

RESEARCH ARTICLE

Initial Newton polynomial of the discriminant

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Abstract

Let $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a holomorphic mapping with an isolated zero. We show that the initial Newton polynomial of its discriminant is determined, up to rescaling variables, by the ideals (f) and (g) .

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1 | INTRODUCTION

Let $\mathbb{R}_{\geq 0}$ ($\mathbb{Z}_{\geq 0}$) be the set of all nonnegative real (integer) numbers. For a power series $f = \sum_{(i,j) \in \mathbb{Z}_{\geq 0}^2} a_{i,j} x^i y^j \in \mathbb{C}[[x, y]]$ we define its *Newton diagram* $\Delta(f)$ as the convex hull of the union $\bigcup_{a_{i,j} \neq 0} ((i, j) + \mathbb{R}_{\geq 0}^2)$. If S is the union of all compact edges of $\Delta(f)$, then the polynomial $f|_S := \sum_{(i,j) \in S} a_{i,j} x^i y^j$ is called the *initial Newton polynomial* of f . We say that power series $f_1, f_2 \in \mathbb{C}[[x, y]]$ are *equal up to rescaling variables* if $f_2(x, y) = f_1(ax, by)$ for some nonzero constants a, b .

Let $\phi = (f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping with an isolated zero at the origin. To any germ ξ of an analytic curve in $(\mathbb{C}^2, 0)$ one associates its *direct image* $\phi_*(\xi)$, see, for example, [3, 4]. The direct image of ξ by ϕ is an analytic curve germ in the target space uniquely determined by the following two properties.

- (i) If $\xi \subset (\mathbb{C}^2, 0)$ is an irreducible curve then $\phi_*(\xi)$ is the curve of equation $H^d = 0$, where $H = 0$ is a reduced equation of the curve $\phi(\xi)$ in the target space and d is the topological degree of the restriction $\phi|_{\xi} : \xi \rightarrow \phi(\xi)$.
- (ii) If $h = h_1 \cdots h_s$ is a factorization of a power series h to the product of irreducible factors in $\mathbb{C}\{x, y\}$, then $\phi_*(\{h = 0\})$ is the curve $H_1 \cdots H_s = 0$, where the curves $H_i = 0$ are the direct images of the branches $h_i = 0$ for $i = 1, \dots, s$.

The direct image can be also characterized as follows. The direct image of a curve germ $h = 0$ is the only curve germ $H = 0$ that satisfies the projection formula: For any analytic curve $w = 0$ in the target space we have the equality of intersection multiplicities $i_0(w \circ \phi, h) = i_0(w, H)$.

Let $H = 0$ be the direct image of $h = 0$. Any factorization $H_1 \cdots H_r$ of H such that H_i and H_j are coprime for $i \neq j$ induces a factorization $h_1 \cdots h_r$ of h such that $H_i = 0$ is the direct image of $h_i = 0$ for $i = 1, \dots, r$.

Suppose that the Newton diagram of H has r edges which are not contained in the coordinate axes. Then H can be written as a product $H_1 \cdots H_r$, where the Newton diagram of each H_i is elementary, that is, has exactly one edge not contained in the coordinate axes. It follows from the projection formula that for every irreducible factor p of h_i , the *Hironaka quotient* $i_0(g, p)/i_0(f, p)$ is the inclination of the edge of $\Delta(H_i)$. Moreover the intersection multiplicities $i_0(f, h_i)$, $i_0(g, h_i)$ determine and are determined by $\Delta(H_i)$. In this case we will call $h_1 \cdots h_r$ a *minimal Hironaka factorization* of h and any finer factorization of h will be called a *Hironaka factorization* of h .

In this article, we deal with the problem of factorization of $\text{Jac}(\phi) = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$. The curve $\text{Jac}(\phi) = 0$ is called the *Jacobian curve* of ϕ and the direct image of the Jacobian curve is called the *discriminant curve*. The Newton diagram of the discriminant, called the *Jacobian Newton diagram* of (f, g) and denoted $Q(f, g)$ was introduced by Teissier in [15, 16].

The Jacobian curve in the case of $f = 0$ smooth and transverse to $g = 0$ is called the generic polar curve of g . In this case the Hironaka quotients $i_0(g, h)/i_0(f, h)$ where h is an irreducible factor of $\text{Jac}(\phi)$, are called the *polar quotients*.

The polar case has been widely studied. The polar quotients are invariants of singularity as shown in [8]. Teissier [15, 16] proved that the Jacobian Newton diagram in the polar case is also a singularity invariant. Merle [13] found the minimal Hironaka decomposition of a polar curve for g irreducible. Eggers [5] found a Hironaka decomposition of a polar curve for any g .

The case where both curves $f = 0$ and $g = 0$ are singular is more complicated. The set of Hironaka quotients for the Jacobian curve was characterized in [11, 12]. Michel [14] found a certain Hironaka factorization of $\text{Jac}(\phi)$ and gave formulas for $i_0(f, h_i)$, $i_0(g, h_i)$ using topological methods. Hence, $Q(f, g)$ depends only on the equisingularity type of pair of curves $f = 0$, $g = 0$. This property was proved independently in [7] using Casas-Alvero formula.

Recall that pairs of curves $f = 0$, $g = 0$ and $\tilde{f} = 0$, $\tilde{g} = 0$ are *equisingular* if there exist factorizations $f = h_1 \cdots h_s$, $g = h_{s+1} \cdots h_r$, $\tilde{f} = \tilde{h}_1 \cdots \tilde{h}_{\tilde{s}}$, $\tilde{g} = \tilde{h}_{\tilde{s}+1} \cdots \tilde{h}_{\tilde{r}}$ into the product of irreducible factors in $\mathbb{C}\{x, y\}$ such that

- $s = \tilde{s}$, $r = \tilde{r}$;
- for $i = 1, \dots, r$, the semigroups $\Gamma(h_i) := \{i_0(h_i, w) : w \notin (h_i)\}$ and $\Gamma(\tilde{h}_i) := \{i_0(\tilde{h}_i, w) : w \notin (\tilde{h}_i)\}$ are equal; and
- $i_0(h_i, h_j) = i_0(\tilde{h}_i, \tilde{h}_j)$ for $1 \leq i < j \leq r$.

In the reduced case, the above definition is equivalent to the definition of topological equisingularity, that is, there exists a homeomorphism $\Phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $\Phi(\{f = 0\}) = \{\tilde{f} = 0\}$ and $\Phi(\{g = 0\}) = \{\tilde{g} = 0\}$.

For such equisingular pairs of curves $Q(f, g) = Q(\tilde{f}, \tilde{g})$. However, the initial Newton polynomials of discriminants do not coincide in general. We prove that they are equal under more restrictive assumptions. The main result of this paper is the following theorem.

Theorem 1.1. *Let $f, g, u', u'' \in \mathbb{C}\{x, y\}$ be convergent power series vanishing at zero such that f and g are coprime and let $\tilde{f} = (1 + u')f$, $\tilde{g} = (1 + u'')g$. Then the initial Newton polynomials of discriminants of mappings $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ and $(\tilde{f}, \tilde{g}) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ are equal.*

Hence, under assumptions of Theorem 1.1 there exists a factorization $D = D_1 \cdots D_r$ of the discriminant D of the mapping $\phi = (f, g)$ (respectively, a factorization $\tilde{D} = \tilde{D}_1 \cdots \tilde{D}_r$ of the

discriminant \tilde{D} of the mapping $\tilde{\phi} = (\tilde{f}, \tilde{g})$ such that the Newton diagram of each D_i is elementary, the initial Newton polynomial of D_i is a power of an irreducible polynomial, $\Delta(D_i) = \Delta(\tilde{D}_i)$, and the initial Newton polynomials of D_i and \tilde{D}_i are equal. This induces Hironaka factorizations of $\text{Jac}(\phi)$ and $\text{Jac}(\tilde{\phi})$ which are usually more subtle than minimal Hironaka factorizations.

The structure of the paper is as follows. In Section 2 we study the initial weighted form of a nonzero power series. We establish in Lemma 2.1 a relation between the intersection multiplicities of this power series with certain test polynomials and the multiplicities of the irreducible factors of its initial weighted form. In Corollary 3.2, thanks to Casas' formula, that we recall for the reader convenience, these multiplicities are expressed in terms of classical equisingularity invariants (Milnor numbers). This allows us in Section 4 to show the key lemma, Lemma 4.2. It states that some specific plane curve singularities, constructed in terms of $f, g, \tilde{f}, \tilde{g}$, and the test polynomials, are equisingular. The key lemma follows from a classical Zariski equisingularity criterion of [18]. The main result, Theorem 1.1, is proven in Section 4.1. Finally, we give several corollaries of the main result. This includes a characterization of the atypical values of the pencils $f^k = tg^l$ and of the asymptotic critical values of the meromorphic functions f^k/g^l . We also propose an alternative proof of [6, Theorem 6.6] by an argument similar to the proof of the main result.

2 | FACTORIZATION OF THE INITIAL WEIGHTED FORM

Let $D(u, v)$ be a nonzero complex power series and let $w = (k, l)$ be a weight vector, where k, l are coprime positive integers. Then D can be written as the sum of quasi-homogeneous polynomials $D = D_m + D_{m+1} + \dots$, where $D_m \neq 0$ and $\deg_w D_i = i$ for $i \geq m$. Write D_m as a product

$$D_m(u, v) = Cu^{v_0}v^{v_{n+1}} \prod_{i=1}^n (v^k - t_i u^l)^{v_i}, \quad (2.1)$$

where $t_i \neq 0$ and $t_i \neq t_j$ for $i \neq j$.

The aim of this section is to express the multiplicities v_i for $1 \leq i \leq n$ by the intersection multiplicities of D with test polynomials.

Lemma 2.1. *Let $H_t = (v^k - tu^l)^N - u^{l(N+1)}$. Then for a sufficiently large integer N and for every $t \in \mathbb{C}^*$ such that $t \neq t_i$ for $1 \leq i \leq n$, one has*

$$v_j k l = i_0(D, H_{t_j}) - i_0(D, H_t).$$

Proof. For any power series $F(u, v)$ denote by $\text{in}_w F$ its weighted initial form with respect to the weight vector w . By the quasi-homogeneous version of Hensel's lemma (see, for example, [1, Lemma A.1]) the power series D factors to a product PQ , where $\text{in}_w P = (v^k - t_j u^l)^{v_j}$ and $\text{in}_w Q$ is not divisible by $v^k - t_j u^l$.

Take any $N > v_j k l$. In order to compute the intersection multiplicity $i_0(P, H_{t_j})$ we will use the classical Zeuten's rule $i_0(P, H_{t_j}) = \sum \text{ord } P(u, \alpha_i(u))$, where the sum runs over all Newton–Puiseux roots $v = \alpha_i(u)$ of $H_{t_j}(u, v) = 0$.

Solving the equation $H_{t_j}(u, v) = 0$ with respect to v we get Nk Newton–Puiseux roots of the form $v = \alpha_i(u) = \omega \sqrt[k]{t_j} u^{l/k} + \text{higher order terms}$, where $\omega^k = 1$ and $\sqrt[k]{t_j}$ is a fixed root of a

polynomial $Y^k - t_j$. We have $\alpha_i(u)^k - t_j u^l = \epsilon u^{l(N+1)/N}$, where $\epsilon^N = 1$. Hence $\text{ord}_{in_w} P(u, \alpha_i(u)) = \nu_j l(N+1)/N$.

For any quasi-homogeneous polynomial $F(u, v)$ with respect to the weight vector w we have $\text{ord} F(u, \alpha_i(u)) \geq (1/k) \text{deg}_w F$. Writing P as a sum $in_w P + \sum_{d=\nu_j kl+1}^{\infty} P_d$ of quasi-homogeneous polynomials and observing that for any $d \geq \nu_j kl + 1$ we have $\text{ord} P_d(u, \alpha_i(u)) \geq (1/k)d \geq \nu_j l(\nu_j kl + 1)/(\nu_j kl) > \nu_j l(N+1)/N$, we get $\text{ord} P(u, \alpha_i(u)) = \nu_j l(N+1)/N$. By Zeuten's rule; $i_0(P, H_{t_j}) = (N+1)\nu_j kl$.

Computing the intersection multiplicity $i_0(Q, H_{t_j})$ is simpler. Since the polynomials $in_w Q$ and $in_w H_{t_j} = (v^k - t_j u^l)^N$ are coprime, we have $\text{ord} Q(u, \alpha_i(u)) = \text{ord}_{in_w} Q(u, \alpha_i(u)) = (1/k) \text{deg}_w(in_w Q)$ for any Newton–Puiseux root $\alpha_i(u)$ of $H_{t_j}(u, v) = 0$. By Zeuten's rule we get $i_0(Q, H_{t_j}) = N \text{deg}_w(in_w Q)$.

Analogously, we obtain $i_0(P, H_t) = N \text{deg}_w(in_w P) = N\nu_j kl$ and $i_0(Q, H_t) = N \text{deg}_w(in_w Q)$. Finally $i_0(D, H_{t_j}) - i_0(D, H_t) = i_0(P, H_{t_j}) + i_0(Q, H_{t_j}) - i_0(P, H_t) - i_0(Q, H_t) = \nu_j kl$ which ends the proof. \square

3 | CASAS' FORMULA

Consider the following result of Casas-Alvero, which provides a very useful formula.

Theorem 3.1 [3, Theorem 3.2]. *Let $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping such that $(f, g)^{-1}(0, 0) = \{(0, 0)\}$. Let $D(u, v) = 0$ be the discriminant of (f, g) . Take any curve germ $H(u, v) = 0$ and let $h(x, y) = H(f(x, y), g(x, y))$. Then*

$$\mu(h) - 1 = i_0(f, g)[\mu(H) - 1] + i_0(D, H),$$

where $\mu(h)$ denotes the Milnor number of the curve $h = 0$ at the origin.

From the above theorem we obtain a crucial corollary, which is used in the proof of the main result of this article.

Corollary 3.2. *Let $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping such that $(f, g)^{-1}(0, 0) = \{(0, 0)\}$. Let $D(u, v) = 0$ be the discriminant curve of (f, g) and $h_t = (g^k - t f^l)^N - f^{l(N+1)}$ for $N > 1$. Then, under the notation of (2.1), for $N \gg 1$ and $t \in \mathbb{C}^*$ different from t_1, \dots, t_n , we have $\nu_j kl = \mu(h_{t_j}) - \mu(h_t)$.*

Proof. It is enough to apply Casas' formula to H_{t_j} and H_t as defined in Lemma 2.1. \square

4 | KEY LEMMA AND THE PROOF OF THE MAIN RESULT

The proof of the main result is based on Lemma 4.2 that says that for N sufficiently large the curves $h = (g - f)^N - f^{N+1} = 0$ and $\tilde{h} = (\tilde{g} - \tilde{f})^N - \tilde{f}^{N+1} = 0$ are equisingular. To show this, it is sufficient to establish that h and \tilde{h} can be resolved or just simplified by combinatorially equivalent modifications in the sense that we now recall.

By Zariski [18, Section 3], two reduced plane curve singularities C and D at the origin are *equisingular*, or *equivalent* in Zariski's terminology, if there is a pairing π between their branches $\pi(\gamma_i) = \delta_i$, where $\gamma_1, \dots, \gamma_s$ are the branches of C and $\delta_1, \dots, \delta_s$ the ones of D , that makes both their proper and total transforms by the blowing-up of the origin equivalent; see [18, Definition 3]. This pairing has to be *tangentially stable*, that is two branches γ_i, γ_j are tangent at the origin if and only if so are the corresponding branches δ_i, δ_j . It is easy to see that this equivalence extends to the nonreduced plane curve singularities just requiring that the multiplicities of branches are preserved by the pairing.

We call a *modification* the composition of a finite sequence of point blowings-up of $(\mathbb{C}^2, 0)$, that is, all the centers live over the origin. Let us order the centers in the order the blowings-up are performed (this order may be not unique, the blowings-up of different points on the same space can be performed in any order), and hence order the irreducible components of the exceptional divisors by their age. We call two such modifications *combinatorially equivalent* if at the process of blowings-up the k th centers of both modifications belong to the divisors with corresponding ages.

Lemma 4.1. *Two plane curve singularities C and D are equisingular if there exists combinatorially equivalent modifications of C and D and a bijection between the singularities of their total transforms (\tilde{C}) and (\tilde{D}) , such that the corresponding singularities $\tilde{C}, p, \tilde{D}, q$ are equivalent by pairings that sends the irreducible components of exceptional divisors to the components of the same age.*

Proof. It follows by the descending induction on the number of blowings-up in the modifications and therefore it suffices to show it for a single blowing-up. Then it follows from [18, Definition 3]. Indeed, a bijection between the singularities of \tilde{C}, \tilde{D} , and the pairings between their branches sending the exceptional divisor to the exceptional divisor, gives a pairing between the branches of C and D . This pairing is tangentially stable by construction, two branches are tangent if and only if their strict transforms belong to the same singularity of the blow-up space. The other conditions of [18, Definition 3] are immediate. \square

Lemma 4.2 (Key lemma). *Let $f, g, u', u'' \in \mathbb{C}\{x, y\}$ be convergent power series vanishing at zero such that f and g are coprime and let $\tilde{f} = (1 + u')f$, $\tilde{g} = (1 + u'')g$. Then for sufficiently large integer N the curves $(g - f)^N - f^{N+1} = 0$ and $(\tilde{g} - \tilde{f})^N - \tilde{f}^{N+1} = 0$ are equisingular.*

Proof. Let $R : M \rightarrow (\mathbb{C}^2, 0)$ be a resolution of singularities of the curve $fg(g - f)(\tilde{g} - \tilde{f}) = 0$. We show, for N sufficiently large, that the total transforms of $h := (g - f)^N - f^{N+1} = 0$ and $\tilde{h} := (\tilde{g} - \tilde{f})^N - \tilde{f}^{N+1} = 0$ by R have locally equivalent singularities by pairings that preserve the components of the exceptional divisor. Then the lemma will follow from Lemma 4.1.

The total transform $R^{-1}(\{fg(g - f)(\tilde{g} - \tilde{f}) = 0\})$ can be written as the union of irreducible components $E_1 \cup \dots \cup E_n \cup E_{n+1} \cup \dots \cup E_m$, where $E = E_1 \cup \dots \cup E_n$ is the exceptional divisor $R^{-1}(0)$ and E_{n+1}, \dots, E_m are the components of the proper transform of the curve $fg(g - f)(\tilde{g} - \tilde{f}) = 0$. By abuse of notation we will use the same symbols for germs of functions on $(\mathbb{C}^2, 0)$ and for their pull-backs to M .

For $i = 1, \dots, m$ we denote the orders of $f, g, g - f, \tilde{f}, \tilde{g}, \tilde{g} - \tilde{f}$ along E_i by $a_i, b_i, c_i, \tilde{a}_i, \tilde{b}_i, \tilde{c}_i$, respectively.

Take any point P of E_i , where $i \in \{1, \dots, m\}$. If P is a smooth point of the total divisor, and E_i in a neighborhood of P has a local equation $x = 0$, then $f = Ax^{a_i}$ and $\tilde{f} = (1 + u')f = \tilde{A}x^{\tilde{a}_i}$, where A

and \tilde{A} do not vanish at P . Hence $\tilde{a}_i = a_i$. Moreover, if E_i is a component of the exceptional divisor, then $\tilde{A}|_{E_i} = A|_{E_i}$, since $u'|_{E_i} = 0$. Similarly, we obtain $\tilde{b}_i = b_i$.

Choose N big enough so that for all $i \in \{1, \dots, n\}$ we have $Nc_i > (N+1)a_i$ if $c_i > a_i$ and $N\tilde{c}_i > (N+1)a_i$ if $\tilde{c}_i > a_i$. This is for instance the case if $N > \max\{a_i : 1 \leq i \leq n\}$. Under this assumption the orders of meromorphic functions $F = (g-f)^N/f^{N+1}$ and $\tilde{F} = (\tilde{g}-\tilde{f})^N/\tilde{f}^{N+1}$ along the components of the exceptional divisor are different from zero. Hence the proper preimage of the curve $h = 0$ (respectively, $\tilde{h} = 0$), which, in the complement of the exceptional divisor, coincides with $F = 1$ (respectively, $\tilde{F} = 1$), does not intersect the exceptional divisor at the smooth points of the total transform.

Take the intersection point P of a component E_i of the exceptional divisor with E_j , where $1 \leq j \leq m$, $j \neq i$. Choose a local analytic coordinate system (x, y) centered at P such that E_i has equation $x = 0$ and E_j has equation $y = 0$. In these coordinates $f = Ax^{a_i}y^{a_j}$, $g = Bx^{b_i}y^{b_j}$, $g-f = Cx^{c_i}y^{c_j}$, $\tilde{f} = \tilde{A}x^{a_i}y^{a_j}$, $\tilde{g} = \tilde{B}x^{b_i}y^{b_j}$, and $\tilde{g}-\tilde{f} = \tilde{C}x^{c_i}y^{c_j}$, where $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$ are germs of holomorphic functions that do not vanish at P . It follows from [20], the first paragraph of proof of Proposition 2.1, see also [2, Lemma 4.7], that the set of pairs $\{(a_i, a_j), (b_i, b_j), (c_i, c_j)\}$ is totally ordered, with respect to the partial order $(a, a') \leq (b, b')$ if $a \leq b$ and $a' \leq b'$, and two of these pairs are equal and are less than or equal to the third one.

In the sequel we denote by ϑ - any nonvanishing germ of a holomorphic function.

Let us write the equation of

$$h = (g-f)^N - f^{N+1} = \vartheta x^{Nc_i}y^{Nc_j} - \vartheta x^{(N+1)a_i}y^{(N+1)a_j}$$

in a neighborhood of P .

We have the following possibilities.

(I) $(b_i, b_j) > (a_i, a_j)$

Then $(c_i, c_j) = (\tilde{c}_i, \tilde{c}_j) = (a_i, a_j)$ and we get

$$h = \vartheta \tilde{h} = \vartheta x^{Na_i}y^{Na_j}.$$

(II) $(b_i, b_j) < (a_i, a_j)$

Then $(c_i, c_j) = (\tilde{c}_i, \tilde{c}_j) = (b_i, b_j)$ and we get

$$h = \vartheta \tilde{h} = \vartheta x^{Nb_i}y^{Nb_j}.$$

(III) $(b_i, b_j) = (a_i, a_j)$

Then $(c_i, c_j) \geq (a_i, a_j)$ and $(\tilde{c}_i, \tilde{c}_j) \geq (a_i, a_j)$ and we get

$$h = x^{Na_i}y^{Na_j}(\vartheta - \vartheta x^{N(c_i-a_i)}y^{N(c_j-a_j)}) - \vartheta x^{a_i}y^{a_j}$$

and similarly for \tilde{h} .

Write $g-f = (B-A)x^{a_i}y^{a_j}$ and $\tilde{g}-\tilde{f} = (\tilde{B}-\tilde{A})x^{a_i}y^{a_j}$.

Let us consider two subcases of (III).

First, assume that E_j is a component of the exceptional divisor. Then $a_j = b_j > 0$. Since A and \tilde{A} are equal on the exceptional divisor, we have $A(0, y) = \tilde{A}(0, y)$ and $A(x, 0) = \tilde{A}(x, 0)$. The same equations hold for B and \tilde{B} . It follows that the Newton diagrams of $B-A$ and $\tilde{B}-\tilde{A}$ have the same

intersection points with the coordinate axes. Moreover, $B - A$ and $\tilde{B} - \tilde{A}$ are factors of $g - f$ or $\tilde{g} - \tilde{f}$ and hence their Newton diagrams have only one vertex.

If $(0,0)$ is the vertex of the Newton diagram Δ of $\tilde{B} - \tilde{A}$, then $c_i = \tilde{c}_i = a_i$ and $c_j = \tilde{c}_j = a_j$, what implies that $h = \vartheta - x^{Na_i}y^{Na_j}$. If $(a, 0)$ is the vertex of Δ for some $a > 0$, then $c_i = \tilde{c}_i > a_i$ and $c_j = \tilde{c}_j = a_j$, what means that $h = x^{(N+1)a_i}y^{Na_j}(\vartheta - x^{N(c_i-a_i)-a_i} - \vartheta - y^{a_j})$. Similarly we have in the case, when $(0, b)$ is the vertex of Δ for some $b > 0$. The last possibility is, when the vertex of Δ is of the form (a, b) for some $a, b > 0$. In this case we have that $c_i > a_i$, $\tilde{c}_i > a_i$, $c_j > a_j$ and $\tilde{c}_j > a_j$. Consequently, by the assumption on N , both h and \tilde{h} are of the form $\vartheta - x^{(N+1)a_i}y^{(N+1)a_j}$, and hence they are equisingular.

Next, assume that E_j is not a component of the exceptional divisor. Then E_j is a component of the proper transform of $(g - f)(\tilde{g} - \tilde{f}) = 0$ and $a_j = b_j = 0$. Without loss of generality we may assume that $E_j \subset \{g - f = 0\}$. Then $c_j > 0$ and $\tilde{c}_j \geq 0$. Write $B - A = \vartheta - x^a y^{c_j}$, where $a = c_i - a_i$, and $\tilde{B} - \tilde{A} = \vartheta - x^{\tilde{a}} y^{\tilde{c}_j}$, where $\tilde{a} = \tilde{c}_i - \tilde{a}_i$. Since $A|_{E_i} = \tilde{A}|_{E_i}$, we have $A(0, y) = \tilde{A}(0, y)$ and similarly $B(0, y) = \tilde{B}(0, y)$. Therefore, $a = 0$ if and only if $\tilde{a} = 0$ and if this is the case $c_j = \tilde{c}_j$. Then $h = x^{Na_i}(\vartheta - y^{Nc_j} - \vartheta - x^{a_i})$ and a similar formula holds for \tilde{h} . Therefore they are equisingular.

If a and \tilde{a} are both strictly positive then, by the assumption on N , both h and \tilde{h} are of the form $x^{(N+1)a_i}(\vartheta - x^{N(c_i-a_i)-a_i}y^{Nc_j} - \vartheta) = \vartheta - x^{(N+1)a_i}$ and hence equisingular. \square

Corollary 4.3. *Let $f, g, u', u'' \in \mathbb{C}\{x, y\}$ be convergent power series vanishing at zero such that f and g are coprime and let $\tilde{f} = (1 + u')f$, $\tilde{g} = (1 + u'')g$. Then for any positive integers k and l , $t \neq 0$, and sufficiently large integer N , depending on k and l , the curves $h_t = (g^k - t f^l)^N - f^{l(N+1)} = 0$ and $\tilde{h}_t = (\tilde{g}^k - t \tilde{f}^l)^N - \tilde{f}^{l(N+1)} = 0$ are equisingular.*

Proof. It is enough to apply Lemma 4.2 to $f_1 = t f^l$, $g_1 = g^k$ and $\tilde{f}_1 = t \tilde{f}^l$, $\tilde{g}_1 = \tilde{g}^k$. \square

Now, we are ready to prove the main result.

4.1 | Proof of Theorem 1.1

Let $D(u, v) = 0$ be the discriminant of $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ and let $\tilde{D}(u, v) = 0$ be the discriminant of $(\tilde{f}, \tilde{g}) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$. Let $w = (k, l)$ be an arbitrary weight vector, where k, l are coprime positive integers. Write $\text{in}_w D = Cu^{\nu_0}v^{\nu_{n+1}} \prod_{i=1}^n (v^k - t_i u^l)^{\nu_i}$ and $\text{in}_w \tilde{D} = \tilde{C}u^{\tilde{\nu}_0}v^{\tilde{\nu}_{n+1}} \prod_{i=1}^n (v^k - \tilde{t}_i u^l)^{\tilde{\nu}_i}$. By [7, Theorem 2.1] the Newton diagrams of D and \tilde{D} coincide. Hence $\nu_0 = \tilde{\nu}_0$, $\nu_{n+1} = \tilde{\nu}_{n+1}$ and it is enough to prove that $\nu_i = \tilde{\nu}_i$ for $1 \leq i \leq n$. This follows from Corollary 3.2 since by Corollary 4.3 for $t \neq 0$ one has $\mu(h_t) = \mu(\tilde{h}_t)$. This ends the proof.

5 | COROLLARIES

As a direct consequence we obtain the following result.

Corollary 5.1. *Under assumptions of Theorem 1.1 the discriminants of (f, g) and (\tilde{f}, \tilde{g}) have the same tangents.*

The next corollary concerns the atypical values of the pencils $g^k - t f^l = 0$ and the asymptotic critical values of the meromorphic functions g^k / f^l .

Corollary 5.2. *Under the assumptions of Theorem 1.1, for every pair of coprime positive integers k, l we have*

- (i) *the pencils $g^k - tf^l = 0$ and $\tilde{g}^k - t\tilde{f}^l = 0$, where $t \in \mathbb{C}$ is a parameter, have the same sets of atypical values;*
- (ii) *the meromorphic functions g^k/f^l and \tilde{g}^k/\tilde{f}^l have the same asymptotic critical values; and*
- (iii) *the generic fibers of g^k/f^l and \tilde{g}^k/\tilde{f}^l are equisingular.*

Proof. To verify (i) we apply Casas' formula to $H_t = v^k - tv^l$ to show that these families are μ -constant for $t \neq t_i, i = 1, \dots, n$.

To show (ii) let us recall after [10, Section 5], that $t_0 \in \mathbb{C} \cup \infty$ is an asymptotic critical value of $F = f/g$ at the origin if there is a sequence of points $(x_k, y_k) \xrightarrow{k \rightarrow \infty} (0, 0), (x_k, y_k) \neq (0, 0)$, such that $\|(x_k, y_k)\| \|\nabla F\| \rightarrow 0$ and $F(x_k, y_k) \rightarrow t_0$. Therefore if $\tilde{F} = uF$, with $u(0) = 1$, then an elementary computation shows that the asymptotic critical values of \tilde{F} and F coincide.

Now we show the equivalence of (i) and (ii) thus providing for both of them alternative proofs. That is, we show that the set of asymptotic critical values of $F = f/g$ coincide with the atypical values t of the pencil $f - tg$. Indeed, an elementary computation, see [10, Proposition 5.1], shows that t_0 is the asymptotic critical value if and only if the Kuo-Verdier condition (w) fails at $(0, 0, t_0)$ for the strata $(\text{Reg}X, T)$, where $\text{Reg}X$ is the regular part of $X = V(f - tg)$ and T is the t -axis. On the other hand, it is well known that for families of isolated plane curve singularities that their equisingularity is equivalent to Whitney equisingularity, see [19, Theorem 8.1], and in the complex domain the Kuo-Verdier condition is equivalent to Whitney's conditions; see [17]. For a direct elementary proof that μ -constant condition is equivalent to Verdier condition for the families of curve singularities $f - tg$; see [9, Theorem 4.1].

Finally, to show (iii) we may again apply Casas' formula to the deformations $s(g^k - tf^l) + (1 - s)(\tilde{g}^k - t\tilde{f}^l)$ with the parameter $s \in \mathbb{C}$. □

Corollary 5.3. *Let $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a holomorphic mapping with an isolated zero. Then the initial Newton polynomial of its discriminant is determined, up to rescaling variables, by the ideals (f) and (g) in $\mathbb{C}\{x, y\}$.*

Proof. Let $u_1, u_2 \in \mathbb{C}\{x, y\}$ be power series with nonzero constant terms. Let $a = u_1(0, 0)$ and $b = u_2(0, 0)$. If D (respectively, D_1) is the discriminant of $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ (respectively, $(af, bg) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$) then $D(au, bv) = D_1(u, v)$. Hence by Theorem 1.1 applied to (af, bg) and (u_1f, u_2g) , the initial Newton polynomials of the discriminants of (f, g) and (u_1f, u_2g) are equal up to rescaling variables. □

As an application of the methods used in this paper we present a new proof of [6, Theorem 6.6].

Theorem 5.4 [6, Theorem 6.6]. *Let $h = 0$ be a unitangent singular curve and let $\ell_1 = 0, \ell_2 = 0$ be smooth curves transverse to $h = 0$. Then there exists a nonzero constant $d \in \mathbb{C}$ such that the initial Newton polynomials of discriminants of mappings $(d\ell_1, h) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ and $(\ell_2, h) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ are equal.*

Keep the assumptions of Theorem 5.4 and let d be the limit at the origin of the meromorphic function ℓ_2/ℓ_1 restricted to the tangent to the curve $h = 0$. Fix positive integers k, l and

a nonzero complex constant t . Let $f = th^l$, $g = (d\ell_1)^k$, and $\tilde{g} = \ell_2^k$. In order to prove Theorem 5.4 it is enough to have the following counterpart of Lemma 4.2 (see the proof of Theorem 1.1).

Lemma 5.5. *For sufficiently large integer N the curves $(g - f)^N - f^{N+1} = 0$ and $(\tilde{g} - f)^N - f^{N+1} = 0$ are equisingular.*

Proof. Let $\sigma : M \rightarrow (\mathbb{C}^2, 0)$ be the blowing-up of \mathbb{C}^2 at the origin. Then the proper transforms of the curves $f = 0$, $g = 0$, $\tilde{g} = 0$ intersect the exceptional divisor E at points P , Q , \tilde{Q} , respectively (if $g = 0$ and $\tilde{g} = 0$ have the same tangent, then $Q = \tilde{Q}$).

Let $R : M_1 \rightarrow M$ be a resolution of singularities of the curve $f g (g - f) (\tilde{g} - f) = 0$ at P . By Zariski's criterion, Lemma 4.1, it is enough to prove that the total and the proper transforms of the curves $(g - f)^N - f^{N+1} = 0$ and $(\tilde{g} - f)^N - f^{N+1} = 0$ by $Ro\sigma$ are equisingular. To prove the equisingularity on $R^{-1}(P) \setminus E$ it is enough to use the arguments from the proof of Lemma 4.2 since the meromorphic function \tilde{g}/g is constant and equal to 1 on the set $R^{-1}(P)$. Thus it remains to show that the equisingularity classes of these curves are the same on the component E of the exceptional divisor.

Denote now by E the strict transform of the original exceptional divisor E of σ and let now P denote the intersection of E and another component E' of the exceptional divisor of $Ro\sigma$. Choose a local analytic coordinate system (x, y) centered at P such that E has equation $x = 0$ and E' has equation $y = 0$. In these coordinates $f = \vartheta x^a y^{a'}$, $g = \vartheta x^b y^{b'}$, $\tilde{g} = \tilde{\vartheta} x^b y^{b'}$, and $g - f = \vartheta x^c y^{c'}$. The set $\{(a, a'), (b, b'), (c, c')\}$ is again totally ordered as in the proof of Key Lemma, and two of these pairs are equal and are less than or equal to the third one.

The case $(a, a') = (b, b')$ is impossible. Indeed, if $(a, a') = (b, b')$ then g/f restricted to E is a meromorphic function which has a zero of order $b > 0$ at Q and has no poles – contradiction.

If $(a, a') < (b, b')$ then $(c, c') = (a, a')$ and we get

$$(g - f)^N - f^{N+1} = \vartheta x^{Na} y^{Na'}.$$

If $(b, b') < (a, a')$ then $(c, c') = (b, b')$ and we get

$$(g - f)^N - f^{N+1} = \vartheta x^{Nb} y^{Nb'}.$$

Now, we will determine the class of equisingularity of $(g - f)^N - f^{N+1} = 0$ at points of E different from P and Q . Choose a local analytic coordinate system (x, y) centered at one of such a point such that E has equation $x = 0$. Then $f = \vartheta x^a$ and $g = \vartheta x^b$.

If $a < b$ then

$$(g - f)^N - f^{N+1} = \vartheta x^{Na}.$$

If $b < a$ then

$$(g - f)^N - f^{N+1} = \vartheta x^{Nb}.$$

If $a = b$ then the meromorphic function $F = g/f$ restricted to E has exactly one zero of order b at Q and exactly one pole at P . Hence every nonzero complex number, in particular 1, is a regular value of $F|_E$. As a consequence the set $(F|_E)^{-1}(1)$ consists of b points P_i , $1 \leq i \leq b$ and at each of

these points the curve $F = 1$ intersects E transversally. Thus, for every $i \in \{1, \dots, b\}$ we may find a local analytic coordinate system (x, y) centered at P_i such that $g - f = x^a y$. We get in the neighborhood of P_i a local equation

$$(g - f)^N - f^{N+1} = x^{Na}(y^N - \vartheta x^a).$$

Finally, we will determine the equisingularity class of $(g - f)^N - f^{N+1} = 0$ at Q . Choose a local analytic coordinate system (x, y) centered at Q such that $f = x^a$ and $g = x^b y^b$ in the neighborhood of Q .

If $a \leq b$ then

$$(g - f)^N - f^{N+1} = \vartheta x^{Na}.$$

If $a > b$ then

$$(g - f)^N - f^{N+1} = x^{Nb}[(y^b - x^{a-b})^N - x^{(a-b)N+a}]$$

and its equisingularity type is uniquely determined by a, b, N . □

Corollary 5.6. *Let $f = 0$ be a unitangent singular curve and let $\ell = 0$ be a smooth curve transverse to $f = 0$. Then the initial Newton polynomial of the discriminant of the mapping $(\ell, f) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is determined, up to rescaling variables by the ideal $(f) \subset \mathbb{C}\{x, y\}$.*

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