Some irrational numbers

Chapter 6

"π is irrational"

Charles Hermite

This was already conjectured by Aristotle, when he claimed that diameter and circumference of a circle are not commensurable. The first proof of this fundamental fact was given by Johann Heinrich Lambert in 1766. Our Book Proof is due to Ivan Niven, 1947: an extremely elegant one-page proof that needs only elementary calculus. Its idea is powerful, and quite a bit more can be derived from it, as was shown by Iwamoto and Koksma, respectively:

- π^2 is irrational (this is a stronger result!) and
- e^r is irrational for rational $r \neq 0$.

Niven's method does, however, have its roots and predecessors: it can be traced back to the classical paper by Charles Hermite from 1873 which first established that e is transcendental, that is, that e is not a zero of a polynomial with rational coefficients.

It is easy to see that $e = \sum_{k\geq 0} \frac{1}{k!}$ is irrational. In fact, from $e = \frac{a}{b}$ (for integers $a, b > 0$) we would get that

$$N := n!\Big(e - \sum_{k=0}^{n} \frac{1}{k!}\Big)$$

is an integer for $n \geq b$, since then $n!e$ and $\frac{n!}{k!}$ (for $0 \leq k \leq n$) are integers. However, this integer can also be written as

$$N = \sum_{k\geq n+1} \frac{n!}{k!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots .$$

Thus N can be compared to a geometric series, yielding

$$0 < N < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots = \frac{1}{n},$$

which is absurd, since N is an integer.

This trick, however, isn't even good enough to prove that e^2 is irrational (which is a stronger statement). For this we use a different method — essentially due to Charles Hermite — whose key is the following simple lemma.

Lemma. *For some fixed $n \geq 1$, let*

$$f(x) = \frac{x^n(1-x)^n}{n!}.$$

(i) *The function $f(x)$ is a polynomial of the form $f(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$, where the coefficients c_i are integers.*

(ii) *For $0 < x < 1$ we have $0 < f(x) < \frac{1}{n!}$.*

(iii) *The derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers for all $k \geq 0$.*

■ **Proof.** Parts (i) and (ii) are clear.
For (iii) note that by (i) the k-th derivative $f^{(k)}$ vanishes at $x = 0$ unless $n \leq k \leq 2n$, and in this range $f^{(k)}(0) = \frac{k!}{n!}c_k$ is an integer. From $f(x) = f(1-x)$ we get $f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$ for all x, and hence $f^{(k)}(1) = (-1)^k f^{(k)}(0)$, which is an integer. □

Theorem 1. *e^r is irrational for every $r \in \mathbb{Q}\backslash\{0\}$.*

■ **Proof.** It suffices to show that e^s cannot be rational for a positive integer s (if $e^{\frac{s}{t}}$ were rational, then $\left(e^{\frac{s}{t}}\right)^t = e^s$ would be rational, too). Assume that $e^s = \frac{a}{b}$ for integers $a, b > 0$, and let n be so large that $n! > as^{2n+1}$. Put

The estimate $n! > e(\frac{n}{e})^n$ yields an explicit n that is "large enough."

$$F(x) := s^{2n}f(x) - s^{2n-1}f'(x) + s^{2n-2}f''(x) \mp \ldots + f^{(2n)}(x),$$

where $f(x)$ is the function of the lemma. $F(x)$ may also be written as

$$F(x) = s^{2n}f(x) - s^{2n-1}f'(x) + s^{2n-2}f''(x) \mp \ldots,$$

since the higher derivatives $f^{(k)}(x)$, for $k > 2n$, vanish. From this we see that the polynomial $F(x)$ satisfies the identity

$$F'(x) = -s\,F(x) + s^{2n+1}f(x).$$

Thus differentiation yields

$$\frac{d}{dx}\left[e^{sx}F(x)\right] = se^{sx}F(x) + e^{sx}F'(x) = s^{2n+1}e^{sx}f(x)$$

and hence

$$N := b\int_0^1 s^{2n+1}e^{sx}f(x)dx = b\left[e^{sx}F(x)\right]_0^1 = aF(1) - bF(0).$$

This is an integer, since part (iii) of the lemma implies that $F(0)$ and $F(1)$ are integers. However, part (ii) of the lemma yields estimates for the size of N from below and from above,

$$0 < N = b\int_0^1 s^{2n+1}e^{sx}f(x)dx < bs^{2n+1}e^s\frac{1}{n!} = \frac{as^{2n+1}}{n!} < 1,$$

which shows that N cannot be an integer: contradiction. □

Now that this trick was so successful, we use it once more.

Theorem 2. π^2 *is irrational.*

■ **Proof.** Assume that $\pi^2 = \frac{a}{b}$ for integers $a, b > 0$. We now use the polynomial

$$F(x) := b^n\Big(\pi^{2n}f(x) - \pi^{2n-2}f^{(2)}(x) + \pi^{2n-4}f^{(4)}(x) \mp \dots\Big),$$

which satisfies $F''(x) = -\pi^2F(x) + b^n\pi^{2n+2}f(x)$.

From part (iii) of the lemma we get that $F(0)$ and $F(1)$ are integers. Elementary differentiation rules yield

$$\begin{aligned}\frac{d}{dx}\big[F'(x)\sin\pi x - \pi F(x)\cos\pi x\big] &= \big(F''(x) + \pi^2F(x)\big)\sin\pi x\\ &= b^n\pi^{2n+2}f(x)\sin\pi x\\ &= \pi^2a^nf(x)\sin\pi x,\end{aligned}$$

and thus we obtain

$$\begin{aligned}N := \pi\int_0^1 a^nf(x)\sin\pi x\,dx &= \Big[\frac{1}{\pi}F'(x)\sin\pi x - F(x)\cos\pi x\Big]_0^1\\ &= F(0) + F(1),\end{aligned}$$

which is an integer. Furthermore N is positive since it is defined as the integral of a function that is positive (except on the boundary). However, if we choose n so large that $\frac{\pi a^n}{n!} < 1$, then from part (ii) of the lemma we obtain

$$0 < N = \pi\int_0^1 a^nf(x)\sin\pi x\,dx < \frac{\pi a^n}{n!} < 1,$$

a contradiction. □

π is not rational, but it does have "good approximations" by rationals — some of these were known since antiquity:

$$\begin{aligned}\tfrac{22}{7} &= 3.142857142857\ldots\\ \tfrac{355}{113} &= 3.141592920353\ldots\\ \tfrac{104348}{33215} &= 3.141592653921\ldots\\ \pi &= 3.141592653589\ldots\end{aligned}$$

This last theorem, together with the following classical result by Euler, proves that the value

$$\zeta(2) = \sum_{n\geq 1}\frac{1}{n^2}$$

of Riemann's zeta function is irrational (see the appendix on page 34).

Theorem 3. $\displaystyle\sum_{n\geq1}\frac{1}{n^2} = \frac{\pi^2}{6}.$

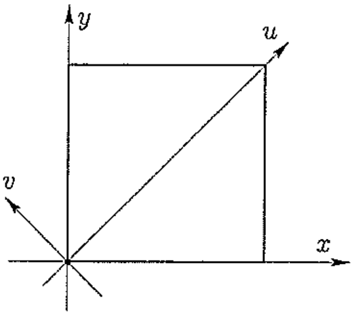
■ **Proof.** The following proof — due to Tom Apostol — consists in two different evaluations of the double integral

$$I := \int_0^1\!\!\int_0^1 \frac{1}{1-xy}\,dx\,dy.$$

For the first one, we expand $\frac{1}{1-xy}$ as a geometric series, decompose the summands as products, and integrate effortlessly:

$$\begin{aligned} I &= \int_0^1\int_0^1 \sum_{n\geq 0}(xy)^n\,dx\,dy = \sum_{n\geq 0}\int_0^1\int_0^1 x^n y^n\,dx\,dy \\ &= \sum_{n\geq 0}\Big(\int_0^1 x^n dx\Big)\Big(\int_0^1 y^n dy\Big) = \sum_{n\geq 0}\frac{1}{n+1}\,\frac{1}{n+1} \\ &= \sum_{n\geq 0}\frac{1}{(n+1)^2} = \sum_{n\geq 1}\frac{1}{n^2} = \zeta(2). \end{aligned}$$

This evaluation also shows that the double integral (over a positive function with a pole at $x = y = 1$) is finite. Note that the computation is also easy and straightforward if we read it backwards — thus the evaluation of $\zeta(2)$ leads one to the double integral I.



The second way to evaluate I makes a change of coordinates: a rotation by 45° leads to coordinates

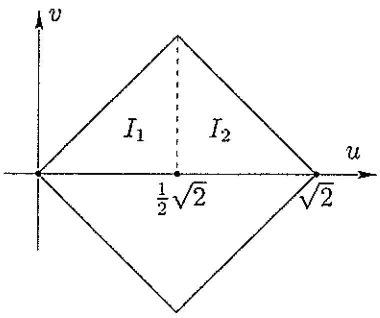
$$u = \frac{y+x}{\sqrt{2}} \quad\text{and}\quad v = \frac{y-x}{\sqrt{2}},$$
$$x = \frac{u-v}{\sqrt{2}} \quad\text{and}\quad y = \frac{u+v}{\sqrt{2}}.$$

Substitution of these new coordinates yields

$$1 - xy = 1 - \frac{u^2 - v^2}{2}$$

and thus

$$\frac{1}{1-xy} = \frac{2}{2-u^2+v^2}.$$



The new domain of integration, and the function to be integrated, are symmetric with respect to the u-axis, so we just need to compute the integral over the upper half domain, which we split into two parts in the most natural way:

$$I = 4\int_0^{\frac{1}{2}\sqrt{2}}\Big(\int_0^{u}\frac{dv}{2-u^2+v^2}\Big)du + 4\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}}\Big(\int_0^{\sqrt{2}-u}\frac{dv}{2-u^2+v^2}\Big)du.$$

Using $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$, this becomes

$$\begin{aligned} I = \quad & 4\int_0^{\frac{1}{2}\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\arctan\Big(\frac{u}{\sqrt{2-u^2}}\Big)\,du \\ + \quad & 4\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\arctan\Big(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\Big)\,du. \end{aligned}$$

Two simple trigonometric substitutions now complete the job. For the first integral, we put $u = \sqrt{2}\sin\theta$. The interval $0 \le u \le \frac{1}{2}\sqrt{2}$ corresponds to $0 \le \theta \le \frac{\pi}{6}$. We compute $du = \sqrt{2}\cos\theta\, d\theta$ and $\sqrt{2-u^2} = \sqrt{2}\sqrt{1-\sin^2\theta} = \sqrt{2}\cos\theta$, and thus

$$\begin{aligned}
4\int_0^{\frac{1}{2}\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}}\, du &= \\
4\int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{2}\cos\theta} \arctan\Big(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\Big)\sqrt{2}\cos\theta\, d\theta &= \\
4\int_0^{\frac{\pi}{6}} \theta\, d\theta \;=\; 4\cdot\frac{1}{2}\Big(\frac{\pi}{6}\Big)^2 \;=\; \frac{1}{3}\frac{\pi^2}{6}.
\end{aligned}$$

For the second integral we use $u = \sqrt{2}\cos 2\theta$. Here $\frac{1}{2}\sqrt{2} \le u \le \sqrt{2}$ translates into $\frac{\pi}{6} \ge \theta \ge 0$. We get $du = -2\sqrt{2}\sin 2\theta\, d\theta$, with

$$\begin{aligned}
\sqrt{2-u^2} &= \sqrt{2}\sqrt{1-\cos^2 2\theta} \;=\; \sqrt{2}\sin 2\theta \;=\; 2\sqrt{2}\cos\theta\sin\theta \\
\sqrt{2}-u &= \sqrt{2}(1-\cos 2\theta) \;=\; 2\sqrt{2}\sin^2\theta
\end{aligned}$$

and thus

$$\begin{aligned}
4\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\, du &= \\
4\int_{\frac{\pi}{6}}^{0} \frac{1}{\sqrt{2}\sin 2\theta} \arctan\Big(\frac{2\sqrt{2}\sin^2\theta}{2\sqrt{2}\cos\theta\sin\theta}\Big)(-2\sqrt{2})\sin 2\theta\, d\theta &= \\
4\int_0^{\frac{\pi}{6}} 2\theta\, d\theta \;=\; 4\Big(\frac{\pi}{6}\Big)^2 \;=\; \frac{2}{3}\frac{\pi^2}{6}.
\end{aligned}$$

Putting both integrals together, we obtain

$$I \;=\; \frac{1}{3}\frac{\pi^2}{6} + \frac{2}{3}\frac{\pi^2}{6} \;=\; \frac{\pi^2}{6}. \qquad \square$$

Wasn't that marvelous? Well, compare it to the following, entirely different proof for $\sum_{n\ge 1}\frac{1}{n^2} = \frac{\pi^2}{6}$. It is completely elementary and it even provides us with an estimate for the speed of convergence of $\sum_{n=1}^{m}\frac{1}{n^2}$ to $\frac{\pi^2}{6}$. Its origin is not quite clear. According to John Scholes the "proof was common knowledge when I was an undergraduate at Cambridge, England in the late 1960s." It appeared in the journal *Eureka* of the Cambridge undergraduate maths society, the Archimedeans, in 1982 [8], there attributed to John Scholes. But as Scholes writes: "I probably heard the proof, at least in outline (one was expected to fill in details oneself) from Peter Swinnerton-Dyer, but he would probably claim to have heard it from someone else."

For $m = 1,2,3$ this yields
$\cot^2\frac{\pi}{3} = \frac{1}{3}$
$\cot^2\frac{\pi}{5} + \cot^2\frac{2\pi}{5} = 2$
$\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7} = 5$

■ **Proof.** The first step is to establish a remarkable relation between values of the (squared) cotangent function. Namely, for all $m \geq 1$ one has

$$\cot^2\left(\tfrac{\pi}{2m+1}\right) + \cot^2\left(\tfrac{2\pi}{2m+1}\right) + \ldots + \cot^2\left(\tfrac{m\pi}{2m+1}\right) \;=\; \tfrac{2m(2m-1)}{6}. \quad (1)$$

To establish this, we start with the relation

$$\cos nx + i\sin nx \;=\; (\cos x + i\sin x)^n$$

and take its imaginary part, which is

$$\sin nx \;=\; \binom{n}{1}\sin x\cos^{n-1}x - \binom{n}{3}\sin^3 x\cos^{n-3}x \pm \ldots \quad (2)$$

Now we let $n = 2m+1$, while for x we will consider the m different values $x = \frac{r\pi}{2m+1}$, for $r = 1,2,\ldots,m$. For each of these values we have $nx = r\pi$, and thus $\sin nx = 0$, while $0 < x < \frac{\pi}{2}$ implies that for $\sin x$ we get m distinct positive values.

In particular, we can divide (2) by $\sin^n x$, which yields

$$0 \;=\; \binom{n}{1}\cot^{n-1}x - \binom{n}{3}\cot^{n-3}x \pm \ldots,$$

that is,

$$0 \;=\; \binom{2m+1}{1}\cot^{2m}x - \binom{2m+1}{3}\cot^{2m-2}x \pm \ldots$$

for each of the m distinct values of x. Thus for the polynomial of degree m

$$p(t) \;:=\; \binom{2m+1}{1}t^m - \binom{2m+1}{3}t^{m-1} \pm \ldots + (-1)^m\binom{2m+1}{2m+1}$$

we know m distinct roots

$$a_r \;=\; \cot^2\left(\tfrac{r\pi}{2m+1}\right) \quad \text{for} \quad r = 1,2,\ldots,m.$$

Hence the polynomial coincides with

$$p(t) \;=\; \binom{2m+1}{1}\left(t - \cot^2\left(\tfrac{\pi}{2m+1}\right)\right)\cdot\ldots\cdot\left(t - \cot^2\left(\tfrac{m\pi}{2m+1}\right)\right).$$

Comparison of coefficients:
If $p(t) = c(t-a_1)\cdots(t-a_m)$, then the coefficient of t^{m-1} is $-c(a_1+\ldots+a_m)$.

Comparison of the coefficients of t^{m-1} in $p(t)$ now yields that the sum of the roots is

$$a_1 + \ldots + a_r \;=\; \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} \;=\; \tfrac{2m(2m-1)}{6},$$

which proves (1).

We also need a second identity, of the the same type,

$$\csc^2\left(\tfrac{\pi}{2m+1}\right) + \csc^2\left(\tfrac{2\pi}{2m+1}\right) + \ldots + \csc^2\left(\tfrac{m\pi}{2m+1}\right) = \tfrac{2m(2m+2)}{6}, \qquad (3)$$

for the cosecant function $\csc x = \frac{1}{\sin x}$. But

$$\csc^2 x = \frac{1}{\sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \cot^2 x + 1,$$

so we can derive (3) from (1) by adding m to both sides of the equation.

Now the stage is set, and everything falls into place. We use that in the range $0 < y < \frac{\pi}{2}$ we have

$$0 \;<\; \sin y \;<\; y \;<\; \tan y,$$

$0 < a < b < c$
implies
$0 < \frac{1}{c} < \frac{1}{b} < \frac{1}{a}$

and thus

$$0 \;<\; \cot y \;<\; \frac{1}{y} \;<\; \csc y,$$

which implies

$$\cot^2 y \;<\; \frac{1}{y^2} \;<\; \csc^2 y.$$

Now we take this double inequality, apply it to each of the m distinct values of x, and add the results. Using (1) for the left-hand side, and (3) for the right-hand side, we obtain

$$\tfrac{2m(2m-1)}{6} \;<\; \left(\tfrac{2m+1}{\pi}\right)^2 + \left(\tfrac{2m+1}{2\pi}\right)^2 + \ldots + \left(\tfrac{2m+1}{m\pi}\right)^2 \;<\; \tfrac{2m(2m+2)}{6},$$

that is,

$$\tfrac{\pi^2}{6}\,\tfrac{2m}{2m+1}\,\tfrac{2m-1}{2m+1} \;<\; \tfrac{1}{1^2} + \tfrac{1}{2^2} + \ldots + \tfrac{1}{m^2} \;<\; \tfrac{\pi^2}{6}\,\tfrac{2m}{2m+1}\,\tfrac{2m+2}{2m+1}.$$

Both the left-hand and the right-hand side converge to $\frac{\pi^2}{6}$ for $m \longrightarrow \infty$: end of proof. □

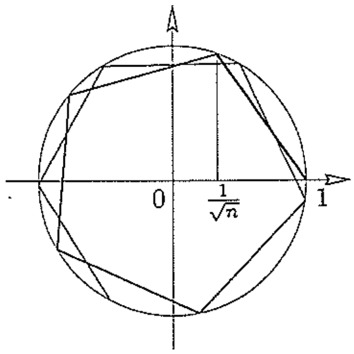
Here comes our final irrationality result.

Theorem 4. *For every odd integer $n \geq 3$, the number*

$$A(n) \;:=\; \frac{1}{\pi}\arccos\left(\frac{1}{\sqrt{n}}\right)$$

is irrational.

We will need this result for Hilbert's third problem (see Chapter 7) in the cases $n = 3$ and $n = 9$. For $n = 2$ and $n = 4$ we have $A(2) = \frac{1}{4}$ and $A(4) = \frac{1}{3}$, so the restriction to odd integers is essential. These values are easily derived by appealing to the diagram in the margin, in which the statement "$\frac{1}{\pi}\arccos\left(\frac{1}{\sqrt{n}}\right)$ is irrational" is equivalent to saying that the polygonal arc constructed from $\frac{1}{\sqrt{n}}$, all of whose chords have the same length, never closes into itself.



We leave it as an exercise for the reader to show that $A(n)$ is rational *only* for $n \in \{1, 2, 4\}$. For that, distinguish the cases when $n = 2^r$, and when n is not a power of 2.

■ **Proof.** We use the addition theorem

$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$

from elementary trigonometry, which for $\alpha = (k+1)\varphi$ and $\beta = (k-1)\varphi$ yields

$$\cos(k+1)\varphi \;=\; 2\cos\varphi\,\cos k\varphi \;-\; \cos(k-1)\varphi. \qquad (4)$$

For the angle $\varphi_n = \arccos\left(\frac{1}{\sqrt{n}}\right)$, which is defined by $\cos\varphi_n = \frac{1}{\sqrt{n}}$ and $0 \le \varphi_n \le \pi$, this yields representations of the form

$$\cos k\varphi_n \;=\; \frac{A_k}{\sqrt{n}^k},$$

where A_k is an integer that is not divisible by n, for all $k \ge 0$. In fact, we have such a representation for $k = 0, 1$ with $A_0 = A_1 = 1$, and by induction on k using (4) we get for $k \ge 1$

$$\cos(k+1)\varphi_n \;=\; 2\frac{1}{\sqrt{n}}\frac{A_k}{\sqrt{n}^k} - \frac{A_{k-1}}{\sqrt{n}^{k-1}} \;=\; \frac{2A_k - nA_{k-1}}{\sqrt{n}^{k+1}}.$$

Thus we obtain $A_{k+1} = 2A_k - nA_{k-1}$. If $n \ge 3$ is odd, and A_k is not divisible by n, then we find that A_{k+1} cannot be divisible by n, either.

Now assume that

$$A(n) \;=\; \frac{1}{\pi}\varphi_n \;=\; \frac{k}{\ell}$$

is rational (with integers $k, \ell > 0$). Then $\ell\varphi_n = k\pi$ yields

$$\pm 1 \;=\; \cos k\pi \;=\; \frac{A_\ell}{\sqrt{n}^\ell}.$$

Thus $\sqrt{n}^\ell = \pm A_\ell$ is an integer, with $\ell \ge 2$, and hence $n \mid \sqrt{n}^\ell$. With $\sqrt{n}^\ell \mid A_\ell$ we find that n divides A_ℓ, a contradiction. □

Appendix: The Riemann zeta function

The *Riemann zeta function* $\zeta(s)$ is defined for real $s > 1$ by

$$\zeta(s) \;:=\; \sum_{n\ge 1}\frac{1}{n^s}.$$

Our estimates for H_n (see page 10) imply that the series for $\zeta(1)$ diverges, but for any real $s > 1$ it does converge. The zeta function has a canonical continuation to the entire complex plane (with one simple pole at $s = 1$), which can be constructed using power series expansions. The resulting complex function is of utmost importance for the theory of prime numbers. Let us mention three diverse connections:

(1) The remarkable identity

$$\zeta(s) \;=\; \prod_p \frac{1}{1-p^{-s}}$$

due to Euler encodes that every natural number has a unique (!) decomposition into prime factors; using this basic fact, it is a simple consequence of the geometric series expansion

$$\frac{1}{1-p^{-s}} \;=\; 1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\ldots$$

(2) The location of the complex zeros of the zeta function is the subject of the "Riemann hypothesis": one of the most famous and important unresolved conjectures in all of mathematics. It claims that all the non-trivial zeros $s \in \mathbb{C}$ of the zeta function satisfy

$$\mathrm{Re}(s) \;=\; \frac{1}{2}.$$

(The zeta function vanishes at all the negative even integers, which are referred to as the "trivial zeros.")

Very recently, Jeff Lagarias showed that, surprisingly, the Riemann hypothesis is equivalent to the following elementary statement: For all $n \geq 1$,

$$\sum_{d\,|\,n} d \;\leq\; H_n \;+\; \exp(H_n)\log(H_n),$$

where H_n is again the n-th harmonic number, with equality only for $n = 1$.

(3) It has been known for a long time that $\zeta(s)$ is a rational multiple of π^s, and hence irrational, if s is an *even* integer $s \geq 2$, see Chapter 19. Here we presented two Book Proofs of $\zeta(2) = \frac{\pi^2}{6}$, a famous identity of Euler from 1734. In contrast, the irrationality of $\zeta(3)$ was proved by Roger Apéry only in 1979 (see the marvelous review in [7]). Despite considerable effort the picture is rather incomplete about $\zeta(s)$ for the other odd integers, $s = 2t+1 \geq 5$. The latest mathematical news about this, a paper by Rivoal, implies that infinitely many of the values $\zeta(2t+1)$ are irrational.

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