z, t) = (\Psi_1(x, t), \Psi_2(x, y, t), \Psi_3(x, y, z, t), t),$$

with $\Psi_1(x, t) = x$. Let $p_0(s)$ denote the lift of $q_0(s)$. Then $p(s) = p(s, t(s)) = \tilde{\Phi}(p_0(s), t(s))$.

The following proposition extends the conclusion of Lemma 7.5 from the complex analytic wing $q(u, t)$ to the quasi-wing $\mathcal{QW}$.

Proposition 7.9. *Let $\mathcal{QW}$ be the quasi-wing containing $p(s)$ given by Proposition 7.7 and let $p(u, v, t) = (q(u, v, t), z(u, v, t))$ be its parameterization. Then for the polar sets C_i parameterized by $p_i(u, t)$ and Σ_k by $p_k(u, t)$,*

$$p(u, v, t) - p_i(u, t) \sim u^{l_i}, \quad p(u, v, t) - p_k(u, t) \sim u^{l_k}.$$

This implies that $\operatorname{dist}(p(u, v, t), \mathcal{PW}_i) \sim u^{l_i}$ *and* $\operatorname{dist}(p(u, v, t), \Sigma_k) \sim u^{l_k}$.

Proof. It will be convenient in the proof to use the constant ε of Definition 4.1 and for this constant fixed, i.e. for $|b| < \varepsilon$, denote the polar wedges by $\mathcal{PW}_{i,\varepsilon}$ and their closures by $\overline{\mathcal{PW}}_{i,\varepsilon}$. We denote by $\mathcal{PW}_\varepsilon$ (and by $\overline{\mathcal{PW}}_\varepsilon$) the union of $\mathcal{PW}_{i,\varepsilon}$ (respectively of $\overline{\mathcal{PW}}_{i,\varepsilon}$) for all i and the singular set Σ_f.

Lemma 7.10. *$\tilde\Phi$ preserves polar wedges in the following sense. There is a constant L (depending on the Lipschitz constant of Ψ_2 for its partial Lipschitz property, see Remark 7.6) such that*

$$\mathcal{PW}_{i,\varepsilon/L} \subset \tilde\Phi(\mathcal{PW}_{i,\varepsilon}) \subset \mathcal{PW}_{i,L\varepsilon}.$$

Proof. By construction $\tilde\Phi$ preserves the polar set and the singular locus. Therefore the lemma follows from the partial Lipschitz property of Ψ_2 and parameterization (15). ■

Lemma 7.11. *The following holds:*

$$\operatorname{dist}(\tilde\Phi(p_0(s),t),\mathcal{PW}_i) \sim s^{l_i}, \qquad \operatorname{dist}(\tilde\Phi(p_0(s),t),\Sigma_k) \sim s^{l_k}.$$

Proof. Let $l = \max_{i\in I} l_i$. First for fixed $\varepsilon > 0$ we show that

$$\operatorname{dist}(\tilde\Phi(p_0(s),t),\overline{\mathcal{PW}}_\varepsilon) \sim s^l. \tag{44}$$

It is clear that this distance is $\gtrsim s^l$, this already holds after the projection π. We show the opposite inequality.

Fix $s_0 > 0$. By Lemma 7.5, $\operatorname{dist}(q_0(s_0),\pi(\overline{\mathcal{PW}}_\varepsilon)\cap\{t=0,s=s_0\}) \sim s_0^l$. Let $c(s_0)$ be such that this distance equals exactly $c(s_0)s_0^l$ and let $q_{\min}(s_0)$ be one of the points in $\pi(\overline{\mathcal{PW}}_\varepsilon)\cap\{t=0,s=s_0\}$ realizing this distance. Let τ be the lift of the segment joining $q_0(s_0)=\pi(p_0(s_0))$ and $q_{\min}(s_0)$. Since τ is in the complement of $\mathcal{PW}_\varepsilon$ (unless its endpoint is in Σ_f), by the boundedness of partial derivatives (cf. the proof of Proposition 7.3), its length is comparable to the length of the segment, that is, s_0^l. Denote by $p_{\min}(s_0)$ the other endpoint of this lift, so that $q_{\min}(s_0)=\pi(p_{\min}(s_0))$. Since Ψ_2 is partially Lipschitz and $\tilde\Phi$ preserves the complement of $\mathcal{PW}_\varepsilon$ (see Lemma 7.10), we have, for small t,

$$\operatorname{dist}(\tilde\Phi(p_0(s_0),t),\tilde\Phi(p_{\min}(s_0),t)) \lesssim s_0^l. \tag{45}$$

Since the distance $c(s_0)s_0^l$ is a subanalytic function we may suppose, by a choice of $q_{\min}(s_0)$, that also $q_{\min}(s_0)$ and $p_{\min}(s_0)$ are subanalytic in s_0.

There are three cases to consider: $p_{\min}(s_0)\in\overline{\mathcal{PW}}_\varepsilon\setminus\Sigma_f$, $p_{\min}(s_0)\in\Sigma_f$, and $p_{\min}(s_0)\notin\overline{\mathcal{PW}}_\varepsilon$.

If $p_{\min}(s_0)$ is in $\overline{\mathcal{PW}}_\varepsilon\setminus\Sigma_f$ then so is $\tilde\Phi(p_{\min}(s_0),t)$, since $\tilde\Phi$ preserves the polar set, and the claim follows from (45). A similar argument applies if $p_{\min}(s_0)\in\Sigma_f$.

If $p_{\min}(s_0)\notin\overline{\mathcal{PW}}_\varepsilon$ then there is another point in $\pi^{-1}(q_{\min}(s_0))$ that is in $\overline{\mathcal{PW}}_\varepsilon$. Suppose that it is in $\overline{\mathcal{PW}}_{j,\varepsilon}$ and denote it by $p_j(s_0)$. By the assumptions $l_j=\tilde l_j=\tilde l=l$ and by the partial Lipschitz property the magnitude of $\operatorname{dist}(\tilde\Phi(p_j(s_0),t),\tilde\Phi(p_{\min}(s_0),t))$ is independent of t, say $\sim s_0^\alpha$. If $\alpha\ge l$ then (44) follows from (45). If $\alpha<l$ then we have $\operatorname{dist}(\tilde\Phi(p_j(s_0),t),\tilde\Phi(p_{\min}(s_0),t)) \sim \operatorname{dist}(\overline{\mathcal{PW}}_j,\tilde\Phi(p_{\min}(s_0),t))$ and therefore

$\mathrm{dist}(\tilde\Phi(p_j(s_0),t),\tilde\Phi(p_0(s_0),t)) \sim \mathrm{dist}(\overline{\mathcal{PW}}_j,\tilde\Phi(p_0(s_0),t))$. But, by assumption on the curve $p(s)=\tilde\Phi(p_0(s),t(s))$,

$$\begin{aligned}&\mathrm{dist}(\tilde\Phi(p_j(s_0),t(s_0)),\tilde\Phi(p_{\min}(s_0),t(s_0)))\\&\qquad\le \mathrm{dist}(\tilde\Phi(p_j(s_0),t(s_0)),p(s_0))+\mathrm{dist}(p(s_0),\tilde\Phi(p_{\min}(s_0),t(s_0)))\le Cs_0^l,\end{aligned}$$

for a universal constant C. This shows that the case $\alpha<l$ is impossible.

Now we show that (44) implies the claim of the lemma. Again, it is enough to show $\lesssim$ since the opposite inequality is already known for the sets projected by π. Firstly, the distance on the left-hand side of (44) has to be attained on $\overline{\mathcal{PW}}_{j,\varepsilon}$ or on Σ_k. Suppose, for simplicity, that it occurs on $\overline{\mathcal{PW}}_{j,\varepsilon}$. Then $l=l_j$, which implies the claim for $i=j$. By the above there is a curve $p_j(s)\in\overline{\mathcal{PW}}_j\cap\{t=0\}$ such that

$$\mathrm{dist}(\tilde\Phi(p_0(s),t),\tilde\Phi(p_j(s),t))\sim s^{l_j}. \tag{46}$$

Let $i\neq j$. Then $l_i\le l_j$ and

$$\mathrm{dist}(\tilde\Phi(p_0(s),t),\overline{\mathcal{PW}}_i)\lesssim s^{l_j}+\mathrm{dist}(\tilde\Phi(p_j(s),t),\overline{\mathcal{PW}}_i). \tag{47}$$

To complete the proof we note that $\mathrm{dist}(\tilde\Phi(p_j(s),t),\overline{\mathcal{PW}}_i)\sim s^{k_{ij}}$ and k_{ij} is also the order of contact between the discriminant branches Δ_i and Δ_j. If $l_i<l_j$ then $\mathrm{dist}(q(s,t),\Delta_i)\sim\mathrm{dist}(\Delta_i,\Delta_j)\sim s^{k_{i,j}}$, and by (43), $l_i=\tilde l_i=k_{i,j}$.

If $l_i=l_j$ then $k_{i,j}<l_i=l_j$ is impossible. Thus $k_{i,j}\ge l_j$ and the RHS of (47) is bounded by $s^{l_i}=s^{l_j}$ as claimed. This ends the proof of Lemma 7.11. ∎

To prove Proposition 7.9 we note that $y_i(u,t)-y(u,t)\sim u^{l_i}$ by Lemma 7.5 and $z_i(s,t)-z(s,t)$ is divisible by s^{l_i} for s real and hence $z_i(u,t)-z(u,t)$ is divisible by u^{l_i}. ∎

Corollary 7.12. *Under the assumption of Proposition 7.9, we have*

$$(y_i(u,t)-y(u,t))\sim u^{l_i}\quad \textit{and}\quad z_i(u,t)-z(u,t)=O(u^{l_i})$$

for all $i\in I=I_C\cup I_\Sigma$. ∎

8. Lipschitz vector fields on quasi-wings

Let the quasi-wings $\mathcal{QW}_\tau$ over a fixed regular wedge $\mathcal{W}_q$ parameterized by (35) be given by

$$p_\tau(u,v,t)=(u^n,y(u,v,t),z_\tau(u,v,t),t),\quad y(u,v,t)=y(u,t)+u^lv. \tag{48}$$

We consider such parameterizations for u in an allowable sector $\Xi=\Xi_I=\{u\in\mathbb{C};\ \arg u\in I\}$. Then we may write these parameterizations in terms of t,x,v assuming implicitly that we work over the sector Ξ, and moreover that $z_\tau(x,v,t)$ is a single valued function. Again, in order to avoid heavy notation we do not use special symbols for the restriction of a quasi-wing parameterization to an allowable sector.

Even if the parameterizations of quasi-wings bear many similarities to the parameterizations of polar wedges, the boundedness of partial derivatives (property (4) at the beginning of the previous section) is opposite to the very definition of polar set, the vertical tangent versus the horizontal tangents. This boundedness and the fact that the projection π restricted to a quasi-wing is bi-Lipschitz make the work with the Lipschitz geometry of quasi-wings in principle simpler.

Proposition 8.1. *For all τ and for all $x_1, x_2, v_1, v_2, t_1, t_2$ sufficiently small,*

$$\begin{aligned} \|p_\tau(x_1, v_1, t_1) - p_\tau(x_2, v_2, t_2)\| &\sim \|(x_1, y_1, t_1) - (x_2, y_2, t_2)\| \\ &\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |v_1 - v_2|\, |x_2|^{l/n}\}. \end{aligned} \tag{49}$$

For every pair of parameterizations p_τ, p_ν

$$\begin{aligned} &\|p_\tau(x_1, v_1, t_1) - p_\nu(x_2, v_2, t_2)\| \\ &\quad\sim \|p_\tau(x_1, v_1, t_1) - p_\tau(x_2, v_2, t_2)\| + \|p_\tau(x_2, v_2, t_2) - p_\nu(x_2, v_2, t_2)\| \\ &\quad\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |x_2|^{r_{\tau\nu}/n}, |v_1 - v_2|\, |x_2|^{l/n}\}, \end{aligned} \tag{50}$$

where $r_{\tau\nu}$ is given by (37). ■

By Proposition 8.1, $h_\tau(x, v, t)$ defines a Lipschitz function on the quasi-wing $\mathcal{QW}_\tau$ if and only if

$$\begin{aligned} |h_\tau(x_1, v_1, t_1) - h_\tau(x_2, v_2, t_2)| &\lesssim \|(x_1, y_1, t_1) - (x_2, y_2, t_2)\| \\ &\sim |t_1 - t_2| + |x_1 - x_2| + |v_1 - v_2|\, |x_2|^{l/n}. \end{aligned} \tag{51}$$

Given two nicely-situated quasi-wings, let h be a function defined on a subset of $\mathcal{QW}_\tau \cup \mathcal{QW}_\nu$. Its restrictions to $\mathcal{QW}_\tau$, $\mathcal{QW}_\nu$ are denoted by $h_\tau(x, v, t) = h \circ p_\tau$, $h_\nu(x, v, t) = h \circ p_\nu$ respectively. Then, by Proposition 8.1, h is Lipschitz iff so are its restrictions h_τ, h_ν and

$$|h_\tau(x_1, v_1, t_1) - h_\nu(x_2, v_2, t_2)| \lesssim |t_1 - t_2| + |x_1 - x_2| + |x_2|^{r_{ij}/n} + |v_1 - v_2|\, |x_2|^{l/n}. \tag{52}$$

Proposition 8.2. *The vector fields given on $\mathcal{QW}_\tau \cup \mathcal{QW}_\nu$ by $p_{k*}(v)$, $k = \tau, \nu$, where v is $\frac{\partial}{\partial t}$, $x\frac{\partial}{\partial x}$, or $\frac{\partial}{\partial v}$, are Lipschitz.*

This result is analogous to Proposition 5.1. The only difference comes from the fact that $b\frac{\partial}{\partial b}$ is replaced by $\frac{\partial}{\partial v}$, since we do not require the vector field to be tangent to the set given by $v = 0$. The proof we sketch below is simpler that the one of Proposition 5.1 thanks to the above mentioned bi-Lipschitz property.

Proof of Proposition 8.2. First we check that the partial derivatives $\frac{\partial}{\partial t}$, $x\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ of the coefficients of these vector fields are bounded. Since $nx\frac{\partial}{\partial x} = u\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial y} = u^{-l}\frac{\partial}{\partial v}$, for the latter two it is more convenient to check that $u\frac{\partial}{\partial u}$ is bounded by $x = u^n$, and $\frac{\partial}{\partial v}$ is bounded by u^l. Then the claim follows from the facts that $y(u,v,t), z_\tau(u,v,t)$ are analytic and divisible by u^n, and $\frac{\partial}{\partial v}y(u,v,t)$, $\frac{\partial}{\partial v}z_\tau(u,v,t)$ are divisible by u^l. This shows that these vector fields are Lipschitz on each wing $\mathcal{QW}_\tau$, $\mathcal{QW}_\nu$.

To obtain the Lipschitz property between the points of $\mathcal{QW}_\tau$ and $\mathcal{QW}_\nu$ we use a similar argument. Namely, we use formula (37) to show that $\frac{\partial}{\partial t}(z_\tau - z_\nu)$, $\frac{\partial}{\partial u}(z_\tau - z_\nu)$, $\frac{\partial}{\partial v}(z_\tau - z_\nu)$ are bounded (up to a constant) by $z_\tau - z_\nu$, and we conclude the proof using formulas (49) and (50). ∎

Let $p_{\tau *}(w)$ be a vector field on $\mathcal{QW}_\tau$, where

$$w(x,v,t) = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial v}. \tag{53}$$

We always suppose the vector field $p_{\tau *}(w)$ is well defined on $\mathcal{QW}_\tau$, that is, independent of v if $x = 0$, and it is stratified, that is, tangent to T. The independence from v if $x = 0$ implies that both $\alpha(0,v,t)$ and $\beta(0,v,t)$ are independent of v, and the tangency to T ensures that in fact $\beta(0,v,t) = 0$. Note also that $p_{i*}(\frac{\partial}{\partial v})$ is always zero on $x = 0$.

The next results easily follow from (51). Their proofs are similar to (and simpler than) the proofs of Propositions 5.2 and 5.5.

Proposition 8.3. *A vector field on $\mathcal{QW}_\tau$ of the form $p_*(w)$ is stratified Lipschitz iff:*

(1) *α satisfies (51);*

(2) *$|\beta| \lesssim |x|$ and β satisfies (51);*

(3) *$\gamma x^{l/n}$ satisfies (51).*

Proposition 8.4. *A vector field on $\mathcal{QW}_\tau \cup \mathcal{QW}_\nu$ given by $p_{\tau *}(w_\tau)$, $p_{\nu *}(w_\nu)$ is stratified Lipschitz iff:*

(0) *$p_{\tau *}(w_\tau)$ and $p_{\nu *}(w_\nu)$ are Lipschitz;*

(1) *α_τ, α_ν satisfy (52);*

(2) *β_τ, β_ν satisfy (52);*

(3) *$\gamma_\tau x^{l/n}, \gamma_\nu x^{l/n}$ satisfy (52).*

We now consider the extension of Lipschitz vector fields on quasi-wings. The classical McShane–Whitney extension theorem ([8, Theorem 1], [24, footnote on p. 63]) says that a Lipschitz function $\varphi : A \to \mathbb{R}$ defined on any nonempty subset A of a metric space B admits a Lipschitz extension $\tilde\varphi$ to B with the same Lipschitz constant. (Such an extension can even be given by a formula: $\tilde\varphi(x) = \inf_{y\in A}(\varphi(x) + \mathrm{Lip}(\varphi)d(x,y))$.) If $B \subset \mathbb{R}^n$, then this theorem gives an extension of Lipschitz vector fields with Lipschitz constant multiplied by $\sqrt{n}$. The Kirszbraun theorem (see e.g. [4, p. 202]) gives the existence of an extension of vector fields with the same Lipschitz constant. In our case we can use any of these results. By Proposition 8.3, $w \mapsto p_*(w)$ gives a one-to-one correspondence between Lipschitz vector fields on $\mathcal{QW}$ and Lipschitz vector fields $w(x,y,t)$ on the wedge $\mathcal{W}$. Hence the McShane–Whitney extension theorem implies the following.

Corollary 8.5 (Extension of Lipschitz vector fields on a quasi-wing). *Any stratified Lipschitz vector field defined on subset of a quasi-wing $\mathcal{QW}$ containing the stratum $T = \{x = 0\}$ can be extended to a stratified Lipschitz vector field on $\mathcal{QW}$.* ■

Propositions 8.3 and 8.4 imply the following.

Corollary 8.6 (Extension of Lipschitz vector fields between quasi-wings). *Let $\mathcal{QW}_\tau$, $\mathcal{QW}_\nu$ be nicely-situated quasi-wings parameterized by $p_\tau(x, v, t)$ and $p_\nu(x, v, t)$ respectively. Let the vector field w, of the form (53), be such that $p_{\tau *}(w)$ is a stratified Lipschitz vector field defined on the image of p_τ. Then $p_{\tau *}(w)$, $p_{\nu *}(w)$ define a stratified Lipschitz vector field on the union $\mathcal{QW}_\tau \cup \mathcal{QW}_\nu$.* ■

9. Extension of Lipschitz vector fields from $\mathcal{PW}$ to an arc in its complement

Suppose we are given a stratified Lipschitz vector field w on S. By the first part of the proof of Theorem 2.1 (Section 6), we may extend it to a Lipschitz vector field, still called w, onto $\mathcal{PW}$. In this section we show how to extend it further on the image of a real analytic arc germ $p(s)$ of the form (39) not included in $\mathcal{PW}$. For this we use Corollary 7.8 to embed $p(s)$ in a quasi-wing $\mathcal{QW}$ and extend the vector field from $\mathcal{PW}$ to $\mathcal{QW}$. The latter extension is explained in Proposition 9.4. In the process we encounter two problems, discussed below, related to the fact that the construction of Corollary 7.8 gives a quasi-wing after a linear change of coordinates.

If $\mathcal{PW}_i$ is a polar wedge in the original system of coordinates then we may choose the corresponding polar wedge in the new system of coordinates $x, y - b_0 z, z, t$, denoted by $\mathcal{PW}_{i,b_0}$, included in $\mathcal{PW}_i$, but we cannot assume that it contains the spine of $\mathcal{PW}_i$, that is, C_i. Therefore, if we extend $w|\mathcal{PW}_{i,b_0}$ to $\mathcal{QW}$ using Proposition 9.4, a priori there is no guarantee that the resulting vector field is Lipschitz on $\mathcal{PW}_i \cup \mathcal{QW}$. To guarantee it we show that the distances from the arc $p(s)$, and hence from the whole quasi-wing $\mathcal{QW}$, to $\mathcal{PW}_i$ and to $\mathcal{PW}_{i,b_0}$ are of the same order. This will follow from Proposition 9.1.

The second problem comes from the fact that the description of stratified Lipschitz vector fields on a polar wedge, given in conditions (1)–(3) of Proposition 5.2, changes slightly when we pass from $\mathcal{PW}_i$ to $\mathcal{PW}_{i,b_0}$, if $\mathcal{PW}_{i,b_0}$ does not contain C_i. Therefore to prove Proposition 9.4 one should not use condition (3). To solve this problem we replace in the proof of Proposition 9.4 condition (3) by a slightly weaker condition (3′) that is satisfied on $\mathcal{PW}_{i,b_0}$.

9.1. Distance to polar wedges

Proposition 9.1. *Let $\gamma(s) = (x(s), y(s), z(s), t(s))$, $s \in [0, \varepsilon)$, be a real analytic arc at the origin. If $\gamma(s) \not\subset \mathcal{PW}$ then for all j,*

$$\operatorname{dist}(\gamma(s), C_j) \gtrsim \|(x(s), y(s), z(s))\|^{m_j/n}.$$

Remark 9.2. If the arc γ is of the form $\gamma(s) = (s^n, y(s), z(s))$ with $y(s) = O(s^n)$, $z(s) = O(s^n)$, which we may suppose, then we get $\operatorname{dist}(\gamma(s), C_j) \gtrsim |s^{m_j}|$.

For the proof of Proposition 9.1 we need the following lemma.

Lemma 9.3. *If the polar set C_i minimizes the distance of γ to S and if this distance satisfies*

$$\operatorname{dist}(\gamma(s), S) = \operatorname{dist}(\gamma(s), C_i) \ll \|(x(s), y(s), z(s))\|^{m_i/n}, \tag{54}$$

then $\gamma(s)$ is contained in $\mathcal{PW}$ for small s.

By (54) we mean that there is $\delta > 0$ such that

$$\operatorname{dist}(\gamma(s), C_i) \leq \|(x(s), y(s), z(s))\|^{\delta + m_i/n}.$$

We do not claim in the lemma that $\gamma(s)$ has to belong to the polar wedge containing C_i, that is, to $\mathcal{PW}_i$.

Proof. We give the proof in the nonparameterized case. The proof in the parameterized case is similar.

We may suppose that the arc γ is of the form $\gamma(s) = (s^n, y(s), z(s))$ with $y(s) = O(s^n)$, $z(s) = O(s^n)$ and note that in this case $\operatorname{dist}(\gamma(s), C_i) \sim |y(s) - y_i(s)| + |z(s) - z_i(s)|$. Therefore, by (54), $|y(s) - y_i(s)| = o(s^{m_i})$ and $|z(s) - z_i(s)| = o(s^{m_i})$. Complexify γ by setting $\gamma(u) = (u^n, y(u), z(u))$. Then, as in the proof of Corollary 7.8, we construct a quasi-wing $\mathcal{QW}$ containing γ by changing the system of coordinates, that is, replacing y by $Y = y - b_0 z$, for b_0 sufficiently generic. In the new coordinates x, Y, z, t (we do not change the parameter b) the parameterizations of $\mathcal{PW}_i$ and $\mathcal{QW}$ are $x = u^n$ and, respectively,

$$\begin{aligned} Y_i(u,b) &= y_i(u,b) - b_0 z_i(u,b) \\ &= (y_i(u) - b_0 z_i(u)) + u^{m_i}(b^2 \varphi_i(u,b) - b b_0 \psi_i(u,b)), \\ z_i(u,b) &= z_i(u) + b u^{m_i} \psi_i(u,b). \end{aligned} \tag{55}$$

and

$$\begin{aligned} Y(u,v) &= (y(u) - b_0 z(u)) + v u^{m_i}, \\ z(u,v) &= z(u) + v u^{m_i} \tilde{\psi}_i(u,v). \end{aligned} \tag{56}$$

To see that the exponent in the latter formula is m_i, note that if we denote the polar set in $\mathcal{PW}_i$ in the new system of coordinates by C_{i,b_0} then $\operatorname{dist}(\gamma(s), C_{i,b_0}) \sim s^{m_i}$ and we conclude by Corollary 7.12. Now we argue as follows. By Proposition 7.3 the polar wedge $\mathcal{PW}_i$ and the quasi-wing $\mathcal{QW}$ are disjoint (if the constants defining them are small). But if the limits of tangent spaces to $\mathcal{X}$ along C_i and along γ do not coincide then the implicit function theorem forces $\mathcal{PW}_i$ and $\mathcal{QW}$ to intersect along a curve and therefore this case cannot happen. This is the geometric idea behind the computation below.

Note that (54) implies that, for the old system of coordinates, $l_i > m_i$. Therefore the intersection $\mathcal{PW}_i \cap \mathcal{QW}$, defined by $Y_i(u,b) = Y(u,v)$ and $z_i(u,b) = z(u,v)$, is given by the system of equations

$$\begin{aligned} (b^2\varphi_i(u,b) - bb_0\psi_i(u,b)) - v &= O(u), \\ b\psi_i(u,b) - v\tilde{\psi}_i(u,v) &= O(u). \end{aligned} \tag{57}$$

There are two cases:

(i) Suppose the Jacobian determinant of the LHS of (57) with respect to the variables b, v is nonzero at $u = b = v = 0$. Then, by the implicit function theorem there is a solution $(b, v) = (b(u), v(u))$ of (57) such that $b(u) \to 0$ and $v(u) \to 0$ as $u \to 0$. Then the intersection $\mathcal{PW}_i \cap \mathcal{QW}$ is the curve parameterized by u: $(u^n, Y_i(u, b(u)), z_i(u, b(u))) = (u^n, Y(u, v(u)), z(u, v(u)))$. Therefore, by Proposition 7.3, this case cannot happen.

(ii) Suppose that the Jacobian determinant of the LHS of (57) vanishes at $u = b = v = 0$. Then the partial derivatives

$$\frac{\partial}{\partial b} u^{-m_i}(Y_i(u,b), z_i(u,b)), \quad \frac{\partial}{\partial v} u^{-m_i}(Y(u,v), z(u,v)),$$

which are both nonzero at $u = b = v = 0$, are proportional. This means that the limits of tangent spaces to $\mathcal{X}$ along C_i, i.e. at $(u^n, y_i(u,0), z_i(u,0))$ as $u \to 0$, and at $\gamma(u)$ as $u \to 0$, coincide. This limit is transverse to $H = \{x = 0\}$ since H is not a limit of tangent spaces by the Transversality Assumptions. Hence the tangent spaces to $\mathcal{X}$ at $\gamma(u)$, for small u, contain vectors of the form $(0, b, 1)$ with $b \to 0$ as $u \to 0$. This shows that $\gamma \in \mathcal{PW}$ (but not necessarily $\gamma \in \mathcal{PW}_i$).

The proof of lemma is now complete. ∎

Proof of Proposition 9.1. The proof is the same in the parameterized and the nonparameterized cases. We may suppose again that $\gamma(s) = (s^n, y(s), z(s))$ with $y(s) = O(s^n)$, $z(s) = O(s^n)$.

If $\mathrm{dist}(\gamma(s), S) = \mathrm{dist}(\gamma(s), C_i)$ then the conclusion for $j = i$ follows directly from Lemma 9.3. Then consider $j \neq i$. If the conclusion is not satisfied then

$$s^{m_i} \lesssim \mathrm{dist}(C_i, \gamma(s)) \le \mathrm{dist}(C_j, \gamma(s)) \ll s^{m_j}.$$

In particular, $m_i > m_j$, and therefore by Remark 3.7, $k_{ij} \le m_j < m_i$. But this is impossible since then

$$s^{m_j} \lesssim s^{k_{ij}} \simeq \mathrm{dist}(p_i(s), p_j(s)) \lesssim \mathrm{dist}(C_j, \gamma(s)) + \mathrm{dist}(C_i, \gamma(s)) \ll s^{m_j},$$

where p_i, p_j denote parameterizations of C_i and C_j respectively. This ends the proof in this case.

If $\mathrm{dist}(\gamma(s), S) = \mathrm{dist}(\gamma(s), \Sigma_k)$ then the conclusion follows by the second part of Lemma 3.8. ∎

9.2. Extension of Lipschitz vector fields from a polar wedge to a quasi-wing

Let the quasi-wing $\mathcal{QW}$ be given by

$$\mathcal{QW} : p(u,v,t) = (u^n, y(u,t) + vu^l, z(u,v,t), t), \quad y(u,v,t) := y(u,t) + vu^l,$$

containing an arc $p(u,t) = p(u,0,t)$.

Fix a polar wedge $\mathcal{PW}_i$ (or Σ_k) closest to $\mathcal{QW}$ and parameterized by

$$\mathcal{PW}_i : p_i(u,b,t) = (u^n, y_i(u,b,t), z_i(u,b,t), t).$$

Recall from Definition 7.4 that $m_i \geq l = l_i$ and then by Corollary 7.12,

$$y_i(u,b,t) - y(u,v,t) \sim u^l, \quad z_i(u,b,t) - z(u,v,t) = O(u^l). \tag{58}$$

Our goal is to extend any Lipschitz stratified vector field on $\mathcal{PW}_i$ onto $\mathcal{QW}$. Recall, from Proposition 5.2, that if $p_{i*}(\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial b})$ is Lipschitz stratified then α, β, and δ satisfy conditions (1)–(3) of that proposition. In what follows we use only a weaker version of condition (3) (see Remark 9.5 for explanation),

(3′) $|\delta|$ is bounded and $\delta x^{m/n}$ satisfies (26).

We note that by (58) and $m_i \geq l$, a vector field is Lipschitz on $\mathcal{PW}_i \cup \mathcal{QW}$ if and only if it is Lipschitz on each $\mathcal{PW}_i$ and $\mathcal{QW}$, and it is Lipschitz on the union of the images of the two arcs $p(u,t)$ and $p_i(u,t)$.

Proposition 9.4 (Extension of Lipschitz vector fields from $\mathcal{PW}_i$ onto $\mathcal{QW}$). *Let $p_{i*}(\alpha(u,b,t)\frac{\partial}{\partial t} + \beta(u,b,t)\frac{\partial}{\partial x} + \delta(u,b,t)\frac{\partial}{\partial b})$ be a stratified Lipschitz vector field on $\mathcal{PW}_i$. Set $\alpha_0(u,v,t) := \alpha(u,0,t)$ and $\beta_0(u,v,t) := \beta(u,0,t)$. Then $p_*(\alpha_0\frac{\partial}{\partial t} + \beta_0\frac{\partial}{\partial x})$ is a stratified Lipschitz vector field on $\mathcal{QW}$ and both fields define a stratified Lipschitz vector field on $\mathcal{PW}_i \cup \mathcal{QW}$.*

Proof. By Proposition 8.3, $p_*(\alpha_0\frac{\partial}{\partial t} + \beta_0\frac{\partial}{\partial x})$ is Lipschitz on $\mathcal{QW}$. To show that both vector fields define a Lipschitz vector field on $\mathcal{PW}_i \cup \mathcal{QW}$ it suffices to show that taking $b = 0$ and $v = 0$ we have

(1) $\alpha(u,0,t)\frac{\partial}{\partial t}(y(u,t) - y_i(u,t)) = O(u^{l_i})$,

(2) $\alpha(u,0,t)\frac{\partial}{\partial t}(z(u,t) - z_i(u,t)) = O(u^{l_i})$,

(3) $\beta(u,0,t)\frac{\partial}{\partial u}(y(u,t) - y_i(u,t)) = O(u^{l_i})$,

(4) $\beta(u,0,t)\frac{\partial}{\partial u}(z(u,t) - z_i(u,t)) = O(u^{l_i})$,

(5) $\delta(u,0,t)u^{m_i} = O(u^{l_i})$.

Items (1)–(4) follow from (58), and (5) follows from $m_i \geq l_i$. ■

Remark 9.5. Since in the above proof we only used condition (3′), we can apply Proposition 9.4 to the quasi-wings constructed in Corollary 7.8, that is, after a change of coordinates to x, Y_{b_0}, z, t, where $Y_{b_0} := y - b_0 z$, which corresponds to a shift in b.

10. Proof of Theorem 2.1: Part II

We complete the proof of Theorem 2.1. Let $\gamma(s), \gamma'(s)$, $s \in [0, \varepsilon)$, be two real analytic arcs in $\mathcal{X}$. We want to show that any stratified Lipschitz vector field v defined on the union of S and γ extends to γ' as stated in the valuative criterion (see the next section). We consider two cases.

Case 1: $\operatorname{dist}(\gamma(s), \gamma'(s)) \gtrsim \operatorname{dist}(\gamma'(s), S)$. Then it is enough to extend $v|_S$ to a Lipschitz vector field on $S \cup \gamma'$, since then such an extension defines a Lipschitz vector field on $S \cup \gamma(s) \cup \gamma'(s)$ for every s sufficiently small, with the Lipschitz constant independent of s.

Case 2: $\operatorname{dist}(\gamma(s), \gamma'(s)) \ll \operatorname{dist}(\gamma'(s), S)$. Then it suffices to extend v from γ to a Lipschitz vector field on $\gamma \cup \gamma'$.

Note that we may suppose that on both arcs γ, γ' we have $y = O(x), z = O(x)$, that is, they are in the form (32). Indeed, by the Transversality Assumptions the variable z restricted to an arc in $\mathcal{X}$ cannot dominate x and y, that is, $x = o(z), y = o(z)$ is not possible. Thus, if $y = O(x), z = O(x)$ is not satisfied, then $x = o(y), z = O(y)$. In this case we change the local coordinate system to $(X_a, y, z, t) = (x - ay, y, z, t)$ for $a \neq 0$ small. This is a change of coordinates in the target of the projection $(x, y, z, t) \mapsto (x, y, t)$ and affects neither the discriminant locus nor Zariski's equisingularity.

To make the proof more precise we will use the constant ε of Definition 4.1 and denote the resulting union of polar wedges and the singular set by $\mathcal{PW}_\varepsilon$. If both $\gamma(s), \gamma'(s)$ belong to $\mathcal{PW}_\varepsilon$ then the claim follows from the first part of the proof (Section 6).

In **Case 1**, given a stratified Lipschitz vector field v onto S we extend it on γ'. By Proposition 9.1 we may suppose that $\operatorname{dist}(\gamma(s), C_j) \gtrsim s^{m_j}$ for every j, and therefore, for b small, say $b \le \varepsilon$, $\operatorname{dist}(\gamma(s), C_j) \sim \operatorname{dist}(\gamma(s), C_{j,b})$, where $C_{j,b}$ denotes the polar set in $\mathcal{PW}_j$ after the change of coordinates to $x, Y_{b_0} = y - b_0 z, z, t$. Then we proceed as follows. First we extend v to a Lipschitz vector field on $\mathcal{PW}_{\varepsilon/2}$ and use Corollary 7.8 to embed γ' in a quasi-wing in this new system of coordinates for a $b_0 \le \varepsilon/2$. Thus there exists a quasi-wing $\mathcal{QW}$ containing γ' and moreover $\operatorname{dist}(\gamma'(s), S) = \operatorname{dist}(\pi_{b_0}(\gamma'(s)), \Delta_{b_0}) \sim s^l$, where $l = \max\{\max l_i, \max r_k\}$ and Δ_{b_0} denotes the discriminant of π_{b_0}. Then there is a Lipschitz extension of v to $\mathcal{QW}$ by Proposition 9.4.

Similarly, in **Case 2** we may suppose $\operatorname{dist}(\gamma(s), C_j) \sim \operatorname{dist}(\gamma'(s), C_j) \gtrsim s^{m_j}$ for every j, since otherwise, by Proposition 9.1, both $\gamma(s), \gamma'(s)$ belong to $\mathcal{PW}_\varepsilon$. Then, choosing b appropriately, we may suppose that

$$\operatorname{dist}(\pi_b(\gamma(s)), \pi_b(\gamma'(s))) \sim \operatorname{dist}(\gamma(s), \gamma'(s)) \ll s^l.$$

Let $\mathcal{QW}$ be a quasi-wing containing γ. It always exists by Corollary 7.8, and γ' is contained either in $\mathcal{QW}$ or in another quasi-wing $\mathcal{QW}'$ such that $\mathcal{QW}$ and $\mathcal{QW}'$ are nicely-situated. Then we apply Corollary 8.6 to extend a Lipschitz vector field v from γ to γ'.

11. Valuative criterion on extension of Lipschitz vector fields

The purpose of this section is to give a precise statement of a valuative criterion on extension of Lipschitz vector fields. In this criterion we formalize our strategy of checking conditions (i) and (ii) of Proposition 2.4 along real analytic arcs.

Let us consider the following more general set-up. Let X be a locally closed subanalytic subset of $\mathbb{R}^n$ with a filtration $\mathcal{F} = (X^j)_{j=l}^d$ by closed subanalytic subsets

$$X = X^d \supset X^{d-1} \supset \cdots \supset X^l \neq \emptyset, \tag{59}$$

such that for every $j = l, \dots, d$, $\mathring{X}^j = X^j \setminus X^{j-1}$ is either empty or a real analytic submanifold of pure dimension j. Here we mean $X^{l-1} = \emptyset$. Note that $\mathcal{F}$ induces a stratification of X by taking the connected components of every $\mathring{X}^j$ as strata. By a stratified Lipschitz vector field (SLVF for short) we mean a Lipschitz vector field defined on a subset of X and tangent to the strata.

Definition 11.1 (Local valuative extension of Lipschitz vector fields). We say that $\mathcal{F}$ satisfies the *LVE condition at* $p \in X$ if for every $j = l, \dots, d$ and every pair of real analytic arc germs $\gamma, \gamma' : [0, \varepsilon) \to X^j$ at p, i.e. $\gamma(0) = \gamma'(0) = p$, every SLVF on $X^{j-1} \cup \gamma([0, \varepsilon))$ can be extended to a vector field on $X^{j-1} \cup \gamma([0, \varepsilon)) \cup \gamma'([0, \varepsilon))$ satisfying the following condition:

- *there is a constant L such that for every s sufficiently small this extension is an SLVF vector field, with Lipschitz constant L, on $X^{j-1} \cup \gamma(s) \cup \gamma'(s)$.*

Remark 11.2. The following, a priori stronger condition, implies the LVE: for every SLVF on $X^{j-1} \cup \gamma([0, \varepsilon))$ there is $\varepsilon' > 0$ such that this vector field admits an extension that is SLVF on $X^{j-1} \cup \gamma([0, \varepsilon')) \cup \gamma'([0, \varepsilon'))$.

We say that $\mathcal{F}$ *induces a Lipschitz stratification at* $p \in X$ if there is an open neighbourhood U of p such that $\mathcal{F}$ restricted to U induces a Lipschitz stratification of $X \cap U$.

Proposition 11.3 (LVE criterion). *$\mathcal{F}$ induces a Lipschitz stratification at $p \in X$ if and only if it satisfies the LVE condition at p.*

Proof. We first recall the notions of a chain and Mostowski's conditions. We follow the approach of [13] simplifying the notation and exposition a little. For slightly different but equivalent conditions see [9, 14]. One can simplify the proof below by using directly the valuative criteria of [6] but we prefer to give a self-contained proof based on elementary computations in [13, proofs of Propositions 1.2 and 1.5].

Fix $c > 1$. A *chain* (more exactly, a *c-chain*) for a point $q \in \mathring{X}^j$ is a strictly decreasing sequence of indices $j = j_1, j_2, \dots, j_r = l$ and a sequence of points $q_m \in \mathring{X}^{j_m}$ such that $q_1 = q$ and j_m is the greatest integer for which

$$\begin{aligned} &\operatorname{dist}(q, X^k) \geq 2c^2 \operatorname{dist}(q, X^{j_m}) \quad \text{for all } k < j_m, \\ &|q - q_m| \leq c \operatorname{dist}(q, X^{j_m}). \end{aligned}$$

The condition $c > 1$ is imposed only to ensure that every point $q \in X$ admits a chain. A chain satisfies the following properties:

(1) $\operatorname{dist}(q, X^{j_{m+1}}) \le 2^n c^{2n} \operatorname{dist}(q, X^{j_m -1})$,

(2) $|q_m - q_{m+1}| \le 2^{n+1} c^{2(n+1)} \operatorname{dist}(q, X^{j_m -1})$,

(3) $2 \operatorname{dist}(q_m, X^{j_m -1}) \ge \operatorname{dist}(q, X^{j_m -1})$.

Let $P_q : \mathbb{R}^n \to T_q \mathring{X}^j$ denote the orthogonal projection onto the tangent space and $P_q^\perp = I - P_q$ the orthogonal projection onto the normal space $T_q^\perp \mathring{X}^j$. We say that $\mathcal{F}$ satisfies *Mostowski's conditions* if there is a constant $C > 0$ such that for all chains $\{q_m\}_{m=1}^r$ and all $2 \le k \le r$,

$$|P_{q_1}^\perp P_{q_2} \cdots P_{q_k}| \le C|q - q_2|/\operatorname{dist}(q, X^{j_k -1}). \tag{M1}$$

If, further, $q' \in \mathring{X}^j$ and $|q - q'| \le \frac{1}{2c} \operatorname{dist}(q, X^{j-1})$ then

$$|(P_q - P_{q'}) P_{q_2} \cdots P_{q_k}| \le C|q - q'|/\operatorname{dist}(q, X^{j_k -1}), \tag{M2}$$

in particular,

$$|P_q - P_{q'}| \le C|q - q'|/\operatorname{dist}(q, X^{j_1 -1}), \tag{M3}$$

where $\operatorname{dist}(\cdot, \emptyset) \equiv 1$.

By [13, Proposition 1.5], $\mathcal{F}$ induces a Lipschitz stratification if and only if any of the two equivalent conditions (i) and (ii) of Proposition 2.4 holds. In particular, the definition of Mostowski's stratification is independent of the choice of the constant $c > 1$ used to define the chains.

Clearly by Proposition 2.4 a Lipschitz stratification satisfies the LVE condition at any point of X.

Suppose that $\mathcal{F}$ satisfies the LVE condition at p. We show by induction on j that $\mathcal{F}$ induces a Lipschitz stratification of X^j at p, the case $j = l$ being obvious because X^l is nonsingular. Thus we suppose it for X^{j-1} and prove for X^j. Suppose the latter does not hold. Then by a fairly straightforward application of the curve selection lemma there are real analytic arcs $q_m(s) : [0, \varepsilon) \to X^{j_m}$, $m = 1, \ldots, r$, $j_1 = j$, at p, that are c-chains of $q(s) = q_1(s)$ for $s \ne 0$, and possibly another arc $q'(s) : [0, \varepsilon) \to X^j$ satisfying $|q(s) - q'(s)| \le \frac{1}{2c} \operatorname{dist}(q(s), X^{j-1})$ for $s \ne 0$, for which one of the conditions (M1), (M2) fails, that is, it holds with the constant $C(s) \to \infty$ as $s \to 0$. Indeed, it follows from [9, Lemma 6.2], stated in the complex analytic set-up, or from the valuative criteria of [6], where the authors even managed to get rid of the constant c defining the chains.

We will show that the existence of such arcs contradicts the LVE condition. We may assume that the index k, given by the length of the expression on the left-hand side of (M1), (M2) for which one of these conditions fails, is minimal. Suppose that this is condition (M1). Put $\gamma'(s) := q(s)$ and $\gamma(s) := q_2(s)$. Then adapting the proofs of [13, Propositions 1.2 and 1.5] and using the LVE condition we show that there is a constant $C > 0$, independent of s, such that (M1) holds along the family of arcs q_m, $m = 1, \ldots, k$, which gives a contradiction.

Let $V_0 = \lim_{s\to 0} T_{q_k(s)} \mathring{X}^{j_k}$. Then $\dim V_0 = j_k$. Let $v \in V_0$, $|v| = 1$. Then $x \mapsto \operatorname{dist}(x, X^{j_k-1})v$ is a Lipschitz vector field (on a neighbourhood of p) with Lipschitz constant 1. By [13, proof of Proposition 1.2] (extension of Lipschitz vector fields on a Lipschitz stratification), $x \mapsto P_x(\operatorname{dist}(x, X^{j_k-1})v)$ defines a Lipschitz vector field on X^{j_k}. By inductive assumption on j, we extend it to an SLVF, denoted by w, on X^{j-1} and then by the LVE condition to the image of γ'. This gives, together with (M1) for $m < k$ and the standard inequalities (1)–(3) satisfied by chains,

$$\begin{aligned} |P^{\perp}_{q_1(s)} P_{q_2(s)} \cdots P_{q_k(s)} w(q_k(s))| &= |P^{\perp}_{q_1(s)} P_{q_2(s)} \cdots P_{q_{k-1}(s)} w(q_k(s))| \\ &\le |P^{\perp}_{q_1(s)} P_{q_2(s)} \cdots P_{q_{k-1}(s)} w(q_{k-1}(s))| \\ &\quad + |P^{\perp}_{q_1(s)} P_{q_2(s)} \cdots P_{q_{k-1}(s)} (w(q_k(s) - w(q_{k-1}(s)))| \\ &\cdots \\ &\le \sum_{1\le s<k} |P^{\perp}_{q_1(s)} P_{q_2(s)} \cdots P_{q_s(s)} (w(q_s(s)) - w(q_{s+1}(s)))| \\ &\le C \sum_{1\le s<k} \frac{|q(s)-q_2(s)|}{\operatorname{dist}(q, X^{j_s-1})} |q_s(s) - q_{s+1}(s)| \le C'|q(s)-q_2(s)|. \end{aligned}$$

Note that if $k = 2$ the first term of the RHS of the first inequality does not appear, otherwise everything is the same.

Since $w(q_k(s)) = \operatorname{dist}(q_k(s), X^{j_k-1}) P_{q_k(s)} v$ we get, by property (3) of chains,

$$|P^{\perp}_{q_1(s)} P_{q_2(s)} \cdots P_{q_k(s)} v| \le C'|q(s) - q_2(s)|/\operatorname{dist}(q(s), X^{j_k-1}).$$

Applying the above to a finite set of v from an orthonormal basis of V_0, and taking into account that $|P_{q_k(s)} v - v| \le C|q_k(s)| \to 0$ as $s \to 0$, we show that (M1) holds along this family of arcs, contrary to our assumptions. A similar argument, based on [13, proof of Proposition 1.5, second part], applies to condition (M2). This ends the proof. ∎

Remark 11.4. Proposition 11.3 holds in a more general o-minimal set-up when one assumes that every X^j is definable, every $\mathring{X}^j$ is a C^2 submanifold, and the arcs are continuous and definable. One can also restrict the LVE condition of Definition 11.1 to definable vector fields, because the construction of extension of Lipschitz vector fields of [13, Proposition 1.2] preserves definability (see [14, Remark 1.4]).

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