6.5 BERNOULLI NUMBERS

The next important sequence of numbers on our agenda is named after Jakob Bernoulli (1654–1705), who discovered curious relationships while working out the formulas for sums of mth powers [26]. Let's write

$$S_m(n) \;=\; 0^m + 1^m + \cdots + (n-1)^m \;=\; \sum_{k=0}^{n-1} k^m \;=\; \sum\nolimits_0^n x^m\,\delta x\,. \quad (6.77)$$

(Thus, when $m > 0$ we have $S_m(n) = H_{n-1}^{(-m)}$ in the notation of generalized harmonic numbers.) Bernoulli looked at the following sequence of formulas and spotted a pattern:

$$\begin{aligned}
S_0(n) &= n\\
S_1(n) &= \tfrac{1}{2}n^2 - \tfrac{1}{2}n\\
S_2(n) &= \tfrac{1}{3}n^3 - \tfrac{1}{2}n^2 + \tfrac{1}{6}n\\
S_3(n) &= \tfrac{1}{4}n^4 - \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2\\
S_4(n) &= \tfrac{1}{5}n^5 - \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n\\
S_5(n) &= \tfrac{1}{6}n^6 - \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2\\
S_6(n) &= \tfrac{1}{7}n^7 - \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n\\
S_7(n) &= \tfrac{1}{8}n^8 - \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2\\
S_8(n) &= \tfrac{1}{9}n^9 - \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n\\
S_9(n) &= \tfrac{1}{10}n^{10} - \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{3}{20}n^2\\
S_{10}(n) &= \tfrac{1}{11}n^{11} - \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - n^7 + n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n
\end{aligned}$$

Can you see it too? The coefficient of n^{m+1} in $S_m(n)$ is always $1/(m+1)$. The coefficient of n^m is always $-1/2$. The coefficient of n^{m-1} is always … let's see … $m/12$. The coefficient of n^{m-2} is always zero. The coefficient of n^{m-3} is always … let's see … hmmm … yes, it's $-m(m-1)(m-2)/720$. The coefficient of n^{m-4} is always zero. And it looks as if the pattern will continue, with the coefficient of n^{m-k} always being some constant times $m^{\underline{k}}$.

That was Bernoulli's empirical discovery. (He did not give a proof.) In modern notation we write the coefficients in the form

$$\begin{aligned}
S_m(n) &= \frac{1}{m+1}\left(B_0\,n^{m+1} + \binom{m+1}{1}B_1\,n^m + \cdots + \binom{m+1}{m}B_m\,n\right)\\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\,n^{m+1-k}\,. \qquad (6.78)
\end{aligned}$$

Bernoulli numbers are defined by an implicit recurrence relation,

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = [m=0], \qquad \text{for all } m \geqslant 0. \tag{6.79}$$

For example, $\binom{2}{0}B_0 + \binom{2}{1}B_1 = 0$. The first few values turn out to be

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$

(All conjectures about a simple closed form for B_n are wiped out by the appearance of the strange fraction $-691/2730$.)

We can prove Bernoulli's formula (6.78) by induction on m, using the perturbation method (one of the ways we found $S_2(n) = \Box_n$ in Chapter 2):

$$\begin{aligned} S_{m+1}(n) + n^{m+1} &= \sum_{k=0}^{n-1} (k+1)^{m+1} \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{m+1} \binom{m+1}{j} k^j = \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n). \end{aligned} \tag{6.80}$$

Let $\widehat{S}_m(n)$ be the right-hand side of (6.78); we wish to show that $S_m(n) = \widehat{S}_m(n)$, assuming that $S_j(n) = \widehat{S}_j(n)$ for $0 \leqslant j < m$. We begin as we did for $m = 2$ in Chapter 2, subtracting $S_{m+1}(n)$ from both sides of (6.80). Then we expand each $S_j(n)$ using (6.78), and regroup so that the coefficients of powers of n on the right-hand side are brought together and simplified:

$$\begin{aligned} n^{m+1} &= \sum_{j=0}^{m} \binom{m+1}{j} S_j(n) = \sum_{j=0}^{m} \binom{m+1}{j} \widehat{S}_j(n) + \binom{m+1}{m} \Delta \\ &= \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j+1} \sum_{k=0}^{j} \binom{j+1}{k} B_k n^{j+1-k} + (m+1)\Delta \\ &= \sum_{0 \leqslant k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j+1}{k} \frac{B_k}{j+1} n^{j+1-k} + (m+1)\Delta \\ &= \sum_{0 \leqslant k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j+1}{j-k} \frac{B_{j-k}}{j+1} n^{k+1} + (m+1)\Delta \\ &= \sum_{0 \leqslant k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j+1}{k+1} \frac{B_{j-k}}{j+1} n^{k+1} + (m+1)\Delta \\ &= \sum_{0 \leqslant k \leqslant m} \frac{n^{k+1}}{k+1} \sum_{k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j}{k} B_{j-k} + (m+1)\Delta \end{aligned}$$

$$\begin{aligned}
&= \sum_{0\le k\le m} \frac{n^{k+1}}{k+1}\binom{m+1}{k} \sum_{k\le j\le m} \binom{m+1-k}{j-k} B_{j-k} + (m+1)\,\Delta \\
&= \sum_{0\le k\le m} \frac{n^{k+1}}{k+1}\binom{m+1}{k} \sum_{0\le j\le m-k} \binom{m+1-k}{j} B_{j} + (m+1)\,\Delta \\
&= \sum_{0\le k\le m} \frac{n^{k+1}}{k+1}\binom{m+1}{k} [m-k=0] + (m+1)\,\Delta \\
&= \frac{n^{m+1}}{m+1}\binom{m+1}{m} + (m+1)\,\Delta \\
&= n^{m+1} + (m+1)\,\Delta\,, \qquad \text{where } \Delta = S_m(n) - \widehat{S}_m(n).
\end{aligned}$$

(This derivation is a good review of the standard manipulations we learned in Chapter 5.) Thus $\Delta = 0$ and $S_m(n) = \widehat{S}_m(n)$, QED.

Here's some more neat stuff that you'll probably want to skim through the first time.
—Friendly TA

In Chapter 7 we'll use generating functions to obtain a much simpler proof of (6.78). The key idea will be to show that the Bernoulli numbers are the coefficients of the power series

$$\frac{z}{e^z-1} = \sum_{n\ge 0} B_n \frac{z^n}{n!}\,. \tag{6.81}$$

Start Skimming

Let's simply assume for now that equation (6.81) holds, so that we can derive some of its amazing consequences. If we add $\frac12 z$ to both sides, thereby cancelling the term $B_1 z/1! = -\frac12 z$ from the right, we get

$$\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\,\frac{e^z+1}{e^z-1} = \frac{z}{2}\,\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2}\,. \tag{6.82}$$

Here coth is the "hyperbolic cotangent" function, otherwise known in calculus books as $\cosh z/\sinh z$; we have

$$\sinh z = \frac{e^z - e^{-z}}{2}\,; \qquad \cosh z = \frac{e^z + e^{-z}}{2}\,. \tag{6.83}$$

Changing z to $-z$ gives $\left(\frac{-z}{2}\right)\coth\left(\frac{-z}{2}\right) = \frac{z}{2}\coth\frac{z}{2}$; hence every odd-numbered coefficient of $\frac{z}{2}\coth\frac{z}{2}$ must be zero, and we have

$$B_3 = B_5 = B_7 = B_9 = B_{11} = B_{13} = \cdots = 0\,. \tag{6.84}$$

Furthermore (6.82) leads to a closed form for the coefficients of coth:

$$z\coth z = \frac{2z}{e^{2z}-1} + \frac{2z}{2} = \sum_{n\ge 0} B_{2n}\frac{(2z)^{2n}}{(2n)!} = \sum_{n\ge 0} 4^n B_{2n}\frac{z^{2n}}{(2n)!}\,. \tag{6.85}$$

But there isn't much of a market for hyperbolic functions; people are more interested in the "real" functions of trigonometry. We can express ordinary trigonometric functions in terms of their hyperbolic cousins by using the rules

$$\sin z = -i \sinh iz, \qquad \cos z = \cosh iz; \tag{6.86}$$

the corresponding power series are

$$\sin z = \frac{z^1}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad \sinh z = \frac{z^1}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots;$$

$$\cos z = \frac{z^0}{0!} - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \qquad \cosh z = \frac{z^0}{0!} + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots.$$

Hence $\cot z = \cos z / \sin z = i \cosh iz / \sinh iz = i \coth iz$, and we have

I see, we get "real" functions by using imaginary numbers.

$$z \cot z = \sum_{n \geqslant 0} B_{2n} \frac{(2iz)^{2n}}{(2n)!} = \sum_{n \geqslant 0} (-4)^n B_{2n} \frac{z^{2n}}{(2n)!}. \tag{6.87}$$

Another remarkable formula for $z \cot z$ was found by Euler (exercise 73):

$$z \cot z = 1 - 2 \sum_{k \geqslant 1} \frac{z^2}{k^2\pi^2 - z^2}. \tag{6.88}$$

We can expand Euler's formula in powers of z^2, obtaining

$$\begin{aligned} z \cot z &= 1 - 2 \sum_{k \geqslant 1} \left(\frac{z^2}{k^2\pi^2} + \frac{z^4}{k^4\pi^4} + \frac{z^6}{k^6\pi^6} + \cdots \right) \\ &= 1 - 2 \left(\frac{z^2}{\pi^2} H_\infty^{(2)} + \frac{z^4}{\pi^4} H_\infty^{(4)} + \frac{z^6}{\pi^6} H_\infty^{(6)} + \cdots \right). \end{aligned}$$

Equating coefficients of z^{2n} with those in our other formula, (6.87), gives us an almost miraculous closed form for infinitely many infinite sums:

$$\zeta(2n) = H_\infty^{(2n)} = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n} B_{2n}}{(2n)!}, \qquad \text{integer } n > 0. \tag{6.89}$$

For example,

$$\zeta(2) = H_\infty^{(2)} = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \cdots = \pi^2 B_2 = \pi^2/6; \tag{6.90}$$

$$\zeta(4) = H_\infty^{(4)} = 1 + \tfrac{1}{16} + \tfrac{1}{81} + \cdots = -\pi^4 B_4/3 = \pi^4/90. \tag{6.91}$$

Formula (6.89) is not only a closed form for $H_\infty^{(2n)}$, it also tells us the approximate size of B_{2n}, since $H_\infty^{(2n)}$ is very near 1 when n is large. And it tells us that $(-1)^{n-1} B_{2n} > 0$ for all $n > 0$; thus the nonzero Bernoulli numbers alternate in sign.

And that's not all. Bernoulli numbers also appear in the coefficients of the tangent function,

Start Skipping

$$\tan z = \frac{\sin z}{\cos z} = \sum_{n\geqslant 0}(-1)^{n-1}4^n(4^n-1)B_{2n}\frac{z^{2n-1}}{(2n)!}, \qquad (6.92)$$

as well as other trigonometric functions (exercise 72). Formula (6.92) leads to another important fact about the Bernoulli numbers, namely that

$$T_{2n-1} = (-1)^{n-1}\frac{4^n(4^n-1)}{2n}B_{2n} \quad \text{is a positive integer.} \qquad (6.93)$$

We have, for example:

n	1	3	5	7	9	11	13
T_n	1	2	16	272	7936	353792	22368256

(The T's are called *tangent numbers.*)

One way to prove (6.93), following an idea of B. F. Logan, is to consider the power series

$$\frac{\sin z + x\cos z}{\cos z - x\sin z} = x + (1+x^2)z + (2x^3+2x)\frac{z^2}{2} + (6x^4+8x^2+2)\frac{z^3}{6} + \cdots$$
$$= \sum_{n\geqslant 0} T_n(x)\frac{z^n}{n!}, \qquad (6.94)$$

When $x = \tan w$, this is $\tan(z+w)$. Hence, by Taylor's theorem, the nth derivative of $\tan w$ is $T_n(\tan w)$.

where $T_n(x)$ is a polynomial in x; setting $x = 0$ gives $T_n(0) = T_n$, the nth tangent number. If we differentiate (6.94) with respect to x, we get

$$\frac{1}{(\cos z - x\sin z)^2} = \sum_{n\geqslant 0} T_n'(x)\frac{z^n}{n!};$$

but if we differentiate with respect to z, we get

$$\frac{1+x^2}{(\cos z - x\sin z)^2} = \sum_{n\geqslant 1} T_n(x)\frac{z^{n-1}}{(n-1)!} = \sum_{n\geqslant 0} T_{n+1}(x)\frac{z^n}{n!}.$$

(Try it — the cancellation is very pretty.) Therefore we have

$$T_{n+1}(x) = (1+x^2)T_n'(x), \qquad T_0(x) = x, \qquad (6.95)$$

a simple recurrence from which it follows that the coefficients of $T_n(x)$ are nonnegative integers. Moreover, we can easily prove that $T_n(x)$ has degree $n+1$, and that its coefficients are alternately zero and positive. Therefore $T_{2n+1}(0) = T_{2n+1}$ is a positive integer, as claimed in (6.93).

Recurrence (6.95) gives us a simple way to calculate Bernoulli numbers, via tangent numbers, using only simple operations on integers; by contrast, the defining recurrence (6.79) involves difficult arithmetic with fractions.

If we want to compute the sum of nth powers from a to $b-1$ instead of from 0 to $n-1$, the theory of Chapter 2 tells us that

$$\sum_{k=a}^{b-1} k^m = \sum\nolimits_a^b x^m \,\delta x = S_m(b) - S_m(a) . \tag{6.96}$$

This identity has interesting consequences when we consider negative values of k: We have

$$\sum_{k=-n+1}^{-1} k^m = (-1)^m \sum_{k=0}^{n-1} k^m , \qquad \text{when } m > 0,$$

hence

$$S_m(0) - S_m(-n+1) = (-1)^m \big(S_m(n) - S_m(0)\big) .$$

But $S_m(0) = 0$, so we have the identity

$$S_m(1-n) = (-1)^{m+1} S_m(n) , \qquad m > 0. \tag{6.97}$$

Johann Faulhaber implicitly used (6.97) in 1635 [119] to find simple formulas for $S_m(n)$ as polynomials in $n(n+1)/2$ when $m \leqslant 17$; see [222].)

Therefore $S_m(1) = 0$. If we write the polynomial $S_m(n)$ in factored form, it will always have the factors n and $(n-1)$, because it has the roots 0 and 1. In general, $S_m(n)$ is a polynomial of degree $m+1$ with leading term $\frac{1}{m+1} n^{m+1}$. Moreover, we can set $n = \frac{1}{2}$ in (6.97) to deduce that $S_m(\frac{1}{2}) = (-1)^{m+1} S_m(\frac{1}{2})$; if m is even, this makes $S_m(\frac{1}{2}) = 0$, so $(n - \frac{1}{2})$ will be an additional factor. These observations explain why we found the simple factorization

$$S_2(n) = \tfrac{1}{3} n (n - \tfrac{1}{2})(n-1)$$

in Chapter 2; we could have used such reasoning to deduce the value of $S_2(n)$ without calculating it! Furthermore, (6.97) implies that the polynomial with the remaining factors, $\hat{S}_m(n) = S_m(n)/(n - \frac{1}{2})$, always satisfies

$$\hat{S}_m(1-n) = \hat{S}_m(n) , \qquad m \text{ even}, \quad m > 0.$$

It follows that $S_m(n)$ can always be written in the factored form

$$S_m(n) = \begin{cases} \dfrac{1}{m+1} \displaystyle\prod_{k=1}^{\lceil m/2 \rceil} (n - \tfrac{1}{2} - \alpha_k)(n - \tfrac{1}{2} + \alpha_k) , & m \text{ odd}; \\[2ex] \dfrac{(n - \frac{1}{2})}{m+1} \displaystyle\prod_{k=1}^{m/2} (n - \tfrac{1}{2} - \alpha_k)(n - \tfrac{1}{2} + \alpha_k) , & m \text{ even}. \end{cases} \tag{6.98}$$

Here $\alpha_1 = \frac{1}{2}$, and $\alpha_2, \ldots, \alpha_{\lceil m/2 \rceil}$ are appropriate complex numbers whose values depend on m. For example,

$$\begin{aligned} S_3(n) &= n^2(n-1)^2/4\,; \\ S_4(n) &= n(n-\tfrac{1}{2})(n-1)(n-\tfrac{1}{2}+\sqrt{7/12}\,)(n-\tfrac{1}{2}-\sqrt{7/12}\,)/5\,; \\ S_5(n) &= n^2(n-1)^2(n-\tfrac{1}{2}+\sqrt{3/4}\,)(n-\tfrac{1}{2}-\sqrt{3/4}\,)/6\,; \\ S_6(n) &= n(n-\tfrac{1}{2})(n-1)(n-\tfrac{1}{2}+\alpha)(n-\tfrac{1}{2}-\alpha)(n-\tfrac{1}{2}+\overline{\alpha})(n-\tfrac{1}{2}-\overline{\alpha})\,, \\ &\qquad \text{where } \alpha = 2^{-5/2}\,3^{-1/2}\,31^{1/4}\bigl(\sqrt{\sqrt{31}+\sqrt{27}}+i\sqrt{\sqrt{31}-\sqrt{27}}\,\bigr). \end{aligned}$$

Stop Skipping

If m is odd and greater than 1, we have $B_m = 0$; hence $S_m(n)$ is divisible by n^2 (and by $(n-1)^2$). Otherwise the roots of $S_m(n)$ don't seem to obey a simple law.

Let's conclude our study of Bernoulli numbers by looking at how they relate to Stirling numbers. One way to compute $S_m(n)$ is to change ordinary powers to falling powers, since the falling powers have easy sums. After doing those easy sums we can convert back to ordinary powers:

$$\begin{aligned} S_m(n) = \sum_{k=0}^{n-1} k^m = \sum_{k=0}^{n-1}\sum_{j\geq 0}\left\{ {m \atop j} \right\} k^{\underline{j}} &= \sum_{j\geq 0}\left\{ {m \atop j} \right\}\sum_{k=0}^{n-1} k^{\underline{j}} \\ &= \sum_{j\geq 0}\left\{ {m \atop j} \right\}\frac{n^{\underline{j+1}}}{j+1} \\ &= \sum_{j\geq 0}\left\{ {m \atop j} \right\}\frac{1}{j+1}\sum_{k\geq 0}(-1)^{j+1-k}\left[{j+1 \atop k} \right] n^k . \end{aligned}$$

Therefore, equating coefficients with those in (6.78), we must have the identity

$$\sum_{j\geq 0}\left\{ {m \atop j} \right\}\left[{j+1 \atop k} \right]\frac{(-1)^{j+1-k}}{j+1} = \frac{1}{m+1}\binom{m+1}{k} B_{m+1-k}\,. \tag{6.99}$$

It would be nice to prove this relation directly, thereby discovering Bernoulli numbers in a new way. But the identities in Tables 264 or 265 don't give us any obvious handle on a proof by induction that the left-hand sum in (6.99) is a constant times $m^{\underline{k-1}}$. If $k = m+1$, the left-hand sum is just $\left\{ {m \atop m} \right\}\left[{m+1 \atop m+1} \right]/(m+1) = 1/(m+1)$, so that case is easy. And if $k = m$, the left-hand side sums to $\left\{ {m \atop m-1} \right\}\left[{m \atop m} \right]m^{-1} - \left\{ {m \atop m} \right\}\left[{m+1 \atop m} \right](m+1)^{-1} = \frac{1}{2}(m-1) - \frac{1}{2}m = -\frac{1}{2}$; so that case is pretty easy too. But if $k < m$, the left-hand sum looks hairy. Bernoulli would probably not have discovered his numbers if he had taken this route.