GLOBAL SMOOTHING OF A SUBANALYTIC SET

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Abstract

We give rather simple answers to two long-standing questions in real-analytic geometry, on global smoothing of a subanalytic set, and on transformation of a proper real-analytic mapping to a mapping with equidimensional fibers by global blowings-up of the target. These questions are related: a positive answer to the second can be used to reduce the first to the simpler semianalytic case. We show that the second question has a negative answer, in general, and that the first problem nevertheless has a positive solution.

1. Introduction

Semialgebraic and subanalytic sets have become ubiquitous in mathematics since their introduction by Łojasiewicz [7] in the 1960s, following the celebrated Tarski–Seidenberg theorem on quantifier elimination. In this article, we give rather simple answers to two long-standing questions in real-analytic geometry, on global smoothing of a subanalytic set (an analogue of resolution of singularities), and on transformation of a proper real-analytic mapping to a mapping with locally equidimensional fibers by global blowings-up of the target (a classical result of Hironaka [6] in the complex-analytic case).

These questions are related: a positive answer to the second can be used to reduce the first to the simpler semianalytic case. We show that the second question has a negative answer, in general, and that the first problem nevertheless has a positive solution.

1.1. Global smoothing

Throughout the article, all spaces and mappings are assumed to be defined over the field of real numbers, unless stated otherwise. The results stated in this section will be proved in Section 2 below.

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THEOREM 1.1 (Nonembedded global smoothing)
Let V be an analytic manifold of dimension n, and let X denote a closed subanalytic subset of V, $\dim X = k$. Then there is a proper analytic mapping $\varphi : X' \to V$, where X' is an analytic manifold of pure dimension k, and a smooth open subanalytic subset U of X, where $\dim X \setminus U < k$, such that

(1) $\varphi(X') \subset X$;
(2) $\varphi^{-1}(X \setminus U)$ *is a simple normal crossings hypersurface* $B' \subset X'$;
(3) *for each connected component W of U, $\varphi^{-1}(W)$ is a finite union of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto W by φ.*

There is an analogous semialgebraic version of Theorem 1.1. Condition (3) of the theorem is an analogue for subanalytic (or semialgebraic) sets of the bimeromorphic (or birational) property of resolution of singularities. The example of a closed half-line in $\mathbb{R}$ shows that the finite-to-one property in (3) is needed. The fact that U is not required to be the entire k-dimensional smooth part of X in Theorem 1.1 means there is freedom in the construction of the mapping φ that can be exploited to prove the global smoothing result by essentially local means.

THEOREM 1.2 (Embedded global smoothing)
Let V be an analytic manifold of dimension n, and let X denote a closed subanalytic subset of V, $\dim X = k$. Then there is a proper analytic mapping $\varphi : V' \to V$, where V' is an analytic manifold of dimension n, together with a smooth closed analytic subset $X' \subset V'$ of dimension k, and a simple normal crossings hypersurface $B' \subset V'$ transverse to X' (i.e., the components of B' are smooth and simultaneously transverse to X'), such that

(1) $\dim \varphi(B') < k$;
(2) $\varphi|_{V' \setminus B'}$ *is finite-to-one and of constant rank n;*
(3) *φ induces an isomorphism from a union of components of $X' \setminus B'$ to a smooth open subanalytic subset U of X such that* $\dim X \setminus U < k$.

The union in (3) is necessarily finite if X is compact; in general, X itself may have infinitely many components. The following example shows that the finite-to-one property (2) is again needed. In the case that X is a closed semialgebraic subset of $\mathbb{R}^n$, there is an analogue of Theorem 1.2 where the mapping in (2) is one-to-one (see Remark 2.6).

Example 1.3
Let

$$g(x) := \sin\Big(\frac{1}{\delta x - 1/\pi}\Big), \tag{1.1}$$

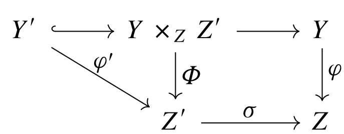
where $\delta > 0$ is a constant. Then $g(x)$ is analytic on the open interval $(-\infty, 1/\delta\pi)$. Let $X = \{(x, y) \in \mathbb{R}^2 : y = g(x), x \leq 1/2\delta\pi\}$. A mapping $\varphi : V' \to \mathbb{R}^2$ as in Theorem 1.2 must be at least two-to-one on $V' \setminus B'$. (Otherwise, the image of X' would provide an extension of X to a closed analytic curve in $\mathbb{R}^2$.)

We believe that Theorems 1.1 and 1.2 are not, in general, true with the stronger condition that U is the entire smooth part of X of dimension k, but we do not have a counterexample. The following example in the algebraic case is illustrative.

Example 1.4
Let X be the algebraic subset of $\mathbb{R}^3$ defined by $z^4 = x^3 + wxz^2$ (cf. [1, Remark 7.3]); X can be obtained as a blowing-down ($u = x/z$) of the smooth hypersurface $X' \subset \mathbb{R}^3$ given by $z = u^3 + uw$. The smooth part of X (as an algebraic *set*) is the complement in X of the half-line $\{x = z = 0, w \leq 0\}$. The blowing-up $\varphi : X' \to X$ satisfies Theorem 1.1 with U equal to the complement in X of the w-axis, but the inverse image of $\operatorname{Sing} X$ in X' is a "T-shaped" set including only the nonpositive w-axis. We can get a mapping as in Theorem 1.1, where U is the entire smooth part of X, by following the blowing-up with an additional (generically) two-to-one covering.

1.2. Simplification of an analytic morphism
Let $\varphi : Y \to Z$ denote a proper morphism of *analytic spaces*. We say that φ is *finite* if, for every $a \in Y$, the local ring $\mathcal{O}_{Y,a}$ is a finite $\mathcal{O}_{Z,\varphi(a)}$-module, via the ring homomorphism $\varphi^* : \mathcal{O}_{Z,\varphi(a)} \to \mathcal{O}_{Y,a}$. If φ is finite, then $\varphi(Y)$ is a closed semianalytic subset of Z [5, Lemma 7.3.6].

Let $\sigma : Z' \to Z$ denote a morphism given as a composite of blowings-up (more precisely, for every relatively compact open subset V of Z, $\sigma|_{\sigma^{-1}(V)} : \sigma^{-1}(V) \to V$ is the composite of a finite sequence of blowings-up over V). Given a proper morphism $\varphi : Y \to Z$, let $\Phi : Y \times_Z Z' \to Z'$ denote the canonical morphism from the fiber-product. There is an induced morphism $\varphi' : Y' \to Z'$, where Y' denotes the smallest closed analytic subspace of $Y \times_Z Z'$ containing $Y \times_Z Z' \setminus \Phi^{-1}(B')$, where $B' \subset Z'$ is the exceptional divisor of σ (i.e., the critical set of σ, in the case that Z is smooth). The morphism φ' is called the *strict transform* of φ.



If σ is a blowing-up with center $C \subset Z$, then $Y' \to Y$ can be identified with the blowing-up of the pullback ideal $\varphi^*(\mathcal{I}_C)$ (where $\mathcal{I}_C \subset \mathcal{O}_Z$ is the ideal of C, and $\varphi^*(\mathcal{I}_C) \subset \mathcal{O}_Y$ denotes the coherent ideal generated by the pullbacks of all local sections of $\mathcal{I}_C$). This follows essentially from the definitions (cf. [5, Chapter 4]).

Question 1.5
Given $\varphi : Y \to Z$, can we find a composite of blowings-up $\sigma : Z' \to Z$ such that φ' has fibers that are equidimensional in some neighborhood of every point of Y'?

Any closed subanalytic subset X of Z is the image of a proper morphism $\varphi : Y \to Z$ with fibers that generically are finite [5, Chapter 7], [1, Theorem 0.1], so a positive answer would provide a composite of blowings-up σ such that $X' := \varphi'(Y')$ is semianalytic (cf. Lemma 2.1 below). In Section 3 below, we will use the function (1.1) to construct examples showing that the answer to Question 1.5 is *no*, in general.

Remark 1.6
In the complex-analytic case, the answer is *yes* and, in fact, there is a stronger result due to Hironaka [6]: φ can be transformed to a flat morphism by a composite of blowings-up σ. Hironaka's proof is based on successively blowing up *local flatteners* of the morphism. Remarkably, Hironaka shows that φ can be flattened by global blowings-up of X although a global flattener does not exist, in general, even in the complex case (cf. [5, Chapter 4]). Equidimensionality of fibers as a substitute for the stronger flatness condition is studied in [8].

As a final remark (Remark 3.4), we note that a construction similar to that in Examples 3.1 and 3.3 can be used to show that, in the real-analytic category, it is not true, in general, that a composite of blowings-up is also a blowing-up. It follows that a characterization of blow-analytic mappings claimed by Fukui [4, Section 2] is not true as stated.

2. Global smoothing theorems

2.1. Lemma of Hironaka

The proofs of our global smoothing theorems (Theorems 1.1 and 1.2) use the following local lemma due essentially to Hironaka [5, Proposition 7.3] (see also [3, Theorem A.4.1]). The lemma is a consequence of Hironaka's local flattening theorem [5, Chapter 4], using resolution of singularities to dominate each blowing-up of a local flattener by a sequence of blowings-up with smooth centers. We recall that a *local blowing-up* $\sigma : V' \to V$ means a composite $V' \to W \hookrightarrow V$, where $W \hookrightarrow V$ is the inclusion of an open subset, and $V' \to W$ is a blowing-up.

LEMMA 2.1

Let V be an analytic manifold, and let X denote a closed subanalytic subset of V. Let K be a compact subset of V. Then there exists a finite collection of analytic mappings $\pi_\lambda : V_\lambda \to V$, where each V_λ is an analytic manifold of dimension equal to $\dim V$, *and a compact subset K_λ of V_λ, for each λ, with the following properties.*

(1) *$\bigcup \pi_\lambda(K_\lambda)$ is a neighborhood of $X \cap K$ in V.*

(2) *For each λ, π_λ is the composite of a finite sequence of local blowings-up with smooth centers. The union of the inverse images of these centers in V_λ is a closed analytic hypersurface B_λ of V_λ, so that π_λ induces an open embedding $V_\lambda \backslash B_\lambda \to V$. Moreover,* $\dim \bigcup \pi_\lambda(B_\lambda) < \dim X$.

(3) *(The closure of) $(\pi_\lambda)^{-1}(X) \backslash B_\lambda$ is semianalytic, for every λ.*

Let p denote the longest length of the sequence of local blowings-up involved in π_λ, for any λ, in Lemma 2.1. We will call $\{\pi_\lambda\}$ a *semianalytic covering* of $X \cap K$ of *depth p*. We will prove Theorem 1.2 first in the case that X is semianalytic, and reduce the subanalytic to the semianalytic case by induction on the depth of a semianalytic covering, for suitable K.

2.2. Smoothing of a semianalytic n-cell

Let V be an analytic manifold of dimension n, and let C denote the closure of a relatively compact open semianalytic subset of V. We will say that C is a *semianalytic n-cell* if there are finitely many analytic functions f_i, $i = 1, \dots, q$, defined in a neighborhood W of C, such that $C = \bigcup_{j=1}^r C_j$, where each

$$C_j = \left\{x \in W : f_i(x) \geq 0, i \in I_j\right\}, \quad \text{for some } I_j \subset \{1, \dots, q\};$$

in particular, the boundary of C, $\operatorname{bdry} C \subset \bigcup_i \{f_i(x) = 0\}$. Note that the *boundary hypersurfaces* $\{f_i(x) = 0\}$ may include interior points of C.

LEMMA 2.2

Let C denote a semianalytic n-cell in V, as above. Then there is an analytic mapping $\xi : S \to V$, where S is a compact analytic manifold of dimension n, a simple normal crossings hypersurface $D \subset S$, and a dense open semianalytic subset U of C, such that $\xi(S) = C$, $S \backslash D = \xi^{-1}(U)$ equals a finite union of open and closed subsets, each projecting isomorphically onto U.

Proof

We can assume that $\dim C_{j_1} \cap C_{j_2} < n$, for all $j_1 \neq j_2$, and can thus reduce to the case that C is of the form

$$C = \left\{x \in W : f_i(x) \geq 0, i = 1, \dots, q\right\}.$$

Define

$$Z := \{(x,t) \in W \times \mathbb{R}^q : t_i^2 = f_i(x), i = 1, \dots, q\},$$

where $t = (t_1, \dots, t_q)$. Then Z is a compact analytic subset of $W \times \mathbb{R}^q$. Let $\pi : Z \to W$ denote the restriction of the projection $W \times \mathbb{R}^q \to W$. Then $\pi(Z) = C$. Moreover, there is a closed analytic subset Y of Z, with $\dim Y < n = \dim Z$, and an open dense semianalytic subset U of X, such that $Z \backslash Y = \pi^{-1}(U)$ equals a finite union of open and closed subsets, each projecting isomorphically onto U. The result then follows by composing π with a mapping $S \to Z$ given by resolution of singularities of $Y \subset Z$ (cf. [5, Theorem 5.10], [2, Theorem 1.6]). □

Remark 2.3

(1) In the case that X is a semialgebraic subset of $\mathbb{R}^n$, the same proof gives an analogue of Lemma 2.2 where the mapping ξ is algebraic.

(2) Our proof of Theorem 1.2 involves Lemma 2.2 for a covering of X by semianalytic n-cells with disjoint interiors. In the case that X is a compact subanalytic subset of $V = \mathbb{R}^n$, Lemma 2.2 is needed only in the case of a cube $C \subset \mathbb{R}^n$ (see Remark 2.5). In this case, a smoothing $\xi : S \to V$ can be constructed more efficiently as follows. Suppose that $C = [-1,1]^n$. Then the projection $(x,y) \mapsto x$ of the unit circle S^1 in $\mathbb{R}^2$ onto the closed interval $[-1,1] \subset \mathbb{R}$ induces a real-analytic mapping $\xi : S := (S^1)^n \to \mathbb{R}^n$ onto C, such that ξ induces a 2^n-sheeted covering of the open cube $(-1,1)^n$, and the inverse image of the boundary is a simple normal crossings hypersurface D in S.

2.3. Partition into semianalytic cells

Let V denote an analytic manifold (assumed to be countable at infinity), $\dim V = n$. A locally finite (hence countable) collection of semianalytic n-cells C_λ in V will be called a *partition* of V into semianalytic n-cells if $V = \bigcup C_\lambda$ and the interiors of the C_λ are disjoint. We will say that such a partition $\mathcal{P}$ is *subordinate* to a covering $\mathcal{C}$ of V by open subsets W_μ if each $C_\lambda \in \mathcal{P}$ lies in some W_μ. We will say that $\mathcal{P}$ is *compatible* with a semianalytic subset Y of V if the interior $\operatorname{int} C_\lambda$ of every $C_\lambda \in \mathcal{P}$ lies in either the interior or exterior of Y.

Given a subanalytic subset X of V, we will say that a partition into semianalytic n-cells C_λ is *in general position* with respect to X if, for each λ, the boundary hypersurfaces $\{f_i(x) = 0\}$ of C_λ (see Section 2.2) can be chosen so that $\dim(X \cap \{f_i(x) = 0\}) < \dim X$, for all i.

LEMMA 2.4

Let V denote an analytic manifold of dimension n, and let $\mathcal{C}$ be an open covering of V.

(1) *There exists a (locally finite) partition of V into semianalytic n-cells, subordinate to $\mathcal{C}$.*

(2) *If Y is a semianalytic subset of V, then there exists a partition of V into semianalytic cells, subordinate to $\mathcal{C}$ and compatible with Y.*

(3) *If X is a closed subanalytic subset of V, then there is a partition into semianalytic cells, subordinate to $\mathcal{C}$ and in general position with respect to X.*

Proof

Consider a covering of V by a locally finite (hence countable) collection of analytic coordinate charts V_ι, $\iota = 1, 2, \dots$, where each V_ι lies in a member of $\mathcal{C}$. Given ι and a positive integer q_ι, let $(x_1, \dots, x_n)$ denote the coordinates of V_ι, and consider the q_ι-*grid* of V formed by the hyperplanes $\{x_i = j/q_\iota\}$, $j \in \mathbb{Z}$, $i = 1, \dots, n$. Let $\widehat{C}_{\iota\mu}$ denote the closed cubes (of side length $1/q_\iota$) determined by the q_ι-grid. Of course, we can choose the covering $\{V_\iota\}$ and the q_ι with the property that, for each ι, there is a big closed cube $\widehat{Q}_\iota \subset V_\iota$ with sides determined by the q_ι-grid, such that the interiors $\operatorname{int}\widehat{Q}_\iota$ of all $\widehat{Q}_\iota$ cover V; in fact, we can assume that V is covered by smaller open balls (say, with center equal to the center of $\widehat{Q}_\iota$ and diameter equal to half the side length of $\widehat{Q}_\iota$).

Write $Q_1 := \widehat{Q}_1$, and $C_{1\mu} := \widehat{C}_{1\mu}$, for all μ. For each $\iota > 1$, set

$$Q_\iota := \text{closure of } (\operatorname{int}\widehat{Q}_\iota)\setminus \bigcap_{\gamma<\iota} \widehat{Q}_\gamma,$$

$$C_{\iota\mu} := \text{closure of } (\operatorname{int}\widehat{C}_{\iota\mu})\setminus \bigcap_{\gamma<\iota} \widehat{Q}_\gamma, \quad \text{for all } \mu.$$

Replacing each q_ι by a large enough integral multiple, if necessary, we can assume that each $C_{\iota\mu}$ is a semianalytic n-cell (in particular, $\operatorname{bdry} C_{\iota\mu}$ lies in the union of the zero sets of finitely many analytic functions defined in a neighborhood of $C_{\iota\mu}$, given by the boundary hypersurfaces of $\widehat{C}_{\iota\mu}$ and $\widehat{Q}_\gamma$, $\gamma < \iota$). Then the collection of all cells $C_{\iota\mu}$, where $\widehat{C}_{\iota\mu} \subset \widehat{Q}_\iota$, for all ι, form a partition of V subordinate to $\{V_\iota\}$. Assertion (1) follows.

Clearly, if Y is a semianalytic subset of V, then, after taking a large enough multiple of q_ι above, each $C_{\iota\mu}$ can be partitioned into finitely many cells, each with interior in either the interior or exterior of Y, as required by (2).

Given a closed subanalytic subset X of V, $\dim X = k$, we can also assume that, for each ι, every coordinate hyperplane $\{x_i = j/q_\iota\}$ of V_ι intersects X in a subanalytic subset of dimension less than k (by a small linear coordinate change, if necessary; in fact, it is enough that each hyperplane $\{x_i = j/q_\iota\}$ that intersects $\widehat{Q}_\iota$ has this property). It follows that, for each cell $C_{\iota\mu}$ in the resulting partition, the intersection of

X with every boundary hypersurface $\{f_i(x) = 0\}$ has dimension less than k. This proves (3). □

Remark 2.5
The proof of Lemma 2.4 shows that, if X is a compact subanalytic subset of $\mathbb{R}^n$, then, for any open covering $\mathcal{C}$ of $\mathbb{R}^n$, there is a partition of $\mathbb{R}^n$ into cubes, subordinate to $\mathcal{C}$ and in general position with respect to X.

2.4. *Proofs of the main theorems*

Proof of Theorem 1.2 (Embedded global smoothing)
(i) The semianalytic case. Suppose that X is a closed semianalytic subset of V. Then there is a locally finite covering of V by open subsets V_ι such that, for each ι, there are closed analytic subsets $Y_\iota \subset Z_\iota$ of V_ι, $\dim Z_i = k$, and an open and closed subset U_ι of $Z_\iota \backslash Y_\iota$, such that U_ι is an open subset of the smooth part of X of dimension k and $\dim(X \cap V_\iota)\backslash U_\iota < k$.

By Lemma 2.4, there is a partition $\mathcal{P}$ of V into semianalytic n-cells C, subordinate to $\{V_\iota\}$ and in general position with respect to X. It is enough to show that, for each $C \in \mathcal{P}$ such that $C \cap X \neq \emptyset$, there is a mapping $\varphi_C : V'_C \to V$ onto C, satisfying the conclusion of the theorem with respect to $X \cap C$. Indeed, we can then simply let V' be the disjoint union of the V'_C and let $\varphi : V' \to V$ be the mapping given by φ_C on each V'_C.

Consider such a cell C. Choose ι so that $C \subset V_\iota$. Take $\xi : S \to V$ onto C, and $D \subset S$, as in Lemma 2.2. By resolution of singularities, there exist an analytic manifold V'_C of dimension n, a proper analytic surjection $\rho : V'_C \to S$, and a smooth closed analytic subset X' of V'_C of pure dimension k (X' is the strict transform of Z_ι), such that $B' := \rho^{-1}(\xi^{-1}(Y_\iota) \cup D)$ is a simple normal crossings hypersurface in V'_C transverse to X', and $\varphi_C := \xi \circ \rho : V'_C \to V$, together with X' and B', satisfy the conclusions of the theorem with respect to $X \cap C$ (see [5, Theorems 5.10, 5.11], [2, Theorems 1.6, 1.10]).

(ii) The general subanalytic case. Consider a locally finite covering of V by relatively compact open subsets V_ι. By Lemma 2.1, for each ι, there is a semianalytic covering $\{\pi_{\iota\lambda}\}$ of $X \cap \overline{V_\iota}$, of depth p_ι, say.

Each $\pi_{\iota\lambda}$ is a composite of local blowings-up

$$\pi_{\iota\lambda} = \pi^1_{\iota\lambda} \circ \pi^2_{\iota\lambda} \circ \cdots \circ \pi^{p(\iota,\lambda)}_{\iota\lambda}, \quad p(\iota,\lambda) \leq p_\iota;$$

that is,

$$\pi^i_{\iota\lambda} : V^i_{\iota\lambda} \to W^i_{\iota\lambda} \hookrightarrow V^{i-1}_{\iota\lambda}, \quad i = 1, \dots, p(\iota,\lambda),$$

where $W^i_{\iota\lambda} \subset V^{i-1}_{\iota\lambda}$ is an open subset and $\pi^i_{\iota\lambda} : V^i_{\iota\lambda} \to W^i_{\iota\lambda}$ is a blowing-up with smooth center ($V^0_{\iota\lambda} := V$).

By Lemma 2.4, there is a partition $\mathcal{P}$ of V into semianalytic n-cells C, subordinate to $\{V_\iota\}$ and in general position with respect to X. Let $\mathcal{P}_X := \{C \in \mathcal{P} : X \cap C \neq \emptyset\}$. We can assume that

(1) $\mathcal{P}_X = \bigcup_\iota \mathcal{P}_\iota$, where the $\mathcal{P}_\iota$ are disjoint subsets of $\mathcal{P}_X$ and $Q_\iota := \bigcup\{C : C \in \mathcal{P}_\iota\} \subset V_\iota$;

(2) if $C \in \mathcal{P}_\iota$, then $C \in W^1_{\iota\lambda}$, for some $\lambda = \lambda(\iota, C)$.

(This is clear, for example, from the construction of $\mathcal{P}$ in the proof of Lemma 2.4(1), by taking a large enough multiple of q_ι.)

Now, it is enough to prove that, for each ι, there is a mapping $\varphi_\iota : V'_\iota \to V$ (onto Q_ι) satisfying the conclusion of the theorem with respect to $X \cap Q_\iota$. Fix ι. Our proof is by induction on the depth p_ι of the semianalytic covering $\{\pi_{\iota\lambda}\}$. The case $p_\iota = 0$ follows from the theorem in the case that X is semianalytic.

Again, it is enough to prove that, for each $C \in \mathcal{P}_\iota$, there is a mapping $\varphi_C : V'_C \to V$ (onto C) satisfying the conclusion of the theorem with respect to $X \cap C$. Fix $C \in \mathcal{P}_\iota$. Let B^1 denote the exceptional divisor of $\pi^1_{\iota\lambda}$, where $\lambda = \lambda(\iota, C)$, and let X^1 denote the closure in $V^1_{\iota\lambda}$ of $(\pi^1_{\iota\lambda})^{-1}(X \cap C)\backslash B^1$. Then $X^1 \subset V^1_{\iota\lambda}$ has a semianalytic covering of depth $< p_\iota$.

By induction, there is a proper analytic mapping $\psi : T \to V^1_{\iota\lambda}$, where T is an analytic manifold of dimension n, together with a smooth closed analytic subset Z of T, $\dim Z = k$, and a simple normal crossings hypersurface $E \subset T$ transverse to Z, satisfying the conclusions of the theorem with respect to $X^1 \subset V^1_{\iota\lambda}$. In particular, ψ induces an isomorphism of a union of components of $Z\backslash E$ with a smooth open subanalytic subset of X^1 whose complement in X^1 has dimension less than k.

Set $\eta := \pi^1_{\iota\lambda} \circ \psi : T \to V$. Let $\xi : S \to V$ denote an analytic mapping onto C, with a simple normal crossings hypersurface $D \subset S$, satisfying Lemma 2.2. Consider the fiber-product $S \times_V T$ of $\xi : S \to V$ and η, and let π_S, π_T denote the projections of $S \times_V T$ to S, T, respectively. By resolution of singularities, there is a surjective analytic mapping $\rho : V'_C \to S \times_V T$, where V'_C is a compact analytic manifold of dimension n, such that the strict transform X'_C of $\pi_T^{-1}(Z)$ is smooth, and the union in V'_C of the inverse images of B^1, D, and E is a simple normal crossings hypersurface B'_C transverse to X'_C. Then the mapping $\varphi_C : V'_C \to V$ given by ρ followed by the projection to V satisfies the conclusions of the theorem with respect to $X \cap C$, as required. □

Remark 2.6

In the case that X is a closed semialgebraic subset of $V = \mathbb{R}^n$, there are global closed algebraic subsets $Y \subset Z$ of $\mathbb{R}^n$, where $Z\backslash Y$ is smooth, and an open and closed subset

U of $Z \backslash Y$, such that U is an open subset of the smooth part of X of dimension k, and $\dim X \backslash U < k$ (cf. case (i) of the proof above). By resolution of singularities, there is a sequence of blowings-up with smooth algebraic centers over Y, after which the strict transform Z' of Z is smooth, and the inverse image of Y is a simple normal crossings hypersurface transverse to Z'. We thus get a semialgebraic analogue of Theorem 1.2, where the mapping in condition (2) of the theorem is one-to-one.

Proof of Theorem 1.1 (Nonembedded smoothing)
By Theorem 1.2, we can assume that X is the closure of an open semianalytic subset of V. By Lemma 2.4(2), there is a partition of V into semianalytic n-cells compatible with X. In particular, X is a locally finite union of semianalytic n-cells, so the result follows from the special case that X is itself a semianalytic n-cell—this is the result of Lemma 2.2. □

The semialgebraic version of Theorem 1.1 can be proved in the same way (see Remark 2.3(1)).

3. Examples

We begin with two examples of a proper analytic mapping $\varphi : V \to W$, where V is an analytic space of dimension 3 and $W = \mathbb{R}^3$, with the property that there is no mapping $\sigma : W' \to W$ given as the composite of a sequence of global blowings-up such that the strict transform φ' of φ by σ has all fibers finite (or empty). Each of the examples below involves the function (1.1), where $\delta > 0$ is small.

Example 3.1
Let $S^3 := \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$, and let $C := \{(x, y, z, w) \in S^3 : z = 0, y = g(x)\}$. If $\delta > 0$ is small enough (e.g., $\delta \le 1/3\pi$), then C is a smooth curve. We define $\varphi : V \to W$ as the composite

$$V \xrightarrow{\pi_C} S^3 \xrightarrow{p} \mathbb{R}^3 \overset{\iota}{\hookrightarrow} Z \xrightarrow{\pi_0} W = \mathbb{R}^3,$$

where

- π_0 is the blowing-up of $0 \in \mathbb{R}^3$;
- ι is the inclusion of the z-coordinate chart, so that $\pi_0 \circ \iota : (x, y, z) \mapsto (xz, yz, z)$, and $\{z = 0\}$ represents the exceptional divisor D of π_0 in this chart;
- p is induced by the projection $(x, y, z, w) \mapsto (x, y, z)$;
- π_C is the blowing-up with center C (so π_C has 1-dimensional fibers over C).

Note that $p(C) \subset D$. The required property of φ is a consequence of the fact that $p(C)$ does not extend to a closed analytic curve in Z.

Indeed, suppose that there is a composite of global blowings-up $\sigma : W' \to W$ such that the strict transform $\varphi' : V' \to V$ of φ by σ has all fibers finite. Say that $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$, where each $\sigma_j : W_j \to W_{j-1}$ is a blowing-up with smooth center $C_{j-1} \subset W_{j-1}$ ($W_0 = W$, $W_k = W'$). Then there is a commutative diagram

$$\begin{array}{ccccccccc} Z' = Z_k & \xrightarrow{\sigma'_k} & Z_{k-1} & \longrightarrow & \cdots & \longrightarrow & Z_1 & \xrightarrow{\sigma'_1} & Z_0 = Z \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k-1}} & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\ W' = W_k & \xrightarrow{\sigma_k} & W_{k-1} & \longrightarrow & \cdots & \longrightarrow & W_1 & \xrightarrow{\sigma_1} & W_0 = W \end{array}$$

where each σ'_j is a composite of finitely many blowings-up with smooth centers. This can be proved inductively. Given $\pi_j : Z_j \to W_j$, let $T_{j+1} \to W_{j+1}$ be the strict transform of π_j by σ_{j+1}, and let $\tau_{j+1} : T_{j+1} \to Z_j$ denote the associated mapping; that is, τ_{j+1} is the blowing-up of the pullback ideal $\pi_j^*(\mathcal{I}_{C_j}) \subset \mathcal{O}_{Z_j}$, where $\mathcal{I}_{C_j} \subset \mathcal{O}_{W_j}$ is the ideal of C_j (see Section 1.2). By resolution of singularities, τ_{j+1} can be dominated by a finite sequence of blowings-up with smooth centers. More precisely, there is a composite $\sigma'_{j+1} : Z_{j+1} \to Z_j$ of finitely many blowings-up with smooth centers, which principalizes $\pi_j^*(\mathcal{I}_{C_j})$, and σ'_{j+1} factors through T_{j+1}, by the universal mapping property of the blowing-up τ_{j+1}. So we get $\pi_{j+1} : Z_{j+1} \to T_{j+1} \to W_{j+1}$.

Let $\sigma' := \sigma'_1 \circ \sigma'_2 \circ \cdots \circ \sigma'_k : Z' \to Z$. Write $\psi := \iota \circ p \circ \pi_C$, and let $\psi' : V'' \to Z'$ denote the strict transform of ψ by σ'. Since φ' has all fibers finite, it follows that ψ' has all fibers finite. Indeed, by definition, V' and V'' are closed subspaces of the fiber-products $V \times_W W'$ and $V \times_Z Z' \subset (V \times_W W') \times_{W'} Z'$, respectively, and moreover, $V'' \subset V' \times_{W'} Z'$. This means that each fiber of ψ' is a subset of a fiber of φ'.

For each $j = 0, \ldots, k-1$, let $C'_j \subset Z'_j$ denote the smallest closed analytic subset containing $\pi_j^{-1}(C_j) \backslash \pi_j^{-1}(D_j)$, where D_j denotes the exceptional divisor of σ_j ($D_0 = D$). Then $\dim C'_j \leq 1$ (C'_j may be empty). The curve $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : z = 0, y = g(x), x < 1/\delta\pi\} \subset Z$ cannot lie entirely in C'_0; therefore, it lifts to a unique curve $\Gamma_1 \subset Z_1$. Likewise, Γ_1 does not lie in C'_1, and so on. (Here we use the property that every subanalytic set containing Γ is of dimension at least 2; clearly, this property is inherited by Γ_1, etc.) Finally, Γ lifts to a unique curve $\Gamma' \subset Z'$, and Γ' intersects the union of the inverse images of all $\pi_j^{-1}(D_j)$ in a discrete set. Therefore, ψ' has 1-dimensional fibers over the lifting of a nonempty open subset of $p(C)$, which is a contradiction.

Remark 3.2
In general, consider a proper analytic mapping $\varphi : V \to W$ which factors through a blowing-up $\pi : Z \to W$ of a coherent ideal sheaf in $\mathcal{O}_W$; that is, $\varphi = \pi \circ \psi$, where $\psi : V \to Z$. Suppose that there is a composite σ of global blowings-up over W with smooth centers, such that the strict transform of φ by σ has all fibers finite (or empty).

Then, by the argument in Example 3.1, the strict transform of ψ by a composite of such blowings-up over Z also has all fibers finite.

In Example 3.1, we can replace W by an arbitrarily small open ball in $\mathbb{R}^3$ centered at the origin, and restricting φ over such a ball will not change the preceding property. It is true, however, that φ can be transformed to a morphism with all fibers finite by blowing up at each step with center that is globally defined in some neighborhood of the image of the corresponding morphism (e.g., after the first blowing-up π_0, with center globally defined in a neighborhood of the image of ψ containing $p(C)$). The latter phenomenon does not occur in the following example.

Example 3.3
Let $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ denote the projection $(x, y, z, w) \mapsto (x, y, z)$, and let $S \subset \mathbb{R}^4$ denote the algebraic subset defined by

$$(x^2 + z^2)^2(w^4 + z^2w^2) - (x^2 - z^2)^2 = 0. \tag{3.1}$$

Then S is irreducible, $\{x = z = 0\} \subset S$, and S maps onto $\mathbb{R}^3$ since we can solve (3.1) for w^2 when $x^2 + z^2 \neq 0$. (The closure of $S\backslash\{x = z = 0\}$ maps properly onto $\mathbb{R}^3$.)

Let $\pi_0 : M \to \mathbb{R}^3$ denote the blowing-up of the origin $\{x = y = z = 0\} \subset \mathbb{R}^3$. Then there is a commutative diagram

$$\begin{array}{ccccc} S' & \hookrightarrow & M \times \mathbb{R} & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle \pi_0\times\mathrm{id}} & & \downarrow{\scriptstyle \pi_0} \\ S & \hookrightarrow & \mathbb{R}^3 \times \mathbb{R} & \longrightarrow & \mathbb{R}^3 \end{array}$$

where S' denotes the strict transform of S by the blowing-up $\pi_0 \times \mathrm{id}$ of $\mathbb{R}^3 \times \mathbb{R}$. Let $S' \to \mathbb{R}^3$ denote the induced mapping.

Let U_z denote the z-coordinate chart of π_0; that is, the chart with coordinates (X, Y, Z) in which π_0 is given by $(x, y, z) = (XZ, YZ, Z)$. The mapping $U_z \times \mathbb{R} \to \mathbb{R}^3$ given by the diagram above is $(X, Y, Z, w) \mapsto (XZ, YZ, Z)$, and S' is defined in $U_z \times \mathbb{R}$ by the equation

$$(X^2 + 1)^2(w^4 + Z^2w^2) - (X^2 - 1)^2 = 0.$$

Setting $Z = 0$, this equation splits as

$$\big((X^2 + 1)w^2 - (X^2 - 1)\big)\big((X^2 + 1)w^2 + (X^2 - 1)\big) = 0.$$

Let $C \subset S'$ denote the compact smooth curve defined by

$$Z = 0, \qquad Y = g(X), \qquad (X^2 + 1)w^2 + (X^2 - 1) = 0,$$

and let $\pi_C : V \to S'$ denote the blowing-up of S' with center C. Then the mapping

$$\varphi : V \xrightarrow{\pi_C} S' \longrightarrow \mathbb{R}^3 = W$$

has the required property.

Indeed, suppose that the strict transform of φ by the composite of a sequence of global blowings-up over W has all fibers finite. Let $\psi : V \to M$ be the mapping such that $\varphi = \pi_0 \circ \psi$. By Remark 3.2, there is a composite $\sigma' : M' \to M$ of global blowings-up, such that the strict transform ψ' of ψ by σ' has all fibers finite. Then the curve $\Gamma = \{(X, Y, Z) \in \mathbb{R}^3 : Z = 0, Y = g(X), X < 1/\delta\pi\} \subset M$ can be lifted to M', and this leads to a contradiction by the same argument as in Example 3.1.

Remark 3.4

A construction similar to that in the examples above can be used to show that, in the real-analytic category, it is not necessarily true that a composite of blowings-up is also a blowing-up. For example, let $\pi_0 : Z_1 \to \mathbb{R}^4$ be the blowing-up of the origin, and let H denote a projective hyperplane in the exceptional divisor of π_0. Let $\pi_H : Z_2 \to Z_1$ denote the blowing-up with center H. Consider an affine coordinate chart U_1 of Z_1 with coordinates (x, y, Z, W), where $\{W = 0\}$ is the exceptional divisor of π_0 and $H = \{Z = W = 0\}$. Let U_2 denote the affine chart of Z_2 over U_1 with coordinates (x, y, z, w) such that π_H is given on U_2 by $(x, y, Z, W) = (x, y, zw, w)$. Let $\pi_C : V \to Z_2$ denote the blowing-up with center

$$C = \{(x, y, z, w) \in U_2 : w = 0, x^2 + y^2 + z^2 = 1, y = g(x)\},$$

and set $\varphi := \pi_0 \circ \pi_H \circ \pi_C$.

We claim that φ is not the blowing-up of an ideal. Suppose that φ is the blowing-up of an ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{R}^4}$. Then $\psi = \pi_H \circ \pi_C$ is the blowing-up of $\pi_0^* \mathcal{I}$; therefore, the set $A = \{b \in Z_1 : \dim \psi^{-1}(b) \geq 2\}$ lies in a real-analytic curve (since, for example, ψ admits a proper complexification). But this is impossible, because A contains a nonempty open subset of $\{Z = W = 0, y = g(x)\}$.

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