Arc spaces, motivic measure and Lipschitz geometry of real algebraic sets

Jean-Baptiste Campesato, Toshizumi Fukui, Krzysztof Kurdyka & Adam Parusiński

🐴 Springer

Springer

Mathematische Annalen

Arc spaces, motivic measure and Lipschitz geometry of real algebraic sets

**Jean-Baptiste Campesato[1] · Toshizumi Fukui[2] · Krzysztof Kurdyka[3] ·
Adam Parusiński[4]**

Abstract
We investigate connections between Lipschitz geometry of real algebraic varieties and properties of their arc spaces. For this purpose we develop motivic integration in the real algebraic set-up. We construct a motivic measure on the space of real analytic arcs. We use this measure to define a real motivic integral which admits a change of variables formula not only for the birational but also for generically one-to-one Nash maps. As a consequence we obtain an inverse mapping theorem which holds for continuous rational maps and, more generally, for generically arc-analytic maps. These maps appeared recently in the classification of singularities of real analytic function germs. Finally, as an application, we characterize in terms of the motivic measure, germs of arc-analytic homeomorphism between real algebraic varieties which are bi-Lipschitz for the inner metric.

Mathematics Subject Classification 14P99 (Primary) · 26A16 · 14E18 · 14B05

Contents

✉ Adam Parusiński
  Adam.Parusinski@unice.fr

Extended author information available on the last page of the article

1 Introduction

In this paper we establish relations between the arc space and the Lipschitz geometry of a singular real algebraic variety.

The interest in the Lipschitz geometry of real analytic and algebraic spaces emerged in the 70's of the last century by a conjecture of Siebenmann and Sullivan: there are only countably many local Lipschitz structures on real analytic spaces. Subsequently the Lipschitz geometry of real and complex algebraic singularities attracted much attention and various methods have been developed to study it: stratification theory [34,38], L-regular decompositions [24,26,39,43], Lipschitz triangulations [47], non-archimedean geometry [16], and recently, in the complex case, resolution and low dimensional topology [4]. In the algebraic case Siebenmann and Sullivan's conjecture was proved in [37]. The general analytic case was solved in [48].

In this paper we study various versions of Lipschitz inverse mapping theorems, with respect to the inner distance, for homeomorphisms $f : X \to Y$ between (possibly singular) real algebraic set germs. Recall that a connected real algebraic, and more generally a connected semialgebraic, subset $X \subset \mathbb{R}^N$ is path-connected (by rectifiable curves), so we have an *inner* distance on X, defined by the infimum over the length of rectifiable curves joining two given points in X.

We assume that the homeomorphism f is semialgebraic and generically arc-analytic. For instance the recently studied continuous rational maps [20–22] are of this type.

Arc-analytic mappings were introduced to real algebraic geometry in [23]. Those are the mappings sending by composition real analytic arcs to real analytic arcs. It was shown in [2,40] that the semialgebraic arc-analytic mappings coincide with the blow-Nash mappings. Moreover, by [42], real algebraic sets admit algebraic stratifications with local semialgebraic arc-analytic triviality along each stratum.

What we prove can be stated informally as follows: if f^{-1} is Lipschitz, then so is f itself. The problem is non-trivial even when the germs (X, x) and (Y, y) are non-singular [14]. When these germs are singular, then the problem is much more delicate. In fact we have to assume that the motivic measures of the real analytic arcs drawn on (X, x) and (Y, y) are equal.

Developing a rigorous theory of motivic measure on the space of real analytic arcs for real algebraic sets is another main goal of this paper.

We state below a concise version of our main results. For more precise and more general statements see Theorems 4.13 and 5.10.

Theorem *Let $f : (X, x) \to (Y, y)$ be the germ of a semialgebraic generically arc-analytic homeomorphism between two real algebraic set germs, that are of pure dimension[1] d. Assume that the motivic measures of the real analytic arcs centered at x in X and of the real analytic arcs centered at y in Y are equal (see Sect. 3 for the definition of the motivic measure). Then*

1. *If the Jacobian determinant of f is bounded from below then it is bounded from above and f^{-1} is generically arc-analytic.*

---

[1] For ease of reading, in the introduction we avoid varieties admitting points which have a structure of smooth submanifold of smaller dimension as in the handle of the Whitney umbrella $\{x^2 = zy^2\} \subset \mathbb{R}^3$.

2. *If the inverse f^{-1} of f is Lipschitz with respect to the inner distance then so is f.*

The proof of this theorem is based on motivic integration. Recall that in the case of complex algebraic varieties, motivic integration was introduced by Kontsevitch for non-singular varieties in order to avoid the use of p-adic integrals. Then the theory was developed and extended to the singular case in [1,10,11,29]. The motivic measure is defined on the space of formal arcs drawn on an algebraic variety and takes values in a Grothendieck ring which encodes all the additive invariants of the underlying category. One main ingredient consists in reducing the study to truncated arcs in order to work with finite dimensional spaces. Notice that since the seminal paper of Nash [36], it has been established that the arc space of a variety encodes a great deal of information about its singularities.

In the real algebraic set-up, arguments coming from motivic integration were used in [8,9,12,19] to study and classify the singularities of real algebraic function germs.

In the present paper we construct a motivic measure and a motivic integral for possibly singular real algebraic varieties. Similarly to the complex case, the motivic integral comes together with a change of variables formula which is convenient to do actual computations in terms of resolution of singularities. In our real algebraic set-up this formula holds for generically one-to-one Nash maps and not merely for the birational ones.

A first difference of the present construction compared to the complex one, is that we work with real analytic arcs and not with all formal arcs. However, thanks to Artin approximation theorem, this difference is minor. More importantly, it is not possible to follow exactly the construction of the motivic measure in the complex case because of several additional difficulties arising from the absence in the real set-up of the Nullstellensatz and of the theorem of Chevalley (the image of a Zariski-constructible set by a regular mapping is Zariski-constructible).

The real motivic measure and the real motivic integral are constructed and studied in Sect. 3.

2 Geometric framework

Throughout this paper, we say that a subset $X \subset \mathbb{R}^N$ is an algebraic set if it is closed for the Zariski topology, i.e. X may be described as the intersection of the zero sets of polynomials with real coefficients. We denote by $I(X)$ the ideal of $\mathbb{R}[x_1, \ldots, x_N]$ consisting of the polynomials vanishing on X. By noetherianity, we may always assume that the above intersection is indexed by a finite set[2] and that $I(X) = (f_1, \ldots, f_s)$ is finitely generated. The dimension $\dim X$ of X is the dimension of the ring $\mathcal{P}(X) = \mathbb{R}[x_1, \ldots, x_N]\big/ I(X)$ of polynomial functions on X.

The ring $\mathcal{R}(X)$ of regular functions on X is given by the localization of $\mathcal{P}(X)$ with respect to the multiplicative set $\{h \in \mathcal{P}(X),\ h^{-1}(0) = \varnothing\}$. Regular maps are the morphisms of real algebraic sets.

---

[2] Actually, noticing that $f_1 = \cdots = f_s = 0 \Leftrightarrow f_1^2 + \cdots + f_s^2 = 0$, we may always describe a real algebraic set as the zero-set of only one polynomial.

Unless otherwise stated, we will always use the Euclidean topology and not the Zariski one (for instance for the notions of homeomorphism, map germ or closure).

We say that a d-dimensional algebraic set X is non-singular at $x \in X$ if there exist $g_1, \ldots, g_{N-d} \in I(X)$ and an Euclidean open neighborhood U of x in $\mathbb{R}^N$ such that $U \cap X = U \cap V(g_1, \ldots, g_{N-d})$ and $\operatorname{rank}\left(\frac{\partial g_i}{\partial x_j}(x)\right) = N - d$. Then there exists an open semialgebraic neighborhood of x in V which is a d-dimensional Nash submanifold. Notice that the converse doesn't hold [6, Example 3.3.12.b.]. We denote by $\operatorname{Reg}(X)$ the set of non-singular points of X. We denote by $X_{\text{sing}} = X \setminus \operatorname{Reg}(X)$ the set of singular points of X, it is an algebraic subset of strictly smaller dimension, see [6, Proposition 3.3.14].

A semialgebraic subset of $\mathbb{R}^N$ is the projection of an algebraic subset of $\mathbb{R}^{N+m}$ for some $m \in \mathbb{N}_{\geq 0}$. Actually, by a result of Motzkin [35], we may always assume that $m = 1$. Equivalently, a subset $S \subset \mathbb{R}^N$ is semialgebraic if and only if there exist polynomials $f_i, g_{i,1}, \ldots, g_{i,s_i} \in \mathbb{R}[x_1, \ldots, x_N]$ such that

$$S = \bigcup_{i=1}^{r} \left\{ x \in \mathbb{R}^N, \ f_i(x) = 0, \ g_{i,1}(x) > 0, \ldots, g_{i,s_i}(x) > 0 \right\}.$$

Notice that semialgebraic sets are closed under union, intersection and cartesian product. They are also closed under projection by the Tarski–Seidenberg Theorem. A function is semialgebraic if so is its graph.

We refer the reader to [6] for more details on real algebraic geometry.

Let X be a non-singular real algebraic set and $f : X \to \mathbb{R}$. We say that f is a Nash function if it is C^∞ and semialgebraic. Since a semialgebraic function satisfies a non-trivial polynomial equation and since a smooth function satisfying a non-trivial real analytic equation is real analytic [5,30,46], we obtain that f is Nash if and only if f is real analytic and satisfies a non-trivial polynomial equation.

A subset of a real analytic variety is said to be arc-symmetric in the sense of [23] if, given a real analytic arc, either the arc is entirely included in the set or it meets the set at isolated points only. We are going to work with a slightly different notion defined in [41]. We define $\mathcal{AS}^N$ as the boolean algebra generated by semialgebraic[3] arc-symmetric subsets of $\mathbb{P}_{\mathbb{R}}^N$. We set

$$\mathcal{AS} = \bigcup_{N \in \mathbb{N}_{\geq 0}} \mathcal{AS}^N.$$

Formally, a subset $A \subset \mathbb{P}_{\mathbb{R}}^N$ is an $\mathcal{AS}$-set if it is semialgebraic and if, given a real analytic arc $\gamma : (-1, 1) \to \mathbb{P}_{\mathbb{R}}^N$ such that $\gamma(-1, 0) \subset A$, there exists $\varepsilon > 0$ such that $\gamma(0, \varepsilon) \subset A$.

Notice that closed $\mathcal{AS}$-subsets of $\mathbb{P}_{\mathbb{R}}^N$ are exactly the closed sets of a noetherian topology.

For more on arc-symmetric and $\mathcal{AS}$ sets we refer the reader to [27].

---

[3] A subset of $\mathbb{P}_{\mathbb{R}}^N$ is semialgebraic if it is for $\mathbb{P}_{\mathbb{R}}^N$ seen as an algebraic subset of some $\mathbb{R}^M$, or, equivalently, if the intersection of the set with each canonical affine chart is semialgebraic.

One important property of the $\mathcal{AS}$ sets that we rely on throughout this paper is that it admits an additive invariant richer than the Euler characteristic with compact support, namely the virtual Poincaré polynomial presented later in Sect. 3.2. This is in contrast to the semialgebraic sets, for which, by a theorem of Quarez [45], every additive homeomorphism invariant of semialgebraic sets factorises through the Euler characteristic with compact support.

Let E, B, F be three $\mathcal{AS}$-sets. We say that $p : E \to B$ is an $\mathcal{AS}$ piecewise trivial fibration with fiber F if there exists a finite partition $B = \sqcup B_i$ into $\mathcal{AS}$-sets such that $p^{-1}(B_i) \simeq B_i \times F$ where $\simeq$ means bijection with $\mathcal{AS}$-graph.

Notice that, thanks to the noetherianity of the $\mathcal{AS}$-topology, if $p : E \to B$ is locally trivial with fiber F for the $\mathcal{AS}$-topology,[4] then it is an $\mathcal{AS}$ piecewise trivial fibration.

3 Real motivic integration

This section is devoted to the construction of a real motivic measure. Notice that a first step in this direction was done by Quarez in [45] using the Euler characteristic with compact support for semialgebraic sets. The measure constructed in this section takes advantage of the $\mathcal{AS}$-machinery in order to use the virtual Poincaré polynomial which is a real analogue of the Hodge–Deligne polynomial in real algebraic geometry. This additive invariant is richer than the Euler characteristic since it encodes, for example, the dimension.

Since real algebraic geometry is quite different from complex algebraic geometry as there is, for example, no Nullstellensatz or Chevalley's theorem, the classical construction of the motivic measure does not work as it is in this real context and it is necessary to carefully handle these differences.

3.1 Real arcs and jets

We follow the notations of [8, §2.4].

Definition 3.1 The space of real analytic arcs on $\mathbb{R}^N$ is defined as

$$\mathcal{L}(\mathbb{R}^N) = \left\{ \gamma : (\mathbb{R}, 0) \to \mathbb{R}^N, \ \gamma \text{ real analytic} \right\}$$

Definition 3.2 For $n \in \mathbb{N}_{\geq 0}$, the space of n-jets on $\mathbb{R}^N$ is defined as

$$\mathcal{L}_n(\mathbb{R}^N) = \mathcal{L}(\mathbb{R}^N) / {\sim_n}$$

where $\gamma_1 \sim_n \gamma_2 \Leftrightarrow \gamma_1 \equiv \gamma_2 \mod t^{n+1}$.

Notation 3.3 For $m > n$, we consider the following *truncation maps*:

$$\pi_n : \mathcal{L}(\mathbb{R}^N) \to \mathcal{L}_n(\mathbb{R}^N)$$

---

[4] i.e. for every $x \in B$ there is $U \subset B$ an $\mathcal{AS}$-open subset containing x such that $p^{-1}(U) \simeq U \times F$.

and

$$\pi_n^m : \mathcal{L}_m(\mathbb{R}^N) \to \mathcal{L}_n(\mathbb{R}^N).$$

Definition 3.4 For an algebraic set $X \subset \mathbb{R}^N$, we define the space of real analytic arcs on X as

$$\mathcal{L}(X) = \left\{ \gamma \in \mathcal{L}(\mathbb{R}^N), \ \forall f \in I(X), \ f(\gamma(t)) = 0 \right\}$$

and the space of n-jets on X as

$$\mathcal{L}_n(X) = \left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^N), \ \forall f \in I(X), \ f(\gamma(t)) \equiv 0 \quad \mod t^{n+1} \right\}.$$

The truncation maps induce the maps

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

and

$$\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X).$$

Remark 3.5 Notice that $\mathcal{L}_n(X)$ is a real algebraic variety. Indeed, let $f \in I(X)$ and $a_0, \ldots, a_n \in \mathbb{R}^N$, then we have the following expansion

$$\begin{aligned} &f(a_0 + a_1 t + \cdots + a_n t^n) \\ &= P_0^f(a_0, \ldots, a_n) + P_1^f(a_0, \ldots, a_n)t + \cdots + P_n^f(a_0, \ldots, a_n)t^n + \cdots \end{aligned}$$

where the coefficients P_i^f are polynomials. Hence $\mathcal{L}_n(X)$ is the algebraic subset of $\mathbb{R}^{N(n+1)}$ defined as the zero-set of the polynomials P_i^f for $f \in I(X)$ and $i \in \{0, \ldots, n\}$.
In the same way, we may think of $\mathcal{L}(X)$ as an infinite-dimensional algebraic variety.

Remark 3.6 When X is non-singular the following equality holds:

$$\mathcal{L}_n(X) = \pi_n(\mathcal{L}(X))$$

Indeed, using Hensel's lemma, we may always lift an n-jet to a formal arc on X and then use Artin approximation theorem to find an analytic arc whose expansion coincides up to the degree $n + 1$. However this equality doesn't hold anymore when X is singular as highlighted in [8, Example 2.30]. Hence it is necessary to distinguish the space $\mathcal{L}_n(X)$ of n-jets on X and the space $\pi_n(\mathcal{L}(X)) \subset \mathcal{L}_n(X)$ of n-jets on X which may be lifted to real analytic arcs on X. We have the following exact statement.

Proposition 3.7 ([8, Proposition 2.31]) *Let X be an algebraic subset of $\mathbb{R}^N$. Then the following are equivalent :*

(i) *X is non-singular.*

(ii) $\forall n \in \mathbb{N}_{\geq 0},\ \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ *is surjective.*

(iii) $\forall n \in \mathbb{N}_{\geq 0},\ \pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ *is surjective.*

Proposition 3.8 ([8, Proposition 2.33]) *Let X be a d-dimensional algebraic subset of* $\mathbb{R}^N$. *Then*

(1) *For $m \geq n$, the dimensions of the fibers of*

$$\pi_n^m{}_{|\pi_m(\mathcal{L}(X))} : \pi_m\left(\mathcal{L}(X)\right) \to \pi_n\left(\mathcal{L}(X)\right)$$

are smaller than or equal to $(m - n)d$.

(2) *The fiber $\left(\pi_n^{n+1}\right)^{-1}(\gamma)$ of $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ is either empty or isomorphic to $T_{\gamma(0)}^{\mathrm{Zar}} X$.*

Theorem 3.9 (A motivic corollary of Greenberg Theorem) *Let $X \subset \mathbb{R}^N$ be an algebraic subset. There exists $c \in \mathbb{N}_{>0}$ (depending only on $I(X)$) such that*

$$\forall n \in \mathbb{N}_{\geq 0},\ \pi_n(\mathcal{L}(X)) = \pi_n^{cn}(\mathcal{L}_{cn}(X))$$

Proof Assume that $I(X) = (f_1, \ldots, f_s)$. By the main theorem of [15], there exist $N \in \mathbb{N}_{>0}, l \in \mathbb{N}_{>0}$ and $\sigma \in \mathbb{N}_{\geq 0}$, which depend only on the ideal of $\mathbb{R}\{t\}[x_1, \ldots, x_N]$ generated by $f_i \in \mathbb{R}[x_1, \ldots, x_N] \subset \mathbb{R}\{t\}[x_1, \ldots, x_N]$, and such that $\forall \nu \geq N, \forall \gamma \in \mathbb{R}\{t\}^N$, if $f_1(\gamma(t)) \equiv \cdots \equiv f_s(\gamma(t)) \equiv 0 \mod t^\nu$, then there exists $\eta \in \mathbb{R}\{t\}^N$ such that $\eta(t) \equiv \gamma(t) \mod t^{\lfloor \frac{\nu}{l} \rfloor - \sigma}$ and $f_1(\eta(t)) = \cdots = f_s(\eta(t)) = 0$.

Fix $c = \max (l(\sigma + 2), N)$. We are going to prove that

$$\forall n \in \mathbb{N}_{\geq 0},\ \pi_n(\mathcal{L}(X)) = \pi_n^{cn}(\mathcal{L}_{cn}(X))$$

It is enough to prove that $\pi_n^{cn}(\mathcal{L}_{cn}(X)) \subset \pi_n(\mathcal{L}(X))$ for $n \geq 1$.

Let $n \geq 1$. Let $\tilde{\gamma} \in \mathcal{L}_{cn}(X)$. Then there exists $\gamma \in \mathbb{R}\{t\}^N$ such that $\gamma(t) \equiv \tilde{\gamma}(t)$ mod t^{cn+1} and

$$f_1(\gamma(t)) \equiv \cdots \equiv f_s(\gamma(t)) \equiv 0 \mod t^{cn+1}$$

Notice that $cn + 1 \geq N$ so that there exists $\eta \in \mathbb{R}\{t\}^N$ such that $\eta(t) \equiv \gamma(t)$ mod $t^{\lfloor \frac{cn+1}{l} \rfloor - \sigma}$ and $f_1(\eta(t)) = \cdots = f_s(\eta(t)) = 0$.

Since

$$\left\lfloor \frac{cn + 1}{l} \right\rfloor - \sigma > n$$

we have that $\pi_n^{cn}(\tilde{\gamma}) = \pi_n(\eta) \in \pi_n(\mathcal{L}(X))$. $\qquad\square$

Remark 3.10 By Tarski–Seidenberg theorem, $\pi_n(\mathcal{L}(X)) = \pi_n^{cn}(\mathcal{L}_{cn}(X))$ is semialgebraic as the projection of an algebraic set. However, $\pi_n(\mathcal{L}(X))$ may not be $\mathcal{AS}$ (and thus not Zariski-constructible) as shown in [8, Example 2.32].

This is a major difference with the complex case where $\pi_n(\mathcal{L}(X))$ is Zariski-constructible by Chevalley theorem as the projection of a complex algebraic variety.

Definition 3.11 Let X be an algebraic subset of $\mathbb{R}^N$. We define the ideal H_X of $\mathbb{R}[x_1, \ldots, x_N]$ by

$$H_X = \sum_{f_1, \ldots, f_{N-d} \in I(X)} \Delta(f_1, \ldots, f_{N-d})((f_1, \ldots, f_{N-d}) : I(X))$$

where

- $d = \dim X$
- $\Delta(f_1, \ldots, f_{N-d})$ is the ideal generated by the $N-d$ minors of the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i=1,\ldots,N-d \\ j=1,\ldots,N}}$$

- $((f_1, \ldots, f_{N-d}) : I(X)) = \{g \in \mathbb{R}[x_1, \ldots, x_N], \, gI(X) \subset (f_1, \ldots, f_{N-d})\}$ is the ideal quotient of the ideal $(f_1, \ldots, f_{N-d})$ by the ideal $I(X)$

Remark 3.12 By [8, Lemma 4.1], $V(H_X) = X_{\text{sing}}$.

Definition 3.13 Let $X \subset \mathbb{R}^N$ be an algebraic subset and $e \in \mathbb{N}_{\geq 0}$. We set

$$\mathcal{L}^{(e)}(X) = \left\{ \gamma \in \mathcal{L}(X), \, \exists h \in H_X, \, h(\gamma(t)) \not\equiv 0 \mod t^{e+1} \right\}$$

Remark 3.14 From now on, we set

$$\mathcal{L}(X_{\text{sing}}) = \left\{ \gamma \in \mathcal{L}(\mathbb{R}^N), \, \forall h \in H_X, \, h(\gamma(t)) = 0 \right\}$$

and

$$\mathcal{L}_n(X_{\text{sing}}) = \left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^N), \, \forall h \in H_X, \, h(\gamma(t)) \equiv 0 \mod t^{n+1} \right\}.$$

Notice that

$$\left\{ \gamma \in \mathcal{L}(\mathbb{R}^N), \, \forall h \in H_X, \, h(\gamma(t)) = 0 \right\}$$
$$= \left\{ \gamma \in \mathcal{L}(\mathbb{R}^N), \, \forall f \in I(X_{\text{sing}}), \, f(\gamma(t)) = 0 \right\}$$

but in general

$$\left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^N), \, \forall h \in H_X, \, h(\gamma(t)) \equiv 0 \mod t^{n+1} \right\}$$
$$\neq \left\{ \gamma \in \mathcal{L}_n(\mathbb{R}^N), \, \forall f \in I(X_{\text{sing}}), \, f(\gamma(t)) \equiv 0 \mod t^{n+1} \right\}$$

Notice also that since the proof of Greenberg Theorem 3.9 is algebraic, it holds for $\mathcal{L}(X_{\text{sing}})$ (just use the ideal H_X in the proof).

Remark 3.15 $\mathcal{L}(X) = \left(\bigcup_{e \in \mathbb{N}_{\geq 0}} \mathcal{L}^{(e)}(X) \right) \bigsqcup \mathcal{L}(X_{\text{sing}})$

The following proposition is a real version of [10, Lemma 4.1]. Its proof is quite similar to the one of [8, Lemma 4.5].

Proposition 3.16 *Let X be a d-dimensional algebraic subset of $\mathbb{R}^N$ and $e \in \mathbb{N}_{\geq 0}$. Then, for $n \geq e$,*

(i) $\pi_n \left(\mathcal{L}^{(e)}(X) \right) \in \mathcal{AS}$

(ii) $\pi_n^{n+1} : \pi_{n+1} \left(\mathcal{L}^{(e)}(X) \right) \to \pi_n \left(\mathcal{L}^{(e)}(X) \right)$ *is an $\mathcal{AS}$ piecewise trivial fibration with fiber $\mathbb{R}^d$.*

Proof By [8, Lemma 4.7], $\mathcal{L}^{(e)}(X)$ is covered by finitely many sets of the form

$$A_{\mathbf{f}, h, \delta} = \left\{ \gamma \in \mathcal{L}(\mathbb{R}^N), \ (h\delta)(\gamma(t)) \not\equiv 0 \mod t^{e+1} \right\}$$

where $\mathbf{f} = (f_1, \ldots, f_{N-d}) \in I(X)^{N-d}$, δ is a $N - d$ minor of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i=1,\ldots,N-d \\ j=1,\ldots,N}}$ and $h \in ((f_1, \ldots, f_{N-d}) : I(X))$. Moreover,

$$\mathcal{L}(X) \cap A_{\mathbf{f}, h, \delta} = \left\{ \gamma \in \mathcal{L}(\mathbb{R}^N), \ \begin{array}{l} f_1(\gamma(t)) = \cdots = f_{N-d}(\gamma(t)) = 0, \\ (h\delta)(\gamma(t)) \not\equiv 0 \mod t^{e+1} \end{array} \right\},$$

so that $\mathcal{L}^{(e)}(X) = \mathcal{L}(X) \cap \bigcup_{\text{finite}} A_{\mathbf{f}, h, \delta} = \bigcup_{\text{finite}} \left(\mathcal{L}(X) \cap A_{\mathbf{f}, h, \delta} \right)$.
For $e' \leq e$, we set

$$A_{\mathbf{f}, h, \delta, e'} = \left\{ \gamma \in A_{\mathbf{f}, h, \delta}, \ \begin{array}{l} \text{ord}_t \, \delta(\gamma(t)) = e', \\ \text{ord}_t \, \delta'(\gamma(t)) \geq e', \end{array} \text{ for all } N - d \text{ minor } \delta' \text{ of } \left(\frac{\partial f_i}{\partial x_j} \right) \right\}$$

in order to refine the above cover: $\mathcal{L}^{(e)}(X) = \bigcup_{\text{finite}} \left(\mathcal{L}(X) \cap A_{\mathbf{f}, h, \delta, e'} \right)$.

Fix some set $A = A_{\mathbf{f}, h, \delta, e'} \cap \mathcal{L}(X)$. Notice that if $\pi_n(\gamma) \in \pi_n(A)$ and if $\pi_{n+1}(\eta) \in \pi_{n+1}(\mathcal{L}^{(e)}(X))$ is in the preimage of $\pi_n(\gamma)$ by π_n^{n+1} then $\pi_{n+1}(\eta) \in \pi_{n+1}(A)$. Indeed, $\eta \in \mathcal{L}(X)$ so $f_1(\eta) = \cdots = f_{N-d}(\eta) = 0$ and since $\pi_n(\eta) = \pi_n(\gamma)$, we also get that $(h\delta)(\eta(t)) \not\equiv 0 \mod t^{e+1}$, $\text{ord}_t \, \delta(\eta(t)) = e'$ and $\text{ord}_t \, \delta'(\eta(t)) \geq e'$. Hence it is enough to prove the lemma for $\pi_n^{n+1} : \pi_{n+1}(A) \to \pi_n(A)$.

We are first going to prove that the fibers of $\pi_n^{n+1} : \pi_{n+1}(A) \to \pi_n(A)$ are d-dimensional affine subspaces of $\mathbb{R}^N$. We can reorder the coordinates so that δ is the determinant of the first $N - d$ columns of $\Delta = \left(\frac{\partial f_i}{\partial x_j} \right)$. Then, similarly to the proof of [8, Lemma 4.5], there is a matrix P such that $P\Delta = (\delta I_{N-d}, W)$ and $\forall \gamma \in A$, $W(\gamma(t)) \equiv 0 \mod t^{e'}$.

Fix $\gamma \in A$. The elements of the fiber of $\pi_{n+1}(A) \to \pi_n(A)$ over $\pi_n(\gamma)$, $\gamma \in A$, are exactly the

$$\pi_{n+1} \left(\gamma(t) + t^{n+1} v(t) \right)$$

for $v \in \mathbb{R}\{t\}^d$ such that $\mathbf{f}(\gamma(t) + t^{n+1}v(t)) = 0$.

Using Taylor expansion, this last condition becomes

$$\mathbf{f}(\gamma(t)) + t^{n+1}\Delta(\gamma(t))v(t) + t^{2(n+1)}(\cdots) = 0$$

Or equivalently, since $\gamma \in A$,

$$t^{n+1}\Delta(\gamma(t))v(t) + t^{2(n+1)}(\cdots) = 0$$

Multiplying by $t^{-n-1-e'}P$, we get

$$t^{-e'}\big(\delta(\gamma(t))I_{N-d}, W(\gamma(t))\big)v(t) + t^{n+1-e'}(\cdots) = 0$$

Notice that $\mathrm{ord}_t(\delta(\gamma(t)) = e'$. Hence, by Hensel's lemma and Artin approximation theorem, the sought fiber is the set of

$$\pi_{n+1}\big(\gamma(t)\big) + t^{n+1}v_0$$

with v_0 satisfying the linear system induced by

$$t^{-e'}\big(\delta(\gamma(t))I_{N-d}, W(\gamma(t))\big)v_0 \equiv 0 \mod t$$

Let v_0 be a solution, then its first $N - d$ coefficients are expressed as linear combinations of the last d. Therefore each fiber of $\pi_n^{n+1} : \pi_{n+1}(A) \to \pi_n(A)$ is a d-dimensional affine subspace of $\mathbb{R}^N$.

By Greenberg Theorem 3.9, there is a $c \in \mathbb{N}_{\geq 0}$ such that $\pi_{cn}(A)$ is an $\mathcal{AS}$-set. Then $\pi_n(A)$ is an $\mathcal{AS}$-set as the image of $\pi_n^{cn} : \pi_{cn}(A) \to \pi_n(A)$ whose fibers have odd Euler characteristic with compact support, see [41, Theorem 4.3].

Finally, notice that $\pi_{n+1}(A) \subset \pi_n(A) \times \mathbb{R}^N$ and that $\pi_n^{n+1} : \pi_{n+1}(A) \to \pi_n(A)$ is simply the first projection. Then, according to the following lemma, $\pi_n^{n+1} : \pi_{n+1}(A) \to \pi_n(A)$ is an $\mathcal{AS}$ piecewise trivial fibration. □

Lemma 3.17 *Let A be an $\mathcal{AS}$-set, $\Omega \subset A \times \mathbb{R}^N$ be an $\mathcal{AS}$-set and $\pi : \Omega \to A$ be the natural projection.*

Assume that for all $x \in A$, the fiber $\Omega_x = \pi^{-1}(x)$ is a d-dimensional affine subspace of $\mathbb{R}^N$.

Then $\pi : \Omega \to A$ is an $\mathcal{AS}$ piecewise trivial fibration.

Proof Up to embedding the space of d-dimensional affine subspaces of $\mathbb{R}^N$ into the space of $d + 1$-dimensional vector suspaces of $\mathbb{R}^{N+1}$, we may assume that the fibers are linear subspaces.

Denote by $G = \mathbb{G}_{N,d}$ the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^N$ and let $E \to G$ be the tautological bundle; i.e. for $g \in G$, the fiber E_g is the subspace given by g.

We are first going to prove that the following set is $\mathcal{AS}$,

$$\tilde{A} = \big\{(x, g) \in A \times G, \ \Omega_x = E_g\big\}.$$

Identifying G with the set of symmetric idempotent $(N \times N)$-matrices of trace d, see [6, Proof of Theorem 3.4.4], for $i = 1, \ldots, N$ we define the regular map $w_i : G \to \mathbb{R}^N$ as the projection to the coordinates corresponding to the ith-column of such matrices. Then E_g is linearly spanned by $(w_i(g))$. Hence $L_i = \{(v, g) \in \mathbb{R}^N \times G, \ v = w_i(g)\}$ is $\mathcal{AS}$. Thus

$$\left\{(x, v, g) \in A \times \mathbb{R}^N \times G, \ v = w_i(g) \in \Omega_x\right\} = (\Omega \times G) \cap (A \times L_i)$$

is $\mathcal{AS}$ and its projection

$$X_i = \{(x, g) \in A \times G, \ w_i(g) \in \Omega_x\}$$

is also $\mathcal{AS}$ as the image of an $\mathcal{AS}$-set by an injective $\mathcal{AS}$-map, see [41, Theorem 4.5].
  Then $\tilde{A} = \bigcap_i X_i$ is $\mathcal{AS}$ as claimed.
  Let $x_0 \in A$. Fix a coordinate system on $\mathbb{R}^N$ such that

$$\Omega_{x_0} = \{x_{d+1} = \cdots = x_N = 0\}$$

and fix the projection $\Lambda : \mathbb{R}^N \to \mathbb{R}^d$ defined by $\Lambda(x_1, \ldots, x_N) = (x_1, \ldots, x_d)$. Let $\omega : \tilde{A} \to \mathbb{R}^{\binom{N}{d}}$ be such that the coordinates of $\omega(x, g)$ are the d-minors of $(\Lambda(w_i(g)))_{i=1,\ldots,N}$. Then

$$\tilde{A}_0 = \left\{(x, g) \in \tilde{A}, \ \Lambda : \Omega_x \to \mathbb{R}^d \text{ is of rank } d\right\}$$

is an $\mathcal{AS}$-set as the complement of $\omega^{-1}(0)$. Therefore

$$A_0 = \left\{x \in A, \ \Lambda : \Omega_x \to \mathbb{R}^d \text{ is of rank } d\right\}$$

is $\mathcal{AS}$ as the image of the $\mathcal{AS}$-set $\tilde{A}_0$ by the projection to the first factor which is an injective $\mathcal{AS}$-map.
  Thus $\Phi(x, v) = (x, \Lambda(v))$ is a bijection whose graph is $\mathcal{AS}$.

$$\pi^{-1}(A_0) \xrightarrow{\ \Phi\ } A_0 \times \mathbb{R}^d$$
$$\pi \searrow \qquad \swarrow \mathrm{pr}_{A_0}$$
$$A_0$$
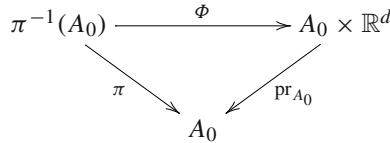
Consequently $\pi : \Omega \to A$ is locally trivial for the $\mathcal{AS}$-topology and hence it is an $\mathcal{AS}$ piecewise trivial fibration. $\qquad\square$

3.2 The Grothendieck ring of $\mathcal{AS}$-sets

Definition 3.18 Let $K_0(\mathcal{AS})$ be the free abelian group generated by $[X]$, $X \in \mathcal{AS}$, modulo

(i) Let $X, Y \in \mathcal{AS}$. If there is a bijection $X \to Y$ with $\mathcal{AS}$-graph, then

$$[X] = [Y].$$

(ii) If $Y \subset X$ are two $\mathcal{AS}$-sets, then

$$[X] = [X \backslash Y] + [Y].$$

We put a ring structure on $K_0(\mathcal{AS})$ by adding the following relation:

(iii) If $X, Y \in \mathcal{AS}$, then

$$[X \times Y] = [X][Y].$$

Notation 3.19 We set $0 = [\varnothing]$, $1 = [\mathrm{pt}]$ and $\mathbb{L} = [\mathbb{R}]$.

Remark 3.20 Notice that 0 is the unit of the addition and 1 the unit of the multiplication.

Remark 3.21 If $p : E \to B$ is an $\mathcal{AS}$ piecewise trivial fibration with fiber F, then $[E] = [B][F]$.

Definition 3.22 We set $\mathcal{M} = K_0(\mathcal{AS})\left[\mathbb{L}^{-1}\right]$.

The authors of [32] proved there exists a unique additive (and multiplicative) invariant of real algebraic varieties up to biregular morphisms which coincides with the Poincaré polynomial for compact non-singular varieties. This construction relies on the weak factorization theorem. Then Fichou [12] extended this construction to $\mathcal{AS}$-sets up to Nash isomorphisms.

Next, in [33], they gave a new construction of the virtual Poincaré polynomial, related to the weight filtration they introduced in real algebraic geometry. They proved it is an invariant of $\mathcal{AS}$-sets up to homeomorphism with $\mathcal{AS}$-graph. Actually, using the additivity, they proved it is an invariant of $\mathcal{AS}$-sets up to $\mathcal{AS}$-bijections (see [9, Remark 4.15]).

Theorem 3.23 [12,32,33] *There is a unique ring morphism*

$$\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u]$$

such that if X is compact and non-singular then

$$\beta([X]) = \sum_{i \geq 0} \dim H_i(X, \mathbb{Z}_2)u^i.$$

We say that $\beta([X])$ is the virtual Poincaré polynomial *of X.*
Moreover, if $X \neq \varnothing$, $\deg \beta(X) = \dim X$ and the leading coefficient of $\beta(X)$ is positive.

Theorem 3.24 ([13, Theorem 1.16]) *The virtual Poincaré polynomial is a ring isomorphism*

$$\beta : K_0(\mathcal{AS}) \xrightarrow{\sim} \mathbb{Z}[u].$$

Remark 3.25 The virtual Poincaré polynomial induces a ring isomorphism

$$\beta : \mathcal{M} \to \mathbb{Z}[u, u^{-1}].$$

Definition 3.26 We define the ring $\widehat{\mathcal{M}}$ as the completion of $\mathcal{M}$ with respect to the ring filtration[5] defined by the following subgroups induced by dimension

$$\mathcal{F}^m \mathcal{M} = \langle [S]\mathbb{L}^{-i}, i - \dim S \geq m \rangle$$

i.e.

$$\widehat{\mathcal{M}} = \varprojlim \mathcal{M} \Big/ \mathcal{F}^m \mathcal{M} .$$

Proposition 3.27 *The virtual Poincaré polynomial induces a ring isomorphism*

$$\beta : \widehat{\mathcal{M}} \to \mathbb{Z}[u][\![u^{-1}]\!].$$

Proof We have to prove that

$$\varprojlim_m \mathbb{Z}[u, u^{-1}] \Big/ \mathcal{F}^m \mathbb{Z}[u, u^{-1}] = \mathbb{Z}[u][\![u^{-1}]\!]$$

where $\mathcal{F}^m \mathbb{Z}[u, u^{-1}] = \langle f \in \mathbb{Z}[u, u^{-1}], \deg f \leq -m \rangle$.

For $n < m$, we define

$$\rho_{m,n} : \mathbb{Z}[u, u^{-1}] \Big/ \mathcal{F}^m \mathbb{Z}[u, u^{-1}] \to \mathbb{Z}[u, u^{-1}] \Big/ \mathcal{F}^n \mathbb{Z}[u, u^{-1}]$$

by

$$\sum_{k=-m+1}^{r} a_k u^k \mapsto \sum_{k=-n+1}^{r} a_k u^k$$

and

$$\rho_m : \mathbb{Z}[u][\![u^{-1}]\!] \to \mathbb{Z}[u, u^{-1}] \Big/ \mathcal{F}^n \mathbb{Z}[u, u^{-1}]$$

by

$$\sum_{k=-\infty}^{r} a_k u^k \mapsto \sum_{k=-m+1}^{r} a_k u^k$$

---

[5] i.e. $\mathcal{F}^{m+1} \mathcal{M} \subset \mathcal{F}^m \mathcal{M}$ and $\mathcal{F}^m \mathcal{M} \cdot \mathcal{F}^n \mathcal{M} \subset \mathcal{F}^{m+n} \mathcal{M}$. The last condition induces a ring structure on the group $\widehat{\mathcal{M}}$.

By construction,

$$\varprojlim_m \mathbb{Z}[u, u^{-1}] / \mathcal{F}^m \mathbb{Z}[u, u^{-1}]$$

$$= \left\{ (f_m) \in \prod_{m \in \mathbb{Z}} \mathbb{Z}[u, u^{-1}] / \mathcal{F}^m \mathbb{Z}[u, u^{-1}] \, , \, n < m \Rightarrow \rho_{m,n}(f_m) = f_n \right\}$$

The morphism

$$\varphi : \mathbb{Z}[u] \llbracket u^{-1} \rrbracket \to \varprojlim_m \mathbb{Z}[u, u^{-1}] / \mathcal{F}^m \mathbb{Z}[u, u^{-1}]$$

defined by $f \mapsto (\rho_m(f))_{m \in \mathbb{Z}}$ is an isomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

Definition 3.28 For $\alpha \in \mathcal{M}$, we define the virtual dimension of α by $\dim \alpha = m$ where m is the only integer such that $\alpha \in \mathcal{F}^{-m}\mathcal{M} \backslash \mathcal{F}^{-m+1}\mathcal{M}$.

Proposition 3.29 $\dim \alpha = \deg(\beta(\alpha))$

Remark 3.30 Notice that for $x \in \mathcal{M}$, $(x + \mathcal{F}^m \mathcal{M})_m$ defines a basis of open neighborhoods. This topology coincides with the one induced by the non-archimedean norm $\|\cdot\| : \mathcal{M} \to \mathbb{R}$ defined by $\|\alpha\| = e^{\dim(\alpha)}$. The completion $\widehat{\mathcal{M}}$ is exactly the topological completion with respect to this non-archimedean norm. Particularly,

- Let $(\alpha_n) \in \mathcal{M}$, then $\alpha_n \to 0$ in $\widehat{\mathcal{M}}$ if and only if $\dim(\alpha_n) \to -\infty$.
- Let $(\alpha_n) \in \mathcal{M}$, then $\sum_n \alpha_n$ converges in $\widehat{\mathcal{M}}$ if and only if $\alpha_n \to 0$ in $\widehat{\mathcal{M}}$.
- The following equality holds in $\widehat{\mathcal{M}}$:

$$(1 - \mathbb{L}^{-p}) \sum_{i=0}^{\infty} \mathbb{L}^{-pi} = 1$$

Definition 3.31 We define an order on $\widehat{\mathcal{M}}$ as follows. For $a, b \in \widehat{\mathcal{M}}$, we set $a \preceq b$ if and only if either $b = a$ or the leading coefficient of the virtual Poincaré polynomial $\beta(b - a)$ is positive.

Remark 3.32 Notice that this real setting has good algebraic properties compared to its complex counterpart:

- $K_0(\mathcal{AS})$ is an integral domain whereas $K_0(\mathrm{Var}_{\mathbb{C}})$ is not [44]. Indeed, there is no zero divisor in $K_0(\mathcal{AS})$ whereas the class of the affine line is a zero divisor of $K_0(\mathrm{Var}_{\mathbb{C}})$ [7] [31].

  Notice that in particular $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}}) \left[\mathbb{L}_{\mathbb{C}}^{-1} \right]$ is not injective.
- The natural map $\mathcal{M} \to \widehat{\mathcal{M}}$ is injective. Indeed its kernel is $\cap_m \mathcal{F}^m \mathcal{M}$ and the virtual Poincaré polynomial allows us to conclude: if $\alpha \in \cap_m \mathcal{F}^m \mathcal{M}$, then, for all $m \in \mathbb{Z}$, $\deg \alpha \leq -m$ and hence $\alpha = 0$. In the complex case, it is not known whether $\mathcal{M}_{\mathbb{C}} \to \widehat{\mathcal{M}}_{\mathbb{C}}$ is injective.

3.3 Real motivic measure

Kontsevitch introduced motivic integration in the non-singular case where the measurable sets were the cylinders by using the fact that they are stable. Still in the non-singular case, V. Batyrev [1, §6] enlarged the collection of measurable sets: a subset of the arc space is measurable if it may be approximated by stable sets.

Concerning the singular case, Denef and Loeser [10] defined a measure and a first family of measurable sets including cylinders. Then, in [11, Appendix], they used the tools they developed in the singular case to adapt the definition of Batyrev to the singular case. See also [29].

From now on we assume that X is a d-dimensional algebraic subset of $\mathbb{R}^N$.

Definition 3.33 A subset $A \subset \mathcal{L}(X)$ is said to be *stable* at level n if:

- For $m \geq n$, $\pi_m(A)$ is an $\mathcal{AS}$-subset of $\mathcal{L}_m(X)$;
- For $m \geq n$, $A = \pi_m^{-1}(\pi_m(A))$;
- For $m \geq n$, $\pi_m^{m+1} : \pi_{m+1}(A) \to \pi_m(A)$ is an $\mathcal{AS}$ piecewise trivial fibration with fiber $\mathbb{R}^d$.

Remark 3.34 Notice that, for the two first points, it is enough to verify that $\pi_n(A) \in \mathcal{AS}$ and that $A = \pi_n^{-1}(\pi_n(A))$ only for n. Indeed, then, for $m \geq n$, $\pi_m(A) = (\pi_n^m)^{-1}(\pi_n(A))$ is an $\mathcal{AS}$-set as inverse image of an $\mathcal{AS}$-set by a projection.

Then the following proposition holds (notice that the condition $A = \pi_m^{-1}(\pi_m(A))$ is quite important).

Proposition 3.35 *If A, B are stable subsets of $\mathcal{L}(X)$, then $A \cup B$, $A \cap B$ and $A \backslash B$ are stable too.*

Remark 3.36 Notice that $\mathcal{L}(X)$ may not be stable when X is singular.

Definition 3.37 For $A \subset \mathcal{L}(X)$ a stable set, we define its measure by

$$\mu(A) = \frac{[\pi_n(A)]}{\mathbb{L}^{(n+1)d}} \in \mathcal{M}, \ n \gg 1.$$

Definition 3.38 The virtual dimension of a stable set is

$$\dim(A) = \dim(\pi_n(A)) - (n+1)d, \ n \gg 1.$$

Remark 3.39 Notice that the previous definitions don't depend on n for n big enough.

Remark 3.40 Notice that $\dim(A) = \dim(\mu(A))$ where the second dimension is the one introduced in Definition 3.28.

Definition 3.41 A subset $A \subset \mathcal{L}(X)$ is measurable if, for every $m \in \mathbb{Z}_{<0}$, there exist

- a stable set $A_m \subset \mathcal{L}(X)$;
- a sequence of stable sets $(C_{m,i} \subset \mathcal{L}(X))_{i \geq 0}$

such that

- $\forall i$, $\dim C_{m,i} < m$;
- $A \triangle A_m \subset \cup C_{m,i}$

Then we define the measure of A by $\mu(A) = \lim_{m \to -\infty} \mu(A_m)$.

Proposition 3.42 *The previous limit is well defined in $\widehat{\mathcal{M}}$ and doesn't depend on the choices.*

The proof of the above Proposition, presented below, relies on the following two lemmas.

Lemma 3.43 *Let $(A_i)_{i \in \mathbb{N}_{\geq 0}}$ be a decreasing sequence of non-empty $\mathcal{AS}$-sets*

$$A_1 \supset A_2 \supset \cdots$$

Then

$$\bigcap_{i \in N} A_i \neq \varnothing.$$

Proof Recall that $\overline{A}^{\mathcal{AS}}$ denotes the smallest closed $\mathcal{AS}$-set containing A. We have the following sequence which stabilizes by noetherianity of the $\mathcal{AS}$-topology:

$$\overline{A_1}^{\mathcal{AS}} \supset \overline{A_2}^{\mathcal{AS}} \supset \cdots \supset \overline{A_k}^{\mathcal{AS}} = \overline{A_{k+1}}^{\mathcal{AS}} = \cdots$$

Recall that $\mathcal{AS}$-sets are exactly the constructible subsets of projective spaces for the $\mathcal{AS}$-topology whose closed sets are the semialgebraic arc-symmetric sets in the sense of [23]. Hence $A_l = \cup_{\text{finite}} (U_i \cap V_i)$ where U_i is $\mathcal{AS}$-open, V_i is $\mathcal{AS}$-closed and $U_i \cap V_i \neq \varnothing$. We may assume that the V_i's are irreducible (up to splitting them) and that $\overline{U_i \cap V_i}^{\mathcal{AS}} = V_i$ (up to replacing V_i by $\overline{U_i \cap V_i}^{\mathcal{AS}}$). Hence we obtain the following decomposition as a union of finitely many irreducible closed subsets $\overline{A_k}^{\mathcal{AS}} = \cup V_i$ (it is not necessarily the irreducible decomposition since we may have $V_i \subset V_j$).

Fix Z an $\mathcal{AS}$-irreducible subset of $\overline{A_k}^{\mathcal{AS}}$. By the previous discussion, for $l \geq k$, there exists U_l an open dense $\mathcal{AS}$-subset of Z such that $U_l \subset A_l$.

By [41, Remark 2.7], $\dim(Z \setminus U_l) < \dim U_l$ so that $Z \setminus U_l$ is a closed subset of Z with empty interior for the Euclidean topology. From Baire theorem, we deduce that the Euclidean interior of $\cup_{l \geq k} Z \setminus U_l$ is empty. Hence $\cap_{l \geq k} U_l$ is non-empty. □

The following lemma is an adaptation to the real context of [1, Theorem 6.6].

Lemma 3.44 *Let $A \subset \mathcal{L}(X)$ be a stable set and $(C_i)_{i \in \mathbb{N}_{\geq 0}}$ be a family of stable sets such that*

$$A \subset \bigcup_{i \in \mathbb{N}_{\geq 0}} C_i$$

Then there exists $l \in \mathbb{N}_{\geq 0}$ such that

$$A \subset \bigcup_{i=0}^{l} C_i$$

Proof Without loss of generality, we may assume that $C_i \subset A$ (up to replacing C_i by $C_i \cap A$).

Set $D_i = A \setminus (C_1 \cup \cdots \cup C_i)$ so that we get a decreasing sequence of stable sets

$$D_1 \supset D_2 \supset D_3 \supset \cdots$$

satisfying

$$\bigcap_{i \in \mathbb{N}_{\geq 0}} D_i = \varnothing$$

Assume by contradiction that A may not be covered by finitely many C_i, then

$$\forall i \in \mathbb{N}_{\geq 0}, \, D_i \neq \varnothing$$

Now assume that A is stable at level n and that D_i is stable at level $n_i \geq n$. Then $\pi_n(D_i) = \pi_n^{n_i}(\pi_{n_i}(D_i)) \in \mathcal{AS}$ as the image of an $\mathcal{AS}$-set by a regular map whose fibers have odd Euler characteristic with compact support, see [41, Theorem 4.3]. Hence, by Lemma 3.43,

$$B_n = \bigcap_{i \in \mathbb{N}_{\geq 0}} \pi_n(D_i) \neq \varnothing$$

Choose $u_n \in B_n$.

Now set

$$B_{n+1} = \bigcap_{i \in \mathbb{N}_{\geq 0}} \pi_{n+1}(D_i) \neq \varnothing$$

As before each $\pi_{n+1}(D_i)$ is a non-empty $\mathcal{AS}$-set. Notice that $(\pi_n^{n+1})^{-1}(u_n)$ is a non-empty $\mathcal{AS}$-subset of $\mathcal{L}_{n+1}(X)$.

Then, by Lemma 3.43, $B_{n+1} \cap (\pi_n^{n+1})^{-1}(u_n) \neq \varnothing$. This way, there exists $u_{n+1} \in B_{n+1}$ such that $\pi_n^{n+1}(u_{n+1}) = u_n$.

Therefore, we may inductively construct a sequence $(u_m \in \mathcal{L}_m(X))_{m \geq n}$ such that:

- $u_m \in B_m = \bigcap_{i \in \mathbb{N}_{\geq 0}} \pi_m(D_i) \neq \varnothing$;
- $\pi_m^{m+1}(u_{m+1}) = u_m$.

This defines an element $u \in \mathcal{L}(X)$ such that for all $m \geq n$, $\pi_m(u) \in B_m$. Hence for $i \in \mathbb{N}_{\geq 0}, \pi_{n_i}(u) \in B_{n_i} \subset \pi_{n_i}(D_i)$. Since D_i is stable at level n_i, $u \in \pi_{n_i}^{-1}(\pi_{n_i}(D_i)) = D_i$.

Therefore $u \in \bigcap D_i$ which is a contradiction. □

Proof of Proposition 3.42 We first prove that the limit is well defined. Let A_m, $C_{m,i}$ be as in the definition. Then for $m_1, m_2 \in \mathbb{Z}_{<0}$,

$$A_{m_1} \Delta A_{m_2} \subset \bigcup_{i \in \mathbb{N}_{\geq 0}} (C_{m_1,i} \cup C_{m_2,i})$$

By Lemma 3.44, there exists $l \in \mathbb{N}_{\geq 0}$ such that

$$A_{m_1} \Delta A_{m_2} \subset \bigcup_{i=0}^{l} (C_{m_1,i} \cup C_{m_2,i})$$

hence $\dim(A_{m_1} \Delta A_{m_2}) \leq \max(m_1, m_2)$. Thus $\mu(A_m)$ is a Cauchy sequence and its limit is well defined in the completion $\widehat{\mathcal{M}}$.

We now check that the limit doesn't depend on the choices. Let A'_m, $C'_{m,i}$ be another choice of data for the measurability of A. Fix $m \in \mathbb{Z}_{<0}$ then

$$A_m \Delta A'_m \subset \bigcup_{i \in \mathbb{N}_{\geq 0}} (C_{m,i} \cup C'_{m,i})$$

By Lemma 3.44, there exists $l \in \mathbb{N}_{\geq 0}$ such that

$$A_m \Delta A'_m \subset \bigcup_{i=0}^{l} (C_{m,i} \cup C'_{m,i})$$

Hence $\dim(A_m \Delta A'_m) < m$ and $\lim_{m \to -\infty} \mu(A_m) = \lim_{m \to -\infty} \mu(A'_m)$.   □

Proposition 3.45 *If A, B are measurable subsets of $\mathcal{L}(X)$, then $A \cup B$, $A \cap B$ and $A \backslash B$ are measurable too.*

Proof Assume that A and B are measurable, respectively with the data A_m, $C_{m,i}$ and B_m, $D_{m,i}$.

- $A \cup B$ is measurable since

$$(A \cup B) \Delta (A_m \cup B_m) \subset \bigcup (C_{m,i} \cup D_{m,i})$$

- In order to prove that $A \backslash B$ is measurable, we may use the previous point and assume that $B \subset A$ up to replacing A by $A \cup B$. Similarly, we may assume that $B_m \subset A_m$. Then

$$(A \backslash B) \Delta (A_m \backslash B_m) \subset \bigcup C_{m,i} \cup D_{m,i}$$

- Using both previous points, we obtain that

$$A \cap B = (A \cup B) \backslash (((A \cup B) \backslash A) \cup ((A \cup B) \backslash B))$$

is measurable.   □

Proposition 3.46 *The measure is additive on disjoint unions:*

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

Proof According to the previous proof we have

$$\mu(A \sqcup B) = \lim_{m \to \infty} (\mu(A_m) + \mu(B_m) - \mu(A_m \cap B_m))$$

and

$$0 = \mu(A \cap B) = \lim_{m \to \infty} \mu(A_m \cap B_m)$$

Hence

$$\mu(A \sqcup B) = \lim_{m \to \infty} \mu(A_m) + \lim_{m \to \infty} \mu(B_m) = \mu(A) + \mu(B)$$

$\square$

Proposition 3.47 *Let $(B_n)_{n \in \mathbb{N}_{\geq 0}}$ be a sequence of measurable sets with*

$$\dim B_n \to -\infty.$$

Then $B = \cup B_n$ is measurable and

$$\mu(B) = \lim_{n \to +\infty} \mu \left(\bigcup_{k \leq n} B_k \right).$$

Furthermore, if the sets B_n are pairwise disjoint, then

$$\mu(B) = \sum_{n=0}^{\infty} \mu(B_k).$$

Proof By Definition 3.41 for each $n \in \mathbb{N}_{\geq 0}$ and $m \in \mathbb{Z}_{<0}$ there are stable sets $A_{n,m}$ and $C_{n,m,i}$, $\dim C_{n,m,i} < m$ such that

$$B_n \Delta A_{n,m} \subset \bigcup_i C_{n,m,i}.$$

For $m \in \mathbb{Z}_{<0}$ choose $N \in \mathbb{N}_{\geq 0}$ such that if $n \geq N$ then $\dim B_n < m$. Note that then $\dim A_{n,m} < m$. Let us set $A_m = \bigcup_{k<N} A_{k,m}$. Then

$$\bigcup_n B_n \Delta A_m \subset \bigcup_{n,i} C_{n,m,i} \cup \bigcup_{n \geq N} A_{n,m}.$$

This shows that B is measurable. The other properties follows easily. $\square$

3.4 Measurability of the cylinders

Lemma 3.48 *Let X be a d-dimensional algebraic subset of $\mathbb{R}^N$. Let $S \subset X$ be an algebraic subset of X with $\dim S < d$. For every $e \in \mathbb{N}_{\geq 0}$, there exists $N \in \mathbb{N}_{\geq 0}$ such that*

$$\forall i, n \in \mathbb{N}_{\geq 0}, \ n \geq i \geq N \Rightarrow \dim\left(\pi_n\left(\pi_i^{-1}\left(\mathcal{L}_i(S)\right)\right)\right) \leq (n+1)d - e - 1$$

where π_n denotes the n-th truncation map for X

$$\forall n \in \mathbb{N}_{\geq 0}, \ \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

and where $\mathcal{L}(S) \subset \mathcal{L}(X)$ and $\forall i \in \mathbb{N}_{\geq 0}, \ \mathcal{L}_i(S) \subset \mathcal{L}_i(X)$.

Proof By Greenberg Theorem 3.9 applied to S, there exists $c \in \mathbb{N}_{\geq 0}$ such that

$$\pi_e\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right) = \pi_e\left(\mathcal{L}(S)\right)$$

Let $N = ce$ and $n \geq N$. By 3.8(1) applied to

$$\pi_n\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right) \to \pi_e\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right)$$

we get that

$$\dim\left(\pi_n\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right)\right) \leq \dim\left(\pi_e\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right)\right) + (n-e)d$$

But

$$\pi_e\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right) = \pi_e\left(\mathcal{L}(S)\right)$$

so that (see [8, Proposition 2.33.(i)])

$$\dim\left(\pi_n\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right)\right) \leq (e+1)(d-1) + (n-e)d = (n+1) - e - 1$$

Now if $n \geq i \geq N(= ce)$, the result derives from the inclusion

$$\pi_n\left(\pi_i^{-1}\left(\mathcal{L}_i(S)\right)\right) \subset \pi_n\left(\pi_{ce}^{-1}\left(\mathcal{L}_{ce}(S)\right)\right)$$

$\square$

Definition 3.49 Let $X \subset \mathbb{R}^N$ be an algebraic subset. For $i \in \mathbb{N}_{>0}$, we set

$$C_i(X) = \mathcal{L}^{(i)}(X) \backslash \mathcal{L}^{(i-1)}(X).$$

Remark 3.50

$$C_i(X) = \left\{ \gamma \in \mathcal{L}(X),\ \forall h \in H_X,\ \text{ord}_t\, h \circ \gamma \geq i,\ \exists \tilde{h} \in H_X,\ \text{ord}_t\, \tilde{h} \circ \gamma = i \right\}$$

Proposition 3.51 *For $i \in \mathbb{N}_{>0}$, $C_i(X)$ is stable and*

$$\lim_{i \to +\infty} \dim C_i(X) = -\infty$$

Proof Fix some $i \in \mathbb{N}_{>0}$. First, $C_i(X)$ is stable at level i since the $\mathcal{L}^{(e)}(X)$ are stable by Proposition 3.16.

Notice that $\pi_{i-1}(C_i(X)) \subset \mathcal{L}_{i-1}(X_{\text{sing}})$. Hence

$$C_i(X) \subset \pi_{i-1}^{-1}(\pi_{i-1}(C_i(X))) \subset \pi_{i-1}^{-1}\left(\mathcal{L}_{i-1}(X_{\text{sing}})\right)$$

and then

$$\pi_i(C_i(X)) \subset \pi_i\left(\pi_{i-1}^{-1}\left(\mathcal{L}_{i-1}(X_{\text{sing}})\right)\right).$$

As explained in Remark 3.14, we may apply Greenberg Theorem 3.9 to H_X so that Lemma 3.48 holds for X_{sing}.

Hence, for all $e \in \mathbb{N}_{\geq 0}$, there exists $N \in \mathbb{N}_{\geq 0}$ so that for $i \geq N$ we have

$$\dim\left(\pi_i(C_i(X))\right) - (i+1)d \leq \dim\left(\pi_i\left(\pi_{i-1}^{-1}\left(\mathcal{L}_{i-1}(X_{\text{sing}})\right)\right)\right) - (i+1)d \leq -e$$

$\square$

Corollary 3.52 *A subset $A \subset \mathcal{L}(X)$ is measurable if and only if $\forall e \gg 0$, $A \cap \mathcal{L}^{(e)}(X)$ is measurable.*

Proof By Proposition 3.16 every $\mathcal{L}^{(e)}(X)$ is stable and therefore if A is measurable so is every $A \cap \mathcal{L}^{(e)}(X)$.

Suppose now that $\forall e \geq N$, $A \cap \mathcal{L}^{(e)}(X)$ is measurable.
Then so are $A \cap C_i(X)$ for $i > N$. Hence

$$A = \left(A \cap \mathcal{L}^{(N)}(X)\right) \cup \bigcup_{i > N} \left(A \cap C_i(X)\right)$$

is measurable by Proposition 3.47. $\square$

Definition 3.53 A cylinder at level n is a subset $A \subset \mathcal{L}(X)$ of the form

$$A = \pi_n^{-1}(C)$$

for C an $\mathcal{AS}$-subset of $\mathcal{L}_n(X)$.

Remark 3.54 A cylinder at level n is a cylinder at level m for $m \geq n$. Indeed $\pi_n = \pi_n^m \circ \pi_m$ so that $\pi_n^{-1}(C) = \pi_m^{-1}\left((\pi_n^m)^{-1}(C)\right)$ where $(\pi_n^m)^{-1}(C) \in \mathcal{AS}$ as the inverse image of an $\mathcal{AS}$-set by a projection.

The following result derives from Proposition 3.16.

Proposition 3.55 *If X is non-singular, a cylinder of $\mathcal{L}(X)$ is stable.*

Proposition 3.56 *A cylinder $A \subset \mathcal{L}(X)$ is measurable and*

$$\mu(A) = \lim_{m \to +\infty} \mu\left(A \cap \mathcal{L}^{(m)}(X)\right)$$

Proof By Proposition 3.51, we may construct by induction an increasing map $\varphi : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ such that

$$i \geq \varphi(s) \Rightarrow \dim C_i(X) < -s$$

Let $m \in \mathbb{Z}_{<0}$. Set $A_m = A \cap \mathcal{L}^{(\varphi(-m))}(X)$. Then A_m is stable by Proposition 3.16 and

$$A \Delta A_m = A \backslash \mathcal{L}^{(\varphi(-m))}(X) = A \cap \pi_{\varphi(-m)}^{-1}\left(\mathcal{L}_{\varphi(-m)}(X_{\text{sing}})\right) \subset \bigcup_{i \geq \varphi(-m)} C_i(X)$$

where $C_i(X)$ is stable with $\dim C_i(X) < m$. Hence A is measurable and

$$\mu(A) = \lim_{m \to +\infty} \mu\left(A \cap \mathcal{L}^{(\varphi(m))}(X)\right)$$

The second part of the statement derives from the fact that

$$\left(\mu\left(A \cap \mathcal{L}^{(m)}(X)\right)\right)_{m \in \mathbb{N}_{>0}}$$

is already a Cauchy sequence. Assume that A is a cylinder at level s then $A \cap \mathcal{L}^{(m)}(X)$ is stable at level $\max(m, s)$. Indeed fix $k \in \mathbb{N}_{\geq 0}$. Then, for $n \geq m' \geq m \geq \max(\varphi(k), s)$, we get

$$\mu\left(A \cap \mathcal{L}^{(m)'}(X)\right) - \mu\left(A \cap \mathcal{L}^{(m)}(X)\right)$$
$$= \frac{\left[\pi_n\left(A \cap \mathcal{L}^{(m')}(X)\right)\right]}{\mathbb{L}^{-(n+1)d}} - \frac{\left[\pi_n\left(A \cap \mathcal{L}^{(m)}(X)\right)\right]}{\mathbb{L}^{-(n+1)d}}$$
$$= \frac{\left[\pi_n\left(A \cap \mathcal{L}^{(m')}(X)\right) \backslash \pi_n\left(A \cap \mathcal{L}^{(m)}(X)\right)\right]}{\mathbb{L}^{-(n+1)d}} \in \mathcal{F}^k \mathcal{M}$$

$\square$

Corollary 3.57 *For $Y \subset X$ an algebraic subset, set*

$$\mathcal{L}(X, Y) = \{\gamma \in \mathcal{L}(X), \ \gamma(0) \in Y\}$$

then

- $\mathcal{L}(X, Y)$ *is a measurable subset of $\mathcal{L}(X)$;*
- *in particular, $\mathcal{L}(X)$ is measurable.*

Proof Indeed, $\mathcal{L}(X) = \pi_0^{-1}(X)$ and $\mathcal{L}(X, Y) = \pi_0^{-1}(Y)$ are cylinders. $\quad\square$

Corollary 3.58 *If $Y \subset X$ is an algebraic subset with $\dim Y < \dim X$, then $\mathcal{L}(Y) \subset \mathcal{L}(X)$ is measurable of measure 0:*

$$\mu_{\mathcal{L}(X)}\left(\mathcal{L}(Y)\right) = 0$$

Proof Notice that $\mathcal{L}(Y)$ is a countable intersection of cylinders:

$$\mathcal{L}(Y) = \bigcap_{n \in \mathbb{N}_{\geq 0}} \pi_n^{-1}(\mathcal{L}_n(Y))$$

Then $\pi_n^{-1}(\mathcal{L}_n(Y))$ is measurable as a cylinder and

$$\dim \mu\left(\pi_n^{-1}(\mathcal{L}_n(Y))\right) \leq (n+1)(\dim Y - \dim X) \xrightarrow[n \to \infty]{} -\infty$$

$$\square$$

3.5 Motivic integral and the change of variables formula

Definition 3.59 Let $X \subset \mathbb{R}^N$ be an algebraic subset. Let $A \subset \mathcal{L}(X)$ be a measurable set. Let $\alpha : A \to \mathbb{N}_{\geq 0} \cup \{\infty\}$ be such that each fiber is measurable and $\mu(\alpha^{-1}(\infty)) = 0$. We say that $\mathbb{L}^{-\alpha}$ is integrable if the following sequence converges in $\widehat{\mathcal{M}}$:

$$\int_A \mathbb{L}^{-\alpha} d\mu = \sum_{n \geq 0} \mu\left(\alpha^{-1}(n)\right) \mathbb{L}^{-n}$$

Definition 3.60 We say that a semialgebraic map $\sigma : M \to X$ between two semialgebraic sets is *generically one-to-one* if there exists a semialgebraic set $S \subset X$ satisfying

- $\dim(S) < \dim(X)$,
- $\dim\left(\sigma^{-1}(S)\right) < \dim(M)$,
- $\forall p \in X \backslash S, \ \#\sigma^{-1}(p) = 1$.

Definition 3.61 Let $\sigma : M \to X$ be a Nash map between a d-dimensional non-singular algebraic set M and an algebraic subset $X \subset \mathbb{R}^N$. For a real analytic arc $\gamma : (\mathbb{R}, 0) \to M$, we set

$$\text{ord}_t \, \text{jac}_\sigma (\gamma(t)) = \min \{ \text{ord}_t \, \delta(\gamma(t)), \, \forall \delta \, d\text{-minor of } \text{Jac}_\sigma \},$$

where the Jacobian matrix Jac_σ is defined using a local system of coordinates around $\gamma(0)$ in M.

The following lemma is a generalization of Denef–Loeser change of variables key lemma [10, Lemma 3.4] to generically one-to-one Nash maps in the real context.

Lemma 3.62 ([8, Lemma 4.5]) *Let* $\sigma : M \to X$ *be a proper generically one-to-one Nash map where* M *is a non-singular d-dimensional algebraic subset of* $\mathbb{R}^p$ *and* X *a d-dimensional algebraic subset of* $\mathbb{R}^N$. *For* $e, e' \in \mathbb{N}_{\geq 0}$ *and* $n \in \mathbb{N}_{\geq 0}$, *set*

$$\Delta_{e,e'} = \left\{ \gamma \in \mathcal{L}(M), \, \text{ord}_t \, \text{jac}_\sigma (\gamma(t)) = e, \, \sigma_*(\gamma) \in \mathcal{L}^{(e')}(X) \right\}$$

and

$$\Delta_{e,e',n} = \pi_n \left(\Delta_{e,e'} \right),$$

where $\sigma_* : \mathcal{L}(M) \to \mathcal{L}(X)$ *is induced by* σ.
Then for $n \geq \max(2e, e')$ *the following holds:*

(i) *Given* $\gamma \in \Delta_{e,e'}$ *and* $\delta \in \mathcal{L}(X)$ *with* $\sigma_*(\gamma) \equiv \delta \mod t^{n+1}$ *there exists a unique* $\eta \in \mathcal{L}(M)$ *such that* $\sigma_*(\eta) = \delta$ *and* $\eta \equiv \gamma \mod t^{n-e+1}$.
(ii) *Let* $\gamma, \eta \in \mathcal{L}(M)$. *If* $\gamma \in \Delta_{e,e'}$ *and* $\sigma_*(\gamma) \equiv \sigma_*(\eta) \mod t^{n+1}$ *then* $\gamma \equiv \eta \mod t^{n-e+1}$ *and* $\eta \in \Delta_{e,e'}$.
(iii) *The set* $\Delta_{e,e',n}$ *is a union of fibers of* σ_{*n}.
(iv) $\sigma_{*n}(\Delta_{e,e',n})$ *is an* $\mathcal{AS}$-*set and* $\sigma_{*n|\Delta_{e,e',n}} : \Delta_{e,e',n} \to \sigma_{*n}(\Delta_{e,e',n})$ *is an* $\mathcal{AS}$ *piecewise trivial fibration with fiber* $\mathbb{R}^e$.

Lemma 3.63 *Let* $\sigma : X \to Y$ *be a Nash map between algebraic sets. If* $A \subset \mathcal{L}(Y)$ *is a cylinder then* $\sigma_*^{-1}(A) \subset \mathcal{L}(X)$ *is also a cylinder.*

Proof Assume that $A = \pi_n^{-1}(C)$ where C is an $\mathcal{AS}$-subset of $\mathcal{L}_n(Y)$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{L}(X) & \xrightarrow{\sigma_*} & \mathcal{L}(Y) \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} \\
\mathcal{L}_n(X) & \xrightarrow{\sigma_{*n}} & \mathcal{L}_n(Y)
\end{array}
$$

Notice that σ_{*n} is polynomial and thus its graph is $\mathcal{AS}$ so that the inverse image of an $\mathcal{AS}$-set by σ_{*n} is also an $\mathcal{AS}$-set. Hence $\sigma_*^{-1}(A) = \pi_n^{-1}(\sigma_{*n}^{-1}(C))$ where $\sigma_{*n}^{-1}(C)$ is $\mathcal{AS}$. □

Proposition 3.64 *Let* $\sigma : M \to X$ *be a proper generically one-to-one Nash map where* M *is a non-singular d-dimensional algebraic subset of* $\mathbb{R}^p$ *and* X *a d-dimensional algebraic subset of* $\mathbb{R}^N$.
If $A \subset \mathcal{L}(X)$ *is a measurable subset, then the inverse image* $\sigma_*^{-1}(A)$ *is also measurable.*

Proof Let

$$S' = \overline{\sigma^{-1}(X_{\text{sing}} \cup S) \cup \Sigma_\sigma}^{\text{Zar}}$$

where $S \subset X$ is as in Definition 3.60 and Σ_σ is the critical set of σ. Notice that the Zariski-closure of a semialgebraic set doesn't change its dimension. Therefore $\mathcal{L}(S')$ is a measurable subset of $\mathcal{L}(M)$ with measure 0.

Hence $\sigma_*^{-1}(A)$ is measurable if and only if $\sigma_*^{-1}(A) \backslash \mathcal{L}(S')$ is measurable and then

$$\mu\left(\sigma_*^{-1}(A)\right) = \mu\left(\sigma_*^{-1}(A) \backslash \mathcal{L}(S')\right)$$

Since A is measurable, there exists A_m and $C_{m,i}$ as in Definition 3.41. Hence for all $m \in \mathbb{Z}_{<0}$,

$$\sigma_*^{-1}(A) \Delta \sigma_*^{-1}(A_m) \subset \bigcup_i \sigma_*^{-1}(C_{m,i})$$

and

$$\left(\sigma_*^{-1}(A) \backslash \mathcal{L}(S')\right) \Delta \left(\sigma_*^{-1}(A_m) \backslash \mathcal{L}(S')\right) \subset \bigcup_i \left(\sigma_*^{-1}(C_{m,i}) \backslash \mathcal{L}(S')\right) \qquad (1)$$

By Lemma 3.63 the sets $\sigma_*^{-1}(A_m)$ and $\sigma_*^{-1}(C_{m,i})$ are cylinders, therefore they are stable sets by Proposition 3.55 since M is non-singular.

By definition of S',

$$\mathcal{L}(M) \backslash \mathcal{L}(S') \subset \bigcup_{e,e'} \Delta_{e,e'}$$

By Lemma 3.44, there exists k such that

$$\mathcal{L}(M) \backslash \mathcal{L}(S') \subset \bigcup_{e,e' \leq k} \Delta_{e,e'}$$

Thus, by Lemma 3.62, $\dim\left(\sigma_*^{-1}(C_{m,i}) \backslash \mathcal{L}(S')\right) < k + m$.

This allows one to prove that $\sigma_*^{-1}(A) \backslash \mathcal{L}(S')$ is measurable by shifting the index m in (1). $\qquad \square$

Proposition 3.65 *Let* $\sigma : M \to X$ *be a proper generically one-to-one Nash map where* M *is a non-singular d-dimensional algebraic subset of* $\mathbb{R}^p$ *and* X *a d-dimensional algebraic subset of* $\mathbb{R}^N$.
If $A \subset \mathcal{L}(M)$ *is a measurable subset, then the image* $\sigma_*(A)$ *is also measurable.*

Proof We use the same S' as in the proof of Proposition 3.64. Then $\mathcal{L}(S')$ and $\sigma_*\left(\mathcal{L}(S')\right)$ have measure 0 so that it is enough to prove that $\sigma_*\left(A\backslash\mathcal{L}(S')\right)$ is measurable.

Lemma 3.66 *There exists k such that for every stable set $B \subset \mathcal{L}(M)\backslash\mathcal{L}(S')$, $\sigma_*(B)$ is stable and $\dim\left(\sigma_*(B)\right) < \dim(B) - k$.*

Proof By definition of S' and Lemma 3.44, there exists k such that

$$B \subset \mathcal{L}(M)\backslash\mathcal{L}(S') \subset \bigcup_{e,e'\leq k} \Delta_{e,e'}$$

Then the lemma derives from Lemma 3.62.   □

Assume that A is measurable with the data $A_m, C_{m,i}$ then

$$A \Delta A_m \subset \bigcup C_{m,i}$$

so that

$$(A\backslash\mathcal{L}(S'))\Delta(A_m\backslash\mathcal{L}(S')) \subset \bigcup C_{m,i}\backslash\mathcal{L}(S')$$

and

$$
\begin{aligned}
\sigma_*&(A\backslash\mathcal{L}(S'))\Delta\sigma_*(A_m\backslash\mathcal{L}(S'))\\
&\subset \sigma_*\left((A\backslash\mathcal{L}(S'))\Delta(A_m\backslash\mathcal{L}(S'))\right)\\
&\subset \bigcup \sigma_*\left(C_{m,i}\backslash\mathcal{L}(S')\right)
\end{aligned}
$$

Then we may conclude using Lemma 3.66.   □

Theorem 3.67 *Let $\sigma : M \to X$ be a proper generically one-to-one Nash map where M is a non-singular d-dimensional algebraic subset of $\mathbb{R}^p$ and X a d-dimensional algebraic subset of $\mathbb{R}^N$.*
Let $A \subset \mathcal{L}(X)$ be a measurable set. Let $\alpha : A \to \mathbb{N}_{\geq 0} \cup \{\infty\}$ be such that $\mathbb{L}^{-\alpha}$ is integrable.
Then $\mathbb{L}^{-(\alpha\circ\sigma_*+\mathrm{ord}_t\,\mathrm{jac}_\sigma)}$ is integrable on $\sigma_*^{-1}(A)$ and

$$\int_{A\cap\mathrm{Im}(\sigma_*)} \mathbb{L}^{-\alpha}\mathrm{d}\mu_{\mathcal{L}(X)} = \int_{\sigma_*^{-1}(A)} \mathbb{L}^{-(\alpha\circ\sigma_*+\mathrm{ord}_t\,\mathrm{jac}_\sigma)}\mathrm{d}\mu_{\mathcal{L}(M)}$$

where $\sigma_* : \mathcal{L}(M) \to \mathcal{L}(X)$ is induced by σ.

Proof Set $\beta = \alpha\circ\sigma_*+\mathrm{ord}_t\,\mathrm{jac}_\sigma$. By Proposition 3.64, $\sigma_*^{-1}(A)$ and the fibers of $\alpha\circ\sigma_*$ are measurable.

Notice that

$$\beta^{-1}(n) = \bigsqcup_{e=0}^{n} \left((\alpha \circ \sigma_*)^{-1}(n-e) \cap (\mathrm{ord}_t \, \mathrm{jac}_\sigma)^{-1}(e) \cap \sigma_*^{-1}(A) \right)$$

so that the fibers of β are measurable.

As in the proof of Proposition 3.64, up to replacing $\sigma_*^{-1}(A)$ by $\sigma_*^{-1}(A) \backslash \mathcal{L}(S')$, we may assume that

$$\sigma_*^{-1}(A) \subset \bigcup_{e,e' \leq k} \Delta_{e,e'}$$

Using Lemma 3.62, we obtain

$$\int_{\sigma_*^{-1}(A)} \mathbb{L}^{-(\alpha \circ \sigma_* + \mathrm{ord}_t \, \mathrm{jac}_\sigma)} \mathrm{d}\mu_{\mathcal{L}(M)}$$

$$= \sum_{e,e' \leq k} \int_{\sigma_*^{-1}(A) \cap \Delta_{e,e'}} \mathbb{L}^{-(\alpha \circ \sigma_* + \mathrm{ord}_t \, \mathrm{jac}_\sigma)} \mathrm{d}\mu_{\mathcal{L}(M)}$$

$$= \sum_{e,e' \leq k} \sum_{n \geq e} \mu \left(\gamma \in \sigma_*^{-1}(A) \cap \Delta_{e,e'}, \, \alpha \circ \sigma_*(\gamma) = n - e \right) \mathbb{L}^{-n}$$

$$= \sum_{e,e' \leq k} \sum_{n \geq e} \mu \left(\gamma \in A \cap \sigma_*(\Delta_{e,e'}), \, \alpha(\gamma) = n - e \right) \mathbb{L}^{-(n-e)}$$

$$= \sum_{e,e' \leq k} \sum_{n \geq 0} \mu \left(\gamma \in A \cap \sigma_*(\Delta_{e,e'}), \, \alpha(\gamma) = n \right) \mathbb{L}^{-n}$$

$$= \sum_{n \geq 0} \sum_{e,e' \leq k} \mu \left(\gamma \in A \cap \sigma_*(\Delta_{e,e'}), \, \alpha(\gamma) = n \right) \mathbb{L}^{-n}$$

$$= \sum_{n \geq 0} \mu \left(\gamma \in A \cap \mathrm{Im}(\sigma_*), \, \alpha(\gamma) = n \right) \mathbb{L}^{-n}$$

$$= \int_{A \cap \mathrm{Im}(\sigma_*)} \mathbb{L}^{-\alpha} \mathrm{d}\mu_{\mathcal{L}(X)}$$

Notice that $\mathrm{Im}(\sigma_*)$ is measurable by Proposition 3.65. $\square$

4 An inverse mapping theorem for blow-Nash maps

4.1 Blow-Nash and generically arc-analytic maps

Definition 4.1 ([23, Définition 4.1]) Let X and Y be two real algebraic sets. We say that $f : X \to Y$ is arc-analytic if for every real analytic arc $\gamma : (-1, 1) \to X$ the composition $f \circ \gamma : (-1, 1) \to Y$ is also real analytic.

Definition 4.2 ([8, Definition 2.22]) Let X and Y be two algebraic sets. We say that $f : X \to Y$ is *generically arc-analytic* if there exists an algebraic subset $S \subset X$ satisfying $\dim S < \dim X$ and such that if $\gamma : (-1, 1) \to X$ is a real analytic arc not entirely included in S, then the composition $f \circ \gamma : (-1, 1) \to Y$ is also real analytic.

Definition 4.3 Let X and Y be two algebraic sets. We say that $f : X \to Y$ is *blow-Nash* if f is semialgebraic and if there exists a finite sequence of algebraic blowings-up with non-singular centers $\sigma : M \to X$ such that $f \circ \sigma : M \to Y$ is real analytic (and hence Nash).

Lemma 4.4 [8, Lemma 2.27] *Let $f : X \to Y$ be a semialgebraic map between two real algebraic sets. Then $f : X \to Y$ is blow-Nash if and only if f is generically arc-analytic.*

Remark 4.5 In the non-singular case, the previous lemma derives from [2] or [40].

Assumption 4.6 For the rest of this section we assume that $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ are two d-dimensional algebraic sets and that $f : X \to Y$ is blow-Nash. Since f is, in particular, semialgebraic, it is real analytic in the complement of an algebraic subset S of X of dimension $< d$. We may choose S sufficiently big so that S contains the singular set of X and the non-analyticity set of f. Because f is blow-Nash we may suppose, moreover, that f is analytic on every analytic arc γ not included entirely in S. Then for every $\gamma \in \mathcal{L}(X) \backslash \mathcal{L}(S)$, $f \circ \gamma \in \mathcal{L}(Y)$.

We say that such *f is generically of maximal rank* if the Jacobian matrix of f is of rank d on a dense semialgebraic subset of $X \backslash S$.

Let $\gamma \in \mathcal{L}(X) \backslash \mathcal{L}(S)$. Then the limit of tangent spaces $T_{\gamma(t)} X$ exists in the Grassmannian $\mathbb{G}_{N,d}$ of d-dimensional linear subspaces of $\mathbb{R}^N$. After a linear change of coordinates we may assume that this limit is equal to $\mathbb{R}^d \subset \mathbb{R}^N$. Then $(x_1, \ldots, x_d)$ is a local system of coordinates at every $\gamma(t)$, $t \neq 0$. Fix $J = \{j_1, \ldots, j_d)$ with $1 \leq j_1 < \cdots < j_d \leq M$. Then, for $t \neq 0$,

$$\mathrm{d} f_{j_1} \wedge \cdots \wedge \mathrm{d} f_{j_d}(\gamma(t)) = \eta_J(t) \, \mathrm{d} x_1 \wedge \cdots \wedge \mathrm{d} x_d,$$

where $\eta_J(t)$ is a semialgebraic function, well-defined for $t \neq 0$. Indeed, let $\Gamma_f \subset \mathbb{R}^{N+M}$ denote the graph of f and let $\tau_{\Gamma_f} : \mathrm{Reg}(\Gamma_f) \to \mathbb{G}_{N+M,d}$ be the Gauss map. It is semialgebraic, see e.g. [6, Proposition 3.4.7], [24]. Denote by $\widetilde{\Gamma}_f$ the closure of its image and by $\pi_f : \widetilde{\Gamma}_f \to \Gamma_f$ the induced projection. Then γ lifts to a semialgebraic arc $\overline{\gamma}$ in $\widetilde{\Gamma}_f$. The limits $\lim_{t \to 0^+} \overline{\gamma}(t)$ and $\lim_{t \to 0^-} \overline{\gamma}(t)$ exist, and as follows from Proposition 4.10 they coincide.

Denote by $E \to \mathbb{G}_{N+M,d}$ the tautological bundle. Thus each fiber of $E \to \mathbb{G}_{N+M,d}$ is a d-dimensional vector subspace of $\mathbb{R}^{N+M}$. We denote by $(x_1, \ldots, x_N, f_1, \ldots f_M)$ the linear coordinates in $\mathbb{R}^{N+M}$. Then the restriction of alternating d-forms to each $V^d \in \mathbb{G}_{N+M,d}$ gives an identity

$$\mathrm{d} f_{j_1} \wedge \cdots \wedge \mathrm{d} f_{j_d} = \eta_J(V^d) \, \mathrm{d} x_1 \wedge \cdots \wedge \mathrm{d} x_d$$

that defines a semialgebraic function $\eta_J(V_d)$ on $\mathbb{G}_{N+M,d}$ with values in $\mathbb{R} \cup \{\pm\infty\}$. Then $\eta_J(t) = \eta_J(\overline{\gamma}(t))$. As follows from Proposition 4.10, $\eta_J(t)$ is meromorphic and $\text{ord}_t \, \eta_J \in \mathbb{Z} \cup \{\infty\}$.

The following notion generalizes the order defined in Definition 3.61.

Definition 4.7 *The order of the Jacobian determinant of f along γ is defined as*

$$\text{ord}_t \, \text{jac}_f(\gamma) = \min_J \{\text{ord}_t \, \eta_J(t)\}.$$

If $\eta(t) \equiv 0$ then we define its order as $+\infty$.

Definition 4.8 We say that *the Jacobian determinant of f is bounded from above (resp. below)* if there exists $S \subset X$ as in 4.6 such that

$$\forall \gamma \in \mathcal{L}(X) \backslash \mathcal{L}(S), \ \text{ord}_t \, \text{jac}_f(\gamma) \geq 0 \ (\text{resp. } \text{ord}_t \, \text{jac}_f(\gamma) \leq 0).$$

4.2 Resolution diagram of *f*

Let $g : M \to X$ be a Nash map where M is a non-singular algebraic set and X is an algebraic subset of $\mathbb{R}^N$. Denote by $\mathcal{O}_M$ the sheaf of Nash functions on M. Assume that $\dim M = \dim X = d$. Then the Jacobian sheaf $\mathcal{J}_g$ of g is the sheaf of $\mathcal{O}_M$-ideals generated, in a local system of coordinates $z_1, \ldots, z_d$ on M, by

$$\mathcal{J}_g = \left\langle \frac{\partial(g_{i_1}, \ldots, g_{i_d})}{\partial(z_1, \ldots, z_d)}, \ 1 \leq i_1 < \cdots < i_d \leq N \right\rangle.$$

Let $D = \cup D_i \subset M$ be a divisor with normal crossings. We say that a local system of coordinates $z_1, \ldots, z_d$ at $p \in M$ is compatible with D if D at p is the zero set of a monomial in z_i or $p \notin D$.

Proposition 4.9 *Let $g : M \to X$ be as in the previous definition. Then there exists $\sigma : \tilde{M} \to M$ the composition of a sequence of blowings-up with smooth algebraic centers and an algebraic divisor with simple normal crossings $D = \cup D_i \subset \tilde{M}$ such that in any local Nash system of coordinates compatible with D, $\mathcal{J}_{g \circ \sigma}$ is generated by a monomial.*

Proof First we fix a regular (in the algebraic sense) differential form ω_M of degree d on M which is not identically zero on every component of M.

There exists a sequence of blowings-up whose Jacobian determinant is a normal crossing divisor and such that the compositions with the coefficients of ω_M are also normal crossings, see for instance [3, Theorem 1.10]. Then the zero set of the pullback of ω_M is a divisor with simple normal crossings.

Up to composing with blowings-up, this allows us to assume that the zero set of ω_M, denoted by $Z(\omega_M)$, is a divisor with simple normal crossings.

Since M, and hence $Z(\omega_M)$, is affine there is a regular function φ on M such that $Z(\omega_M) \subset \text{div} \, \varphi$. By performing additional blowings-up we may assume that $\text{div}(\varphi)$ is a divisor with normal crossings.

For $I = \{i_1, \ldots, i_d\} \subset \{1, \ldots, N\}$, let $\pi_I : X \to \mathbb{R}^d$ be defined by

$$\pi_I(x_1, \ldots, x_N) = (x_{i_1}, \ldots, x_{i_d}).$$

We consider the algebraic differential form $\omega_I = \pi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_d})$. Then

$$\varphi g^* \omega_I = h_I \omega_M,$$

where h_I is a Nash function on M. By [9, Proposition 2.11], we may find a finite composition of blowings-up $\sigma : \tilde{M} \to M$, with smooth algebraic centers, such that $h_I \circ \sigma$ is locally a monomial times a Nash unit. More precisely, let $D \subset \tilde{M}$ be the union of $\sigma^{-1}(\mathrm{div}\,\varphi)$ and the exceptional divisor of σ. We may suppose that D is with simple normal crossings and hence $h_I \circ \sigma$ equals a monomial times a Nash unit, in any local system of coordinates compatible with D.

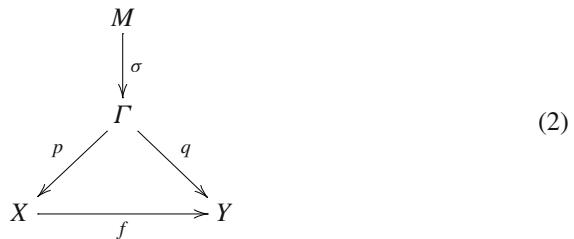Let $z_1, \ldots, z_d$ be such a local system of coordinates and let $\tilde{g} = g \circ \sigma$. Then

$$\tilde{g}^* \omega_I = \frac{\partial(\tilde{g}_{i_1}, \ldots, \tilde{g}_{i_d})}{\partial(z_1, \ldots, z_d)} dz = \varphi^{-1} h_I \sigma^* \omega_M = z^{\alpha_I} u(z) dz,$$

where u is a unit.

We may apply the above procedure to all ω_I and their differences. Then, by [49, Beginning of the proof of Proposition 2.1], see also [2, Lemma 6.5], we conclude that the ideal generated by such $\tilde{g}^* \omega_I$ is, locally, principal and generated by a monomial. □

Let $p : \Gamma \to X$ be a composition of finitely many algebraic blowings-up such that $q = f \circ p : \Gamma \to Y$ is Nash and $\sigma : M \to \Gamma$ be an algebraic resolution of Γ such that $\mathcal{J}_{p \circ \sigma}$ (resp. $\mathcal{J}_{q \circ \sigma}$) is locally generated by a monomial. Notice that M is a non-singular real algebraic variety and that $f \circ p \circ \sigma$ is Nash. Note that if M is not connected then $\mathcal{J}_{p \circ \sigma}$ can vanish identically on a connected component of M if and only if f is not generically of maximal rank.

We call $p : \Gamma \to X$ and $\sigma : M \to \Gamma$ satisfying the above properties *a resolution diagram of* f. By Hironaka's desingularisation theorem [17,18] and Proposition 4.9, such a diagram always exists but is not unique.

$$
\begin{array}{ccc}
& M & \\
& \downarrow{\scriptstyle\sigma} & \\
& \Gamma & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\
X \xrightarrow{\quad f \quad} & & Y
\end{array}
\tag{2}
$$

By choosing the algebraic subset $S \subset X$ bigger (but still with $\dim S < d$) we may assume that $(p \circ \sigma)_*$ induces a bijection $\mathcal{L}(M) \backslash \mathcal{L}(S') \to \mathcal{L}(X) \backslash \mathcal{L}(S)$, where

$S' = (p \circ \sigma)^{-1}(S)$. Note that dim $S' < d$. Thus the diagram (2) induces a diagram

$$
\begin{array}{ccc}
& \mathcal{L}(M) \backslash \mathcal{L}(S') & \\
{\scriptstyle (p \circ \sigma)_*} \swarrow & & \searrow {\scriptstyle (q \circ \sigma)_*} \\
\mathcal{L}(X) \backslash \mathcal{L}(S) \xrightarrow{\hspace{2cm} f_* \hspace{2cm}} & & \mathcal{L}(Y)
\end{array}
$$

where we denote $f_* = (q \circ \sigma)_* \circ (p \circ \sigma)_*^{-1}$.

Now we show how to compute the order of the Jacobian determinant of f along γ using a resolution diagram.

Proposition 4.10 *Let $\gamma \in \mathcal{L}(X) \backslash \mathcal{L}(S)$ and let $\tilde{\gamma} = (p \circ \sigma)_*^{-1}(\gamma)$. Then*

$$
\mathrm{ord}_t \, \mathrm{jac}_f(\gamma) = \mathrm{ord}_t \, \mathrm{jac}_{q \circ \sigma}(\tilde{\gamma}(t)) - \mathrm{ord}_t \, \mathrm{jac}_{p \circ \sigma}(\tilde{\gamma}(t)). \tag{3}
$$

Proof The result derives from the chain rule which holds outside S. $\qquad\square$

Corollary 4.11 *Suppose that f is generically of maximal rank. Then the Jacobian determinant of f is bounded from above, resp. from below, if and only if at every point of M a local generator of $\mathcal{J}_{p \circ \sigma}$ divides a local generator of $\mathcal{J}_{q \circ \sigma}$, resp. a local generator of $\mathcal{J}_{q \circ \sigma}$ divides a local generator of $\mathcal{J}_{p \circ \sigma}$.*

Remark 4.12 We deduce from the previous corollary that if one of the conditions of Definition 4.8 is satisfied for one S, then it holds for every S.

4.3 An inverse mapping theorem

Theorem 4.13 *Let $f : (X, x) \to (Y, y)$ be a germ of semialgebraic homeomorphism between real algebraic sets.*

Assume that $\mu_{\mathcal{L}(X)}(\mathcal{L}(X, x)) = \mu_{\mathcal{L}(Y)}(\mathcal{L}(Y, y))$.

If f is generically arc-analytic and if the Jacobian determinant of f is bounded from below, then the inverse map $f^{-1} : Y \to X$ is also generically arc-analytic and the Jacobian of f is bounded from above.

Remark 4.14 Notice that arc-analyticity is an open condition for semialgebraic continuous maps (see [28, Theorem 3.1] where it is not necessary to assume that f is bounded, up to composing f with a real analytic diffeomorphism $\varphi : \mathbb{R} \to (-1, 1)$). Hence, since the above statement is local, it is enough to use real analytic arcs centered at x for the arc-analyticity condition.

The same holds for the boundedness of the Jacobian of f: we assume that the arcs of Definition 4.8 or Corollary 4.11 are centered at x.

Proof of Theorem 4.13 We have the commutative diagram (2). Notice that $E = (p \circ \sigma)^{-1}(0)$ is algebraic since $p \circ \sigma$ is regular. By Theorem 3.67,

$$\mu_{\mathcal{L}(X)}\left((p \circ \sigma)_*(\mathcal{L}(M, E))\right) = \int_{(p \circ \sigma)_*(\mathcal{L}(M, E))} \mathbb{L}^{-0} d\mu_{\mathcal{L}(X)}$$

$$= \int_{\mathcal{L}(M, E)} \mathbb{L}^{-\operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}} d\mu_{\mathcal{L}(M)}$$

$$= \sum_{n \geq 0} \mu_{\mathcal{L}(M)}\left(\mathcal{L}(M, E) \cap \left(\operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}\right)^{-1}(n)\right) \mathbb{L}^{-n}$$

Thus

$$\mu_{\mathcal{L}(X)}\left((p \circ \sigma)_*(\mathcal{L}(M, E))\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{i \geq 0} \sum_{n \geq 0} \mu_{\mathcal{L}(M)}\left(\mathcal{L}(M, E) \cap \left(\operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}\right)^{-1}(n)\right) \mathbb{L}^{-(i+n)}$$

$$= \sum_{n \geq 0} \mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}(\gamma(t)) \leq n\right) \mathbb{L}^{-n}$$

Similarly

$$\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{q \circ \sigma}(\gamma(t)) \leq n\right) \mathbb{L}^{-n}$$

Hence

$$\left(\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) - \mu_{\mathcal{L}(X)}\left((p \circ \sigma)_*(\mathcal{L}(M, E))\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \left(\mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{q \circ \sigma}(\gamma(t)) \leq n\right)\right.$$

$$\left. - \mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}(\gamma(t)) \leq n\right)\right) \mathbb{L}^{-n}$$

Since we may lift a real analytic arc non-entirely included in the exceptional locus by $p \circ \sigma$, we have

$$\mu_{\mathcal{L}(X)}\left((p \circ \sigma)_*(\mathcal{L}(M, E))\right) = \mu_{\mathcal{L}(X)}\left(\mathcal{L}(X, x)\right)$$

so that

$$\left(\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) - \mu_{\mathcal{L}(X)}\left(\mathcal{L}(X, x)\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \left(\mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{q \circ \sigma}(\gamma(t)) \leq n\right)\right.$$

$$\left. - \mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}(\gamma(t)) \leq n\right)\right)\mathbb{L}^{-n}$$

Since $\mu_{\mathcal{L}(Y)}(\mathcal{L}(Y, y)) = \mu_{\mathcal{L}(X)}(\mathcal{L}(X, x))$, we obtain

$$\left(\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) - \mu_{\mathcal{L}(Y)}\left(\mathcal{L}(Y, y)\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \left(\mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{q \circ \sigma}(\gamma(t)) \leq n\right)\right.$$

$$\left. - \mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}(\gamma(t)) \leq n\right)\right)\mathbb{L}^{-n}$$

Since M is non-singular, we may simply write

$$\left(\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) - \mu_{\mathcal{L}(Y)}\left(\mathcal{L}(Y, y)\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \left(\left[\gamma \in \mathcal{L}_n(M, E), \operatorname{ord}_t \operatorname{jac}_{q \circ \sigma}(\gamma(t)) \leq n\right]\right.$$

$$\left. - \left[\gamma \in \mathcal{L}_n(M, E), \operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}(\gamma(t)) \leq n\right]\right)\mathbb{L}^{-(n+2)d}$$

Since the Jacobian determinant f is bounded from below, using Proposition 4.10, we get that each summand of the RHS is positive or zero (in the sense of Definition 3.31) because the leading coefficient of the virtual Poincaré polynomial of a non-empty $\mathcal{AS}$-set is positive:

$$\left(\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) - \mu_{\mathcal{L}(Y)}\left(\mathcal{L}(Y, y)\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \left(\left[\{\gamma \in \mathcal{L}_n(M, E), \operatorname{ord}_t \operatorname{jac}_{q \circ \sigma}(\gamma(t)) \leq n\}\right.\right.$$

$$\left.\left. \setminus \{\gamma \in \mathcal{L}_n(M, E), \operatorname{ord}_t \operatorname{jac}_{p \circ \sigma}(\gamma(t)) \leq n\}\right]\right)\mathbb{L}^{-(n+2)d}$$

Moreover, the LHS is negative or zero since $(q \circ \sigma)_*(\mathcal{L}(M, E)) \subset \mathcal{L}(Y, y)$.

Assume that f is not bounded from above, then at least one of the summand of the RHS is positive so that we obtain a contradiction. This proves that f is bounded from above.

Furthermore, since the RHS is zero, we obtain that

$$\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) = \mu_{\mathcal{L}(Y)}\left(\mathcal{L}(Y, y)\right) \tag{4}$$

We are now going to prove that f^{-1} is generically arc-analytic so that it is blow-Nash.

Assume by contradiction there exists $\gamma \in \mathcal{L}(Y, y)$ not entirely included in $f(S) \cup Y_{\text{sing}}$ which may not be lifted by $q \circ \sigma$. Nevertheless, by [8, Proposition 2.21],

$$(q \circ \sigma)^{-1}(\gamma(t)) = \sum_{i \geq 0} c_i t^{\frac{i}{a}}, \; t \geq 0$$

and

$$(q \circ \sigma)^{-1}(\gamma(t)) = \sum_{i \geq 0} d_i (-t)^{\frac{i}{b}}, \; t \leq 0.$$

By assumption $(q \circ \sigma)^{-1}(\gamma(t))$ is not analytic so that either these expansions don't coincide or they have a non-integer exponent.

1. We first treat the latter case. Assume that

$$(q \circ \sigma)^{-1}(\gamma(t)) = \sum_{i=0}^{m} c_i t^i + c t^{\frac{a}{b}} + \cdots, \; t \geq 0, \; m < \frac{a}{b} < m+1, \; c \neq 0.$$

Since $(q \circ \sigma)^{-1} : Y \backslash f(S) \to M$ is continuous and subanalytic, it is locally Hölder so that there exists $N \in \mathbb{N}_{\geq 0}$ satisfying for all real analytic arc $\eta(t)$ not entirely included in $f(S) \cup Y_{\text{sing}}$,

$$\eta(t) \equiv \gamma(t) \mod t^N \Rightarrow (q \circ \sigma)^{-1}(\eta(t)) \equiv (q \circ \sigma)^{-1}(\gamma(t)) \mod t^{m+1}.$$

Thus $\pi_N^{-1}(\pi_N(\gamma)) \subset \mathcal{L}(Y, y) \backslash (q \circ \sigma)_*(\mathcal{L}(M, E))$.

Notice that $\pi_N^{-1}(\pi_N(\gamma))$ is measurable as a cylinder. Let $\rho : \tilde{Y} \to Y$ be a resolution of Y. Since γ is not entirely included in the singular set of Y, there exists a unique real analytic arc $\tilde{\gamma}$ on M such that $\gamma = \rho \circ \tilde{\gamma}$. Let $e = \text{ord}_t \, \text{jac}_\rho(\tilde{\gamma}(t))$ and e' be such that $\gamma \in \mathcal{L}^{(e')}(Y)$. We may assume that $N \geq \max(e', 2e)$. Then, by Lemma 3.62 and since $\tilde{Y}$ is non-singular,

$$\mu_{L(Y)}\left(\pi_N^{-1}(\pi_N(\gamma))\right) = \mu_{\mathcal{L}(\tilde{Y})}\left(\pi_N^{-1}(\pi_N(\tilde{\gamma}))\right) \mathbb{L}^{-e}$$
$$= \left[\pi_N\left(\pi_N^{-1}(\pi_N(\tilde{\gamma}))\right)\right] \mathbb{L}^{-(N+1)d-e}$$
$$\neq 0$$

Since $\pi_N^{-1}(\pi_N(\gamma)) \subset \mathcal{L}(Y, y)\backslash(q \circ \sigma)_*(\mathcal{L}(M, E))$, we obtain that

$$\mu\left(\mathcal{L}(Y, y)\backslash(q \circ \sigma)_*(\mathcal{L}(M, E))\right) \neq 0$$

which contradicts (4).

2. We now assume that

$$\tilde{\gamma}^+(t) = (q \circ \sigma)^{-1}(\gamma(t)) = \sum_{i=0}^{m-1} c_i t^i + ct^m + \cdots, \ t \geq 0$$

and

$$\tilde{\gamma}^-(t) = (q \circ \sigma)^{-1}(\gamma(t)) = \sum_{i=0}^{m-1} c_i t^i + dt^m + \cdots, \ t \leq 0$$

with $c \neq d$.

Notice that $(q \circ \sigma)(\gamma^{\pm}(t))$ are analytic so that $\gamma(t) = (f \circ q \circ \sigma)(\gamma^+(t)) = (f \circ q \circ \sigma)(\gamma^-(t))$. Since f is a homeomorphism, we get $(q \circ \sigma)(\gamma^+(t)) = (q \circ \sigma)(\gamma^-(t))$. Since this real analytic arc is not entirely included in S, it may be uniquely lifted by $q \circ \sigma$ so that $\gamma^+(t) = \gamma^-(t)$. Hence $c = d$ and we obtain a contradiction.

Thus, for all $\gamma \in \mathcal{L}(Y, y)\backslash\mathcal{L}(f(S) \cup Y_{\text{sing}})$ there exists $\tilde{\gamma} \in \mathcal{L}(M, E)$ such that $(q \circ \sigma)(\tilde{\gamma}(t)) = \gamma(t)$. Then $f^{-1}(\gamma(t)) = (p \circ \sigma)(\tilde{\gamma}(t))$ which is real analytic. Therefore f^{-1} is generically arc-analytic and so blow-Nash. □

Remark 4.15 Notice that, in the above proof, we do not need a homeomorphism $f : X \to Y$ but only a homeomorphism of $f : \overline{\text{Reg}(X)} \to \overline{\text{Reg}(Y)}$.

Under the assumptions of the previous theorem, we derive the following corollary from Lemma 4.4.

Corollary 4.16 *Let* $f : (X, x) \to (Y, y)$ *be a semialgebraic homeomorphism germ between real algebraic sets with* $\dim X = \dim Y$. *Assume moreover that* $\mu_{\mathcal{L}(X)}(\mathcal{L}(X, x)) = \mu_{\mathcal{L}(Y)}(\mathcal{L}(Y, y))$.
If f *is blow-Nash and if the Jacobian determinant of* f *is bounded from below, then the inverse* f^{-1} *is also blow-Nash and the Jacobian determinant of* f *is bounded from above.*

Remark 4.17 Notice that in the previous results we do not assume that $X = Y$ contrary to [8, Main Theorem 3.5].

Theorem 4.18 *Let* $f : (X, x) \to (Y, y)$ *be a semialgebraic homeomorphism germ between algebraic sets with* $\dim X = \dim Y$. *If* f *is generically arc-analytic and if the Jacobian determinant of* f *is bounded from below, then* $\mu_{\mathcal{L}(X)}(\mathcal{L}(X, x)) \preceq \mu_{\mathcal{L}(Y)}(\mathcal{L}(Y, y))$.

Proof Following the beginning of the proof of Theorem 4.13, we obtain :

$$\left(\mu_{\mathcal{L}(Y)}\left(\mathcal{L}(Y, y)\right) - \mu_{\mathcal{L}(X)}\left(\mathcal{L}(X, x)\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$\succeq \left(\mu_{\mathcal{L}(Y)}\left((q \circ \sigma)_*(\mathcal{L}(M, E))\right) - \mu_{\mathcal{L}(X)}\left(\mathcal{L}(X, x)\right)\right) \sum_{i \geq 0} \mathbb{L}^{-i}$$

$$= \sum_{n \geq 0} \left(\mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \text{ ord}_t \text{ jac}_{q \circ \sigma}(\gamma(t)) \leq n\right) \right.$$

$$\left. - \mu_{\mathcal{L}(M)}\left(\gamma \in \mathcal{L}(M, E), \text{ ord}_t \text{ jac}_{p \circ \sigma}(\gamma(t)) \leq n\right) \right) \mathbb{L}^{-n}$$

$$\succeq 0$$

$\square$

5 An inverse mapping theorem for inner-Lipschitz maps

5.1 Inner distance

Let X be a connected semialgebraic subset of $\mathbb{R}^N$ equipped with the standard Euclidean distance. We denote by d_X the *inner* (also called *geodesic*) distance in X. By definition, for $p, q \in X$, the inner distance $d_X(p, q)$ is the infimum over the length of all rectifiable curves joining p to q in X. By [25], $d_X(p, q)$ is the infimum over the length of continuous semialgebraic curves in X joining p and q. It is proven in [25] that d_X can be approximated uniformly by subanalytic distances.

We recall some results from [25], based on [24]. Let $\varepsilon > 0$, we say that a connected semialgebraic set $\Gamma \subset \mathbb{R}^N$ is K_ε-*regular* if for any $p, q \in \Gamma$ we have

$$d_\Gamma(p, q) \leq (1 + \varepsilon)|p - q|.$$

We state now a semialgebraic version of [25, Proposition 3].

Proposition 5.1 *Let $X \subset \mathbb{R}^N$ be a semialgebraic set and $\varepsilon > 0$. Then there exists a finite decomposition $X = \bigcup_{v \in V} \Gamma_v$ such that:*

1. *each Γ_v is a semialgebraic connected analytic submanifold of $\mathbb{R}^N$,*
2. *each Γ_v is K_ε-regular.*

Remark 5.2 Given a finite family of semialgebraic sets X_i, $i \in I$, we can find a decomposition satisfying the above conditions and such that for any $i \in I$, $v \in V$, we have: either $\Gamma_v \subset X_i$ or $\Gamma_v \cap X_i = \emptyset$.

For a C^1 map $f : X' \to \mathbb{R}^M$ defined on a submanifold X' of $\mathbb{R}^N$ we denote by $D_p f : T_p X \to \mathbb{R}^M$ its differential at $p \in X'$. Then the norm of $D_p f$ is defined by

$$\|D_p f\| = \sup \left\{ |D_p f(v)| : v \in T_p, |v| = 1 \right\}.$$

Lemma 5.3 *Assume that $f_\nu : \Gamma_\nu \to \mathbb{R}^M$ is a C^1-map, such that for any $p \in \Gamma_\nu$ we have $\|D_p f_\nu\| \leq L$. Then f_ν is $(1 + \varepsilon)L$-Lipschitz with respect to the Euclidean distance, hence it extends continuously on $\overline{\Gamma}_\nu$ to a Lipschitz map with the same constant.*

Proof Let $p, q \in \Gamma_\nu$ and $\varepsilon' > \varepsilon$, then, by [25], there exists a C^1-semialgebraic arc $\lambda : [0, 1] \to \Gamma_\nu$ such that $p = \lambda(0), q = \lambda(1)$ of the length $|\lambda| \leq (1 + \varepsilon')|p - q|$. It follows that

$$|f_\nu(p) - f_\nu(q)| \leq L|\lambda| \leq (1 + \varepsilon')L|p - q|.$$

We obtain the conclusion passing to the limit $\varepsilon' \to \varepsilon$.

Notice that, on any metric space, a Lipschitz mapping extends continuously to the closure with the same Lipschitz constant. $\qquad\square$

Let X and Y be locally closed connected semialgebraic subsets respectively of $\mathbb{R}^N$ and $\mathbb{R}^M$. They are equipped with the inner distances d_X and d_Y, respectively. Let

$$f : X \to Y$$

be a continuous semialgebraic map. Then there exists a semialgebraic set $X' \subset X$, which is open and dense in X, such that the connected components of X' are analytic submanifolds of $\mathbb{R}^N$, possibly of different dimensions. Moreover f restricted to each connected component of X' is analytic.

Proposition 5.4 *The following conditions are equivalent:*

(i) $d_Y(f(p), f(q)) \leq Ld_X(p, q)$ *for any* $p, q \in X$,
(ii) $\|D_p f\| \leq L$ *for any* $p \in X'$.

Proof The implication 5.4(i) $\Rightarrow$ 5.4 (ii) is obvious since at a smooth point $p \in X$, the inner and Euclidean distances are asymptotically equal.

To prove the converse let us fix $p, q \in X$. For any $\varepsilon > 0$ there exists a continuous semialgebraic arc $\lambda : [0, 1] \to X$ such that $p = \lambda(0), q = \lambda(1)$ of the length $|\lambda| \leq (1 + \varepsilon)d_X(p, q)$. By Proposition 5.1 there exists a finite decomposition $X' = \bigcup_{\nu \in V} \Gamma_\nu$ into K_ε-regular semialgebraic connected analytic submanifolds of $\mathbb{R}^N$. Let $X'' = \bigcup_{\nu \in V'} \Gamma_\nu$ be the union of those Γ_ν which are open in X'. Note that X'' is dense in X'. It follows that $X \subset \bigcup_{\nu \in V'} \overline{\Gamma}_\nu$. Since the arc λ is semialgebraic there exists a finite sequence $0 = t_0 < \cdots < t_k = 1$ such that each $\lambda([t_i, t_{i+1}]) \subset \overline{\Gamma}_\nu$ for some $\nu \in V'$. By Lemma 5.3 the length of $f(\lambda([t_i, t_{i+1}]))$ is bounded by $(1 + \varepsilon)|\lambda([t_i, t_{i+1}])|$. Hence

$$|f(\lambda([0, 1]))| = \sum_{i=0}^{k-1} |f(\lambda([t_i, t_{i+1}])| \leq (1 + \varepsilon)L \sum_{i=0}^{k-1} |\lambda([t_i, t_{i+1}])| \leq (1 + \varepsilon)L|\lambda|.$$

Thus

$$d_Y(f(p), f(q)) \leq |f(\lambda([0, 1]))| \leq (1 + \varepsilon)L|\lambda| \leq (1 + \varepsilon)^2 Ld_X(p, q)$$

We conclude by taking the limit as $\varepsilon \to 0$. $\qquad\square$

5.2 An inverse mapping theorem

We suppose now that $f : X \to Y$ satisfies Assumption 4.6. Thus it is a blow-Nash map between two real algebraic sets of dimension d.

Let $\gamma \in \mathcal{L}(X)\backslash\mathcal{L}(S)$. Let us adapt the notation introduced in the paragraph after Assumption 4.6. In particular we assume that the limit of tangent spaces $T_{\gamma(t)}X$ in the Grassmannian $\mathbb{G}_{N,d}$ is equal to $\mathbb{R}^d \subset \mathbb{R}^N$. Then, for every $i = 1, \ldots, d$ and every $j = 1, \ldots, M$

$$\eta_{i,j}(t) = \frac{\partial f_j}{\partial x_i}$$

is semialgebraic. Thus the order of $\eta_{i,j}(t)$, as $t \to 0^+$ is a well defined rational number (or $+\infty$ if f_j vanishes identically on γ).

Definition 5.5 *The order of the Jacobian matrix of f along γ is defined as*

$$\mathrm{ord}_{t\to 0^+} \mathrm{Jac}_f(\gamma(t)) = \min_{i,j}\{\mathrm{ord}_{t\to 0^+} \eta_{i,j}(t)\}.$$

Remark 5.6 The above notion should not be confused with the order of the Jacobian *determinant* defined in Definition 4.7.

Remark 5.7 It is likely that $\eta_{i,j}(t)$ is actually meromorphic and it is not necessary, in the above definition, to restrict to $t \to 0^+$. We leave it as an open problem.

Definition 5.8 We say that *the Jacobian matrix of f is bounded from above* if there is an S such that for every $\gamma \in \mathcal{L}(X)\backslash\mathcal{L}(S)$, $\mathrm{ord}_{t\to 0^+} \mathrm{Jac}_f(\gamma(t)) \geq 0$.

One may show again that if the above condition is satisfied for one S they are satisfied for every S.

The following result follows from Proposition 5.4.

Proposition 5.9 *Let $f : (X, x) \to (Y, y)$ be a semialgebraic homeomorphism germ between two real algebraic set germs with $\dim(X, x) = \dim(Y, y)$. Then $f : \overline{\mathrm{Reg}(X)} \to \overline{\mathrm{Reg}(Y)}$ is inner Lipschitz if and only if the Jacobian matrix of f is bounded from above.*

Theorem 5.10 *Let $f : (X, x) \to (Y, y)$ be a semialgebraic homeomorphism germ between two real algebraic set germs with $\dim(X, x) = \dim(Y, y)$. Assume that $\mu_{\mathcal{L}(X)}(\mathcal{L}(X, x)) = \mu_{\mathcal{L}(Y)}(\mathcal{L}(Y, y))$. If f is generically arc-analytic and $f^{-1} : \overline{\mathrm{Reg}(Y)} \to \overline{\mathrm{Reg}(X)}$ is inner Lipschitz, then $f^{-1} : Y \to X$ is also generically arc-analytic and $f : \overline{\mathrm{Reg}(X)} \to \overline{\mathrm{Reg}(Y)}$ is inner Lipschitz.*

Remark 5.11 Notice that both previous results involve the closure of the regular parts of the algebraic sets. The obtained sets $\overline{\mathrm{Reg}(X)}$ and $\overline{\mathrm{Reg}(Y)}$ do not contain any part of smaller dimension but they still may not be smooth submanifolds.
For instance, for the Whitney umbrella $X = \{x^2 = zy^2\}$, $\overline{\mathrm{Reg}(X)}$ consists in the canopy (i.e. the $z \geq 0$ part of X). Therefore $\overline{\mathrm{Reg}(X)}$ is singular along the half-axis

$\{(0, 0, z), \ z \geq 0\}$. However it doesn't contain the handle of the Whitney umbrella (i.e. $\{(0, 0, z), \ z < 0\}$) which is a smooth manifold of dimension 1 whereas $\dim X = 2$.

Proof of Theorem 5.10 To simplify the exposition we suppose that X, and hence Y as well, is pure-dimensional. That is $X = \overline{\mathrm{Reg}(X)}$ and $Y = \overline{\mathrm{Reg}(Y)}$. The proof in the general case is similar.

First we apply Proposition 5.4 to f^{-1}. Hence the Jacobian determinant of f^{-1} is bounded from above. Therefore the Jacobian determinant of f is bounded from below and we can apply to f Theorem 4.13. This shows that f^{-1} is generically arc-analytic and that the Jacobian determinant of f is bounded from above and below.

Now we show that the Jacobian matrix of f is bounded from above. Let $\gamma \in \mathcal{L}(X) \backslash \mathcal{L}(S)$. We may assume, as explained above, that $\mathbb{R}^d \subset \mathbb{R}^N$ is the limit of tangent spaces $T_{\gamma(t)} X$. Similarly by considering the limit of $T_{f(\gamma(t))} Y$ we may assume that it equals $\mathbb{R}^d \subset \mathbb{R}^M$. Then $y_1, \ldots, y_d$ form a local system of coordinates on Y at every $f(\gamma(t))$, $t \neq 0$. By the assumptions the matrix $(\frac{\partial x_i}{\partial y_j})(f(\gamma(t)))$ is bounded and its determinant is a unit. Therefore, by the cofactor formula, its inverse, that is $(\frac{\partial x_i}{\partial y_j})(f(\gamma(t)))$ is bounded. This shows that f is inner Lipschitz by Proposition 5.9. □

Remark 5.12 Notice that, in the above proof, we do not need a homeomorphism $f : X \to Y$ but only a homeomorphism of $f : \overline{\mathrm{Reg}(X)} \to \overline{\mathrm{Reg}(Y)}$.

References

1. Batyrev, V.V.: Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In: Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), pp. 1–32. World Scientific Publishing, River Edge (1998)
2. Bierstone, E., Milman, P.D.: Arc-analytic functions. Invent. Math. **101**(2), 411–424 (1990). https://doi.org/10.1007/BF01231509
3. Bierstone, E., Milman, P.D.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. **128**(2), 207–302 (1997). https://doi.org/10.1007/s002220050141
4. Birbrair, L., Neumann, W.D., Pichon, A.: The thick-thin decomposition and the bilipschitz classification of normal surface singularities. Acta Math. **212**(2), 199–256 (2014). https://doi.org/10.1007/s11511-014-0111-8
5. Bochnak, J.: Analytic functions in Banach spaces. Stud. Math. **35**, 273–292 (1970). https://doi.org/10.4064/sm-35-3-273-292
6. Bochnak, J., Coste, M., Roy, M.F.: Real algebraic geometry. In: Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 36. Springer, Berlin (1998). https://doi.org/10.1007/978-3-662-03718-8. Translated from the 1987 French original, Revised by the authors
7. Borisov, L.A.: The class of the affine line is a zero divisor in the Grothendieck ring. J. Algebraic Geom. **27**(2), 203–209 (2018)
8. Campesato, J.B.: An inverse mapping theorem for blow-Nash maps on singular spaces. Nagoya Math. J. **223**(1), 162–194 (2016). https://doi.org/10.1017/nmj.2016.29
9. Campesato, J.B.: On a motivic invariant of the arc-analytic equivalence. Ann. Inst. Fourier (Grenoble) **67**(1), 143–196 (2017). http://aif.cedram.org/item?id=AIF_2017__67_1_143_0
10. Denef, J., Loeser, F.: Germs of arcs on singular algebraic varieties and motivic integration. Invent. Math. **135**(1), 201–232 (1999). https://doi.org/10.1007/s002220050284
11. Denef, J., Loeser, F.: Motivic integration, quotient singularities and the McKay correspondence. Compos. Math. **131**(3), 267–290 (2002). https://doi.org/10.1023/A:1015565912485
12. Fichou, G.: Motivic invariants of arc-symmetric sets and blow-Nash equivalence. Compos. Math. **141**(3), 655–688 (2005). https://doi.org/10.1112/S0010437X05001168

13. Fichou, G.: On Grothendieck rings and algebraically constructible functions. Math. Ann. **369**(1–2), 761–795 (2017). https://doi.org/10.1007/s00208-017-1564-9

14. Fukui, T., Kurdyka, K., Păunescu, L.: An inverse mapping theorem for arc-analytic homeomorphisms. In: Geometric Singularity Theory, *Banach Center Publ.*, vol. 65, pp. 49–56. Polish Acad. Sci. Inst. Math., Warsaw (2004). https://doi.org/10.4064/bc65-0-3

15. Greenberg, M.J.: Rational points in Henselian discrete valuation rings. Inst. Hautes Études Sci. Publ. Math. (31), 59–64 (1966). http://www.numdam.org/item?id=PMIHES_1966__31__59_0

16. Halupczok, I., Yin, Y.: Lipschitz stratifications in power-bounded o-minimal fields. J. Eur. Math. Soc. (JEMS) **20**(11), 2717–2767 (2018)

17. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. I. Ann. Math. (2) **79**, 109–203 (1964)

18. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. II. Ann. Math. (2) **79**, 205–326 (1964)

19. Koike, S., Parusiński, A.: Motivic-type invariants of blow-analytic equivalence. Ann. Inst. Fourier (Grenoble) **53**(7), 2061–2104 (2003). http://aif.cedram.org/item?id=AIF_2003__53_7_2061_0

20. Kollár, J., Kucharz, W., Kurdyka, K.: Curve-rational functions. Math. Ann. **370**(1–2), 39–69 (2018). https://doi.org/10.1007/s00208-016-1513-z

21. Kollár, J., Nowak, K.: Continuous rational functions on real and *p*-adic varieties. Math. Z. **279**(1–2), 85–97 (2015). https://doi.org/10.1007/s00209-014-1358-7

22. Kucharz, W.: Rational maps in real algebraic geometry. Adv. Geom. **9**(4), 517–539 (2009). https://doi.org/10.1515/ADVGEOM.2009.024

23. Kurdyka, K.: Ensembles semi-algébriques symétriques par arcs. Math. Ann. **282**(3), 445–462 (1988). https://doi.org/10.1007/BF01460044

24. Kurdyka, K.: On a subanalytic stratification satisfying a Whitney property with exponent 1. In: Real Algebraic Geometry (Rennes, 1991), Lecture Notes in Math., vol. 1524, pp. 316–322. Springer, Berlin (1992). https://doi.org/10.1007/BFb0084630

25. Kurdyka, K., Orro, P.: Distance géodésique sur un sous-analytique. Rev. Mat. Univ. Complut. Madrid **10**(Special Issue, suppl.), 173–182 (1997) **(Real algebraic and analytic geometry (Segovia, 1995))**

26. Kurdyka, K., Parusiński, A.: Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture. In: Singularity Theory and its Applications, Advanced Studies in Pure Mathematics, vol. 43, pp. 137–177. Math. Soc. Japan, Tokyo (2006)

27. Kurdyka, K., Parusiński, A.: Arc-symmetric sets and arc-analytic mappings. In: Arc Spaces and Additive Invariants in Real Algebraic and Analytic Geometry. Panor. Synthèses, vol. 24, pp. 33–67. Soc. Math. France, Paris (2007)

28. Kurdyka, K., Păunescu, L.: Arc-analyticity is an open property. In: Singularités Franco-Japonaises, Sémin. Congr., vol. 10, pp. 155–162. Soc. Math. France, Paris (2005)

29. Looijenga, E.: Motivic measures. Astérisque (276), 267–297 (2002). Séminaire Bourbaki, vol. 1999/2000

30. Malgrange, B.: Ideals of differentiable functions. In: Tata Institute of Fundamental Research Studies in Mathematics, No. 3. Tata Institute of Fundamental Research, Bombay; Oxford University Press, London (1967)

31. Martin, N.: The class of the affine line is a zero divisor in the Grothendieck ring: an improvement. C. R. Math. Acad. Sci. Paris **354**(9), 936–939 (2016). https://doi.org/10.1016/j.crma.2016.05.016

32. McCrory, C., Parusiński, A.: Virtual Betti numbers of real algebraic varieties. C. R. Math. Acad. Sci. Paris **336**(9), 763–768 (2003). https://doi.org/10.1016/S1631-073X(03)00168-7

33. McCrory, C., Parusiński, A.: The weight filtration for real algebraic varieties. In: Topology of Stratified Spaces, Math. Sci. Res. Inst. Publ., vol. 58, pp. 121–160. Cambridge University Press, Cambridge (2011)

34. Mostowski, T.: Lipschitz equisingularity. Diss. Math. (Rozpr. Mat.) **243**, 46 (1985)

35. Motzkin, T.S.: The real solution set of a system of algebraic inequalities is the projection of a hypersurface in one more dimension. In: Inequalities, II (Proc. Second Sympos., U.S. Air Force Acad., Colo., 1967), pp. 251–254. Academic Press, New York (1970)

36. Nash Jr., J.F.: Arc structure of singularities. Duke Math. J. **81**(1), 31–38 (1996) (1995). https://doi.org/10.1215/S0012-7094-95-08103-4. **(A celebration of John F. Nash, Jr)**

37. Parusiński, A.: Lipschitz properties of semi-analytic sets. Ann. Inst. Fourier (Grenoble) **38**(4), 189–213 (1988). http://www.numdam.org/item?id=AIF_1988__38_4_189_0

38. Parusiński, A.: Lipschitz stratification. In: Global Analysis in Modern Mathematics (Orono, ME, 1991; Waltham, MA, 1992), pp. 73–89. Publish or Perish, Houston (1993)
39. Parusiński, A.: Lipschitz stratification of subanalytic sets. Ann. Sci. École Norm. Sup. (4) **27**(6), 661–696 (1994). http://www.numdam.org/item?id=ASENS_1994_4_27_6_661_0
40. Parusiński, A.: Subanalytic functions. Trans. Am. Math. Soc. **344**(2), 583–595 (1994). https://doi.org/10.2307/2154496
41. Parusiński, A.: Topology of injective endomorphisms of real algebraic sets. Math. Ann. **328**(1–2), 353–372 (2004). https://doi.org/10.1007/s00208-003-0486-x
42. Parusiński, A., Păunescu, L.: Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity. Adv. Math. **309**, 254–305 (2017). https://doi.org/10.1016/j.aim.2017.01.016
43. Pawłucki, W.: Lipschitz cell decomposition in o-minimal structures. I. Ill. J. Math. **52**(3), 1045–1063 (2008). http://projecteuclid.org/euclid.ijm/1254403731
44. Poonen, B.: The Grothendieck ring of varieties is not a domain. Math. Res. Lett. **9**(4), 493–497 (2002). https://doi.org/10.4310/MRL.2002.v9.n4.a8
45. Quarez, R.: Espace des germes d'arcs réels et série de Poincaré d'un ensemble semi-algébrique. Ann. Inst. Fourier (Grenoble) **51**(1), 43–68 (2001). http://aif.cedram.org/item?id=AIF_2001__51_1_43_0
46. Siciak, J.: A characterization of analytic functions of *n* real variables. Stud. Math. **35**, 293–297 (1970). https://doi.org/10.4064/sm-35-3-293-297
47. Valette, G.: Lipschitz triangulations. Ill. J. Math. **49**(3), 953–979 (2005). http://projecteuclid.org/euclid.ijm/1258138230
48. Valette, G.: On metric types that are definable in an o-minimal structure. J. Symb. Log. **73**(2), 439–447 (2008). https://doi.org/10.2178/jsl/1208359053
49. Zariski, O.: Exceptional singularities of an algebroid surface and their reduction. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **43**, 135–146 (1967)

Affiliations

Jean-Baptiste Campesato[1] · Toshizumi Fukui[2] · Krzysztof Kurdyka[3] · Adam Parusiński[4]

Jean-Baptiste Campesato
Jean-Baptiste.Campesato@univ-amu.fr

Toshizumi Fukui
tfukui@rimath.saitama-u.ac.jp

Krzysztof Kurdyka
Krzysztof.Kurdyka@univ-savoie.fr

[1] Aix Marseille University, CNRS, Centrale Marseille, I2M, Marseille, France

[2] Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Sakura-ku, Saitama 338-8570, Japan

[3] University Grenoble Alpes, University Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France

[4] Université Côte d'Azur, Université Nice Sophia Antipolis, CNRS, LJAD, Nice, France