41. Antipodal Points and Maps: Borsuk's Theorem

The following four assertions are equivalent.

(1) *There is no continuous antipodal map* $S^n \to S^{n-1}$.

(2) *Every continuous map* $S^n \to \mathbb{R}^n$ *sends at least one pair of antipodal points into the same point.*

(3) *Every continuous odd map* $S^n \to \mathbb{R}^n$ *sends at least one point (and so at least one pair of antipodal points) into* 0, *the origin of* $\mathbb{R}^n$.

(4) *For every family of* $n+1$ *closed sets covering* S^n, *one of the sets contains a pair of antipodal points.*

Proof. In this proof we shall not be as economical as we could be: we start by proving that the first three assertions, concerning maps $f : S^n \to \mathbb{R}^n$, are best considered as a single result.

Note first that the implication (2) $\Rightarrow$ (1) is trivial and, in fact, so are the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). Indeed, the assertion that every continuous odd map $f : S^n \to \mathbb{R}^n$ sends at least one point x into 0 means precisely that f cannot map S^n into a set not containing 0; in particular, it cannot map S^n into S^{n-1}. Also, suppose that a map $f : S^n \to \mathbb{R}^n$ sends a pair of antipodal points $x, -x$ into the same point. If f is odd then $f(x) = f(-x) = -f(x)$, so $f(x) = f(-x) = 0$.

To complete the proof of the equivalence of the first three assertions, we show that (1) implies (2). Suppose that $f : S^n \to \mathbb{R}^n$ is a continuous map and $f(-x) \neq f(x)$ for every $x \in S^n$. Then the map sending x into the point of S^{n-1} in the direction of $f(x) - f(-x)$, i.e., the map $x \mapsto (f(x) - f(-x))/\|f(x) - f(-x)\|$, is a continuous antipodal map.

It remains to show that the first three equivalent assertions are also equivalent to the fourth. To this end, we shall show that (2) implies (4) and (4) implies (1).

(2) $\Rightarrow$ (4). Assume that (2) holds, and let $F_1, \ldots, F_{n+1}$ be closed sets covering S^n. Suppose that none of the sets $F_1, \ldots, F_n$ contains a pair of antipodal points. Setting

$$E_i = \{-x : x \in F_i\},$$

we find that none of the closed sets $E_1, \ldots, E_n$ contains a pair of antipodal points, i.e., $E_i \cap F_i = \emptyset$ for $i = 1, \ldots, n$.

Let $f_i(x) = d(x, E_i) - d(x, F_i)$, where $d(x, U)$ is the distance of a point x from a set U. Clearly, each function f_i is continuous, and $f_i(x) > 0 > f_i(-x)$ for $x \in F_i$ (and so $-x \in E_i$). Therefore $f = (f_1, \ldots, f_n)$ is a continuous map $S^n \to \mathbb{R}^n$, so (2) implies that $f(x_0) = f(-x_0)$ for some $x_0 \in S^n$. Then neither x_0 nor $-x_0$ is in $\bigcup_{i=1}^n F_i$. Consequently, both x_0 and $-x_0$ belong to F_{n+1}.

(4) $\Rightarrow$ (1). Assume (4), and let $f : S^n \to S^{n-1}$ be continuous. Let $S^{n-1} \subset \bigcup_{i=1}^{n+1} E_i$, with each E_i closed and diameter less than 2. It is easily seen that there

are such sets E_i. (For example, we may take for E_i the projection of the ith face of a regular simplex inscribed in S^{n-1} from the centre. In fact, we can do considerably better than this: writing $B_1, \ldots, B_n$ for the unit balls of radius 1 centred at the vertices of a regular simplex inscribed in S^{n-1}, if $\varepsilon > 0$ is small enough then the balls $E_i = (1-\varepsilon)B_i$ have diameter $2 - 2\varepsilon < 2$ and cover the entire unit ball B^n with boundary S^{n-1}.)

Setting $F_i = f^{-1}(E_i)$, $i = 1, \ldots, n+1$, we obtain a cover of S^n by the closed sets $F_1, \ldots, F_{n+1}$. By (4), one of the F_i contains antipodal points x_0 and $-x_0$, i.e., $f(x_0), f(-x_0) \in E_i$. As E_i does not contain a pair of antipodal points, $f(x_0)$ and $f(-x_0)$ are not antipodal, so the map f is not antipodal either. □

Notes. The assertions above are classical and very well known results in general and algebraic topology, and can be found in most books on the subject; see, e.g., Dugundji or Spanier. As we noted when posing the problem, these results follow easily from *Borsuk's theorem* that continuous antipodal maps $S^n \to S^n$ have odd degrees (see pp. 189–190 of Massey's book); in particular, they are not null-homotopic. In Borsuk's original paper on *three theorems on the n-dimensional Euclidean sphere*, the first theorem was this assertion, and the other two were assertions (2) and (4).

Any one of these equivalent assertions is usually referred to as *Borsuk's theorem* or *Borsuk's antipodal theorem*. In addition, somewhat surprisingly, assertion (2) is also called the *Borsuk–Ulam theorem*. The reason why Ulam is also given credit for this result is Borsuk's footnote in his paper that assertion (2) had been conjectured by Ulam. The fourth assertion was first proved by Lusternik and Schnirelmann (by a combinatorial method, without any appeal to algebraic topology!), and is frequently called the Lusternik–Schnirelmann–Borsuk theorem. The original formulation was in terms of arbitrary sets: the sphere S^n cannot be partitioned into $n+1$ sets, each of diameter strictly less than 2. Since the closure of a set has the same diameter as the set itself, as the very first step of our proof, we may replace the sets by their closures.

Lusternik and Schnirelmann were from the Soviet Union, so their names are transliterations from the Cyrillic; in English Schnirelmann is better spelled as Shnirelman (although the somewhat artificial spelling Shnirel'man is closer to the original Russian spelling). I have chosen to spell his name with sch and double n since he himself spelt it that way, even when writing in French, in particular, in the relevant pamphlet.

Needless to say, in these results S^n may be replaced by a space homeomorphic to S^n, with an involution coming from the antipodality in S^n. In the simplest case, instead of S^n, the unit sphere of ℓ_2^{n+1}, the $(n+1)$-dimensional Euclidean space, we may take the unit sphere of any $(n+1)$-dimensional normed space, e.g., the unit sphere

$$S(\ell_\infty^{n+1}) = \left\{ x = (x_i)_1^{n+1} : \max_{1 \le i \le n+1} |x_i| = 1 \right\},$$

with x and $-x$ antipodal.

For combinatorial proofs and generalizations of the Borsuk–Ulam–Lusternik–Schnirelmann results, see the papers of Tucker, Jaworowski and Moszyński, Rattray, Ky Fan, Weiss, Fréchet and Fan, Prescott and Su, and many others. For example, Jaworowski, Moszyński and Rattray prove that if $f : S^n \to S^n$ is a continuous map such that $f(-x) \neq -f(x)$ for every $x \in S^n$, then there is a point $x_0 \in S^n$ such that $f(-x_0) = f(x_0)$.

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117. Covering the Sphere

If the n-dimensional unit sphere $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$ is covered by $n + 1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodal points, i.e., points x and $-x$.

Proof. This assertion is just a slight extension of the classical LSB theorem of Lusternik and Schnirelmann (1930) and Borsuk (1933) we presented in Problem 41, in which the $n + 1$ sets are taken to be closed. This extension was noted by Greene (2003).

Let us see first that the case of open cover is an easy consequence of the original LSB theorem. Indeed, suppose that $S^n = \bigcup_{i=1}^{n+1} U_i$, with each U_i open. Let $\lambda > 0$ be a Lebesgue number of this cover, i.e., such that for every point $\mathbf{x} \in S^n$ there is an index $i = i(\mathbf{x})$ such that the closed ball $B_{\mathbf{x}} = \overline{B}(\mathbf{x}, \lambda)$ of centre $\mathbf{x}$ and radius λ is contained in U_i. Since S^n is compact, the closed cover $S^n = \bigcup_{\mathbf{x} \in S^n} B_{\mathbf{x}}$ contains a finite subcover. For $1 \le j \le n + 1$, let V_j be the union of the closed balls in this finite subcover that are contained in U_j. In particular, V_j is a closed subset of U_j and $\bigcup_{j=1}^{n+1} V_j = S^n$. Hence, by the LSB theorem, for some j the set $V_j \subset U_j$ contains a pair of antipodal points, as claimed.

Let us turn to the general case. Let $V_1, \ldots, V_m$ be open subsets of S^n and $U_{m+1}, \ldots, U_{n+1}$ closed subsets with $\bigcup_{i=1}^{m} V_i \cup \bigcup_{j=m+1}^{n+1} U_j = S^n$. Suppose that none of these $n + 1$ sets contains a pair of antipodal points. Since each V_i is compact, for some $\varepsilon > 0$ each of them has diameter less than $2 - \varepsilon$. For $1 \le i \le m$, let U_i be the open subset of S^n consisting of the points at distance less than $\varepsilon/2$ from V_i. Since $S^n = \bigcup_{i=1}^{n+1} U_i$ is an open cover of S^n, as we have just seen, for some i the set U_i contains a pair of antipodal points. Since $U_1, \ldots, U_m$ do not contain antipodal pairs, this index i is greater than m, i.e., one of the original sets $U_{m+1}, \ldots, U_{n+1}$ contains a pair of antipodal points, completing our proof. □

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