Table 272 Stirling convolution formulas.

$$rs\sum_{k=0}^{n}\sigma_k(r+tk)\,\sigma_{n-k}(s+t(n-k)) \;=\; (r+s)\sigma_n(r+s+tn) \qquad (6.46)$$

$$s\sum_{k=0}^{n}k\sigma_k(r+tk)\,\sigma_{n-k}(s+t(n-k)) \;=\; n\sigma_n(r+s+tn) \qquad (6.47)$$

$$\left\{ n \atop m \right\} \;=\; (-1)^{n-m+1}\frac{n!}{(m-1)!}\sigma_{n-m}(-m) \qquad (6.48)$$

$$\left[n \atop m \right] \;=\; \frac{n!}{(m-1)!}\sigma_{n-m}(n) \qquad (6.49)$$

It turns out that these polynomials satisfy two very pretty identities:

$$\left(\frac{ze^z}{e^z-1}\right)^x \;=\; x\sum_{n\geqslant 0}\sigma_n(x)\,z^n\,; \qquad (6.50)$$

$$\left(\frac{1}{z}\ln\frac{1}{1-z}\right)^x \;=\; x\sum_{n\geqslant 0}\sigma_n(x+n)\,z^n\,. \qquad (6.51)$$

And in general, if $\mathcal{S}_t(z)$ is the power series that satisfies

$$\ln\bigl(1-z\mathcal{S}_t(z)^{t-1}\bigr) \;=\; -z\mathcal{S}_t(z)^t\,, \qquad (6.52)$$

then

$$\mathcal{S}_t(z)^x \;=\; x\sum_{n\geqslant 0}\sigma_n(x+tn)\,z^n\,. \qquad (6.53)$$

Therefore we can obtain general convolution formulas for Stirling numbers, as we did for binomial coefficients in Table 202; the results appear in Table 272. When a sum of Stirling numbers doesn't fit the identities of Table 264 or 265, Table 272 may be just the ticket. (An example appears later in this chapter, following equation (6.100). Exercise 7.19 discusses the general principles of convolutions based on identities like (6.50) and (6.53).)

6.3 HARMONIC NUMBERS

It's time now to take a closer look at harmonic numbers, which we first met back in Chapter 2:

$$H_n \;=\; 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \;=\; \sum_{k=1}^{n}\frac{1}{k}\,, \qquad \text{integer } n\geqslant 0. \qquad (6.54)$$

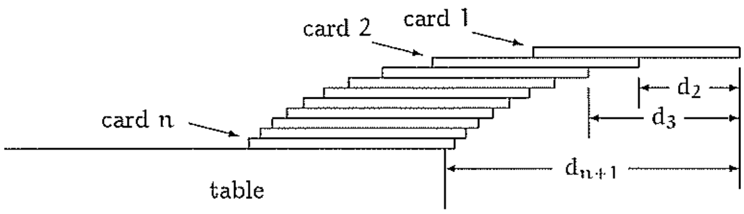
These numbers appear so often in the analysis of algorithms that computer scientists need a special notation for them. We use H_n, the 'H' standing for "harmonic," since a tone of wavelength $1/n$ is called the nth harmonic of a tone whose wavelength is 1. The first few values look like this:

n	0	1	2	3	4	5	6	7	8	9	10
H_n	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$

Exercise 21 shows that H_n is never an integer when $n > 1$.

Here's a card trick, based on an idea by R. T. Sharp [325], that illustrates how the harmonic numbers arise naturally in simple situations. Given n cards and a table, we'd like to create the largest possible overhang by stacking the cards up over the table's edge, subject to the laws of gravity:

This must be Table 273.



To define the problem a bit more, we require the edges of the cards to be parallel to the edge of the table; otherwise we could increase the overhang by rotating the cards so that their corners stick out a little farther. And to make the answer simpler, we assume that each card is 2 units long.

With one card, we get maximum overhang when its center of gravity is just above the edge of the table. The center of gravity is in the middle of the card, so we can create half a cardlength, or 1 unit, of overhang.

With two cards, it's not hard to convince ourselves that we get maximum overhang when the center of gravity of the top card is just above the edge of the second card, and the center of gravity of both cards combined is just above the edge of the table. The joint center of gravity of two cards will be in the middle of their common part, so we are able to achieve an additional half unit of overhang.

This pattern suggests a general method, where we place cards so that the center of gravity of the top k cards lies just above the edge of the $k+1$st card (which supports those top k). The table plays the role of the $n+1$st card. To express this condition algebraically, we can let d_k be the distance from the extreme edge of the top card to the corresponding edge of the kth card from the top. Then $d_1 = 0$, and we want to make d_{k+1} the center of gravity of the first k cards:

$$d_{k+1} = \frac{(d_1+1)+(d_2+1)+\cdots+(d_k+1)}{k}, \quad \text{for } 1 \leqslant k \leqslant n. \qquad (6.55)$$

(The center of gravity of k objects, having respective weights $w_1, \ldots, w_k$ and having respective centers of gravity at positions $p_1, \ldots p_k$, is at position $(w_1p_1 + \cdots + w_kp_k)/(w_1 + \cdots + w_k)$.) We can rewrite this recurrence in two equivalent forms

$$\begin{aligned} kd_{k+1} &= k + d_1 + \cdots + d_{k-1} + d_k\,, & k \geqslant 0; \\ (k-1)d_k &= k - 1 + d_1 + \cdots + d_{k-1}\,, & k \geqslant 1. \end{aligned}$$

Subtracting these equations tells us that

$$kd_{k+1} - (k-1)d_k = 1 + d_k\,, \qquad k \geqslant 1;$$

hence $d_{k+1} = d_k + 1/k$. The second card will be offset half a unit past the third, which is a third of a unit past the fourth, and so on. The general formula

$$d_{k+1} = H_k \tag{6.56}$$

follows by induction, and if we set $k = n$ we get $d_{n+1} = H_n$ as the total overhang when n cards are stacked as described.

Could we achieve greater overhang by holding back, not pushing each card to an extreme position but storing up "potential gravitational energy" for a later advance? No; any well-balanced card placement has

$$d_{k+1} \leqslant \frac{(1+d_1) + (1+d_2) + \cdots + (1+d_k)}{k}\,, \qquad 1 \leqslant k \leqslant n.$$

Furthermore $d_1 = 0$. It follows by induction that $d_{k+1} \leqslant H_k$.

Notice that it doesn't take too many cards for the top one to be completely past the edge of the table. We need an overhang of more than one cardlength, which is 2 units. The first harmonic number to exceed 2 is $H_4 = \frac{25}{12}$, so we need only four cards.

And with 52 cards we have an H_{52}-unit overhang, which turns out to be $H_{52}/2 \approx 2.27$ cardlengths. (We will soon learn a formula that tells us how to compute an approximate value of H_n for large n without adding up a whole bunch of fractions.)

Anyone who actually tries to achieve this maximum overhang with 52 cards is probably not dealing with a full deck—or maybe he's a real joker.

An amusing problem called the "worm on the rubber band" shows harmonic numbers in another guise. A slow but persistent worm, W, starts at one end of a meter-long rubber band and crawls one centimeter per minute toward the other end. At the end of each minute, an equally persistent keeper of the band, K, whose sole purpose in life is to frustrate W, stretches it one meter. Thus after one minute of crawling, W is 1 centimeter from the start and 99 from the finish; then K stretches it one meter. During the stretching operation W maintains his relative position, 1% from the start and 99% from the finish; so W is now 2 cm from the starting point and 198 cm from the goal. After W crawls for another minute the score is 3 cm traveled and 197 to go; but K stretches, and the distances become 4.5 and 295.5. And so on. Does the worm ever reach the finish? He keeps moving, but the goal seems to move away even faster. (We're assuming an infinite longevity for K and W, an infinite elasticity of the band, and an infinitely tiny worm.)

Metric units make this problem more scientific.

Let's write down some formulas. When K stretches the rubber band, the fraction of it that W has crawled stays the same. Thus he crawls 1/100th of it the first minute, 1/200th the second, 1/300th the third, and so on. After n minutes the fraction of the band that he's crawled is

$$\frac{1}{100}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) = \frac{H_n}{100}. \tag{6.57}$$

So he reaches the finish if H_n ever surpasses 100.

We'll see how to estimate H_n for large n soon; for now, let's simply check our analysis by considering how "Superworm" would perform in the same situation. Superworm, unlike W, can crawl 50 cm per minute; so she will crawl $H_n/2$ of the band length after n minutes, according to the argument we just gave. If our reasoning is correct, Superworm should finish before n reaches 4, since $H_4 > 2$. And yes, a simple calculation shows that Superworm has only $33\frac{1}{3}$ cm left to travel after three minutes have elapsed. She finishes in 3 minutes and 40 seconds flat.

A flatworm, eh?

Harmonic numbers appear also in Stirling's triangle. Let's try to find a closed form for $\left[{n \atop 2}\right]$, the number of permutations of n objects that have exactly two cycles. Recurrence (6.8) tells us that

$$\begin{aligned}\left[{n+1 \atop 2}\right] &= n\left[{n \atop 2}\right] + \left[{n \atop 1}\right]\\ &= n\left[{n \atop 2}\right] + (n-1)!\,, \qquad \text{if } n > 0;\end{aligned}$$

and this recurrence is a natural candidate for the summation factor technique of Chapter 2:

$$\frac{1}{n!}\left[{n+1 \atop 2}\right] = \frac{1}{(n-1)!}\left[{n \atop 2}\right] + \frac{1}{n}.$$

Unfolding this recurrence tells us that $\frac{1}{n!}\left[{n+1 \atop 2}\right] = H_n$; hence

$$\left[{n+1 \atop 2}\right] = n!H_n. \tag{6.58}$$

We proved in Chapter 2 that the harmonic series $\sum_k 1/k$ diverges, which means that H_n gets arbitrarily large as $n \to \infty$. But our proof was indirect;

we found that a certain infinite sum (2.58) gave different answers when it was rearranged, hence $\sum_k 1/k$ could not be bounded. The fact that $H_n \to \infty$ seems counter-intuitive, because it implies among other things that a large enough stack of cards will overhang a table by a mile or more, and that the worm W will eventually reach the end of his rope. Let us therefore take a closer look at the size of H_n when n is large.

The simplest way to see that $H_n \to \infty$ is probably to group its terms according to powers of 2. We put one term into group 1, two terms into group 2, four into group 3, eight into group 4, and so on:

$$\underbrace{\frac{1}{1}}_{\text{group 1}} + \underbrace{\frac{1}{2}+\frac{1}{3}}_{\text{group 2}} + \underbrace{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}_{\text{group 3}} + \underbrace{\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}}_{\text{group 4}} + \cdots .$$

Both terms in group 2 are between $\frac{1}{4}$ and $\frac{1}{2}$, so the sum of that group is between $2\cdot\frac{1}{4} = \frac{1}{2}$ and $2\cdot\frac{1}{2} = 1$. All four terms in group 3 are between $\frac{1}{8}$ and $\frac{1}{4}$, so their sum is also between $\frac{1}{2}$ and 1. In fact, each of the 2^{k-1} terms in group k is between 2^{-k} and 2^{1-k}; hence the sum of each individual group is between $\frac{1}{2}$ and 1.

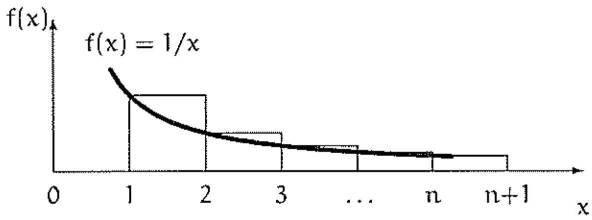
This grouping procedure tells us that if n is in group k, we must have $H_n > k/2$ and $H_n \leqslant k$ (by induction on k). Thus $H_n \to \infty$, and in fact

$$\frac{\lfloor \lg n \rfloor + 1}{2} < H_n \leqslant \lfloor \lg n \rfloor + 1 . \qquad (6.59)$$

We now know H_n within a factor of 2. Although the harmonic numbers approach infinity, they approach it only logarithmically — that is, quite slowly.

We should call them the worm numbers, they're so slow.

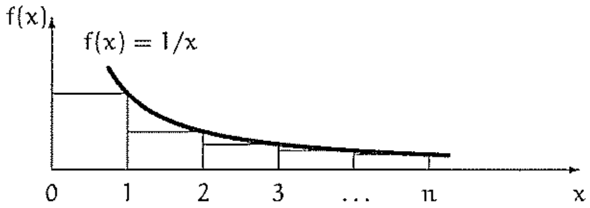
Better bounds can be found with just a little more work and a dose of calculus. We learned in Chapter 2 that H_n is the discrete analog of the continuous function $\ln n$. The natural logarithm is defined as the area under a curve, so a geometric comparison is suggested:



The area under the curve between 1 and n, which is $\int_1^n dx/x = \ln n$, is less than the area of the n rectangles, which is $\sum_{k=1}^n 1/k = H_n$. Thus $\ln n < H_n$; this is a sharper result than we had in (6.59). And by placing the rectangles

"I now see a way too how y^e aggregate of y^e termes of Musicall progressions may bee found (much after y^e same manner) by Logarithms, but y^e calculations for finding out those rules would bee still more troublesom."
— I. Newton [280]

a little differently, we get a similar upper bound:



This time the area of the n rectangles, H_n, is less than the area of the first rectangle plus the area under the curve. We have proved that

$$\ln n \;<\; H_n \;<\; \ln n + 1\,, \qquad \text{for } n > 1. \tag{6.60}$$

We now know the value of H_n with an error of at most 1.

"Second order" harmonic numbers $H_n^{(2)}$ arise when we sum the squares of the reciprocals, instead of summing simply the reciprocals:

$$H_n^{(2)} \;=\; 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \;=\; \sum_{k=1}^{n} \frac{1}{k^2}\,.$$

Similarly, we define harmonic numbers of order r by summing $(-r)$th powers:

$$H_n^{(r)} \;=\; \sum_{k=1}^{n} \frac{1}{k^r}\,. \tag{6.61}$$

If $r > 1$, these numbers approach a limit as $n \to \infty$; we noted in exercise 2.31 that this limit is conventionally called Riemann's zeta function:

$$\zeta(r) \;=\; H_\infty^{(r)} \;=\; \sum_{k\geqslant 1} \frac{1}{k^r}\,. \tag{6.62}$$

Euler [103] discovered a neat way to use generalized harmonic numbers to approximate the ordinary ones, $H_n^{(1)}$. Let's consider the infinite series

$$\ln\left(\frac{k}{k-1}\right) \;=\; \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \frac{1}{4k^4} + \cdots\,, \tag{6.63}$$

which converges when $k > 1$. The left-hand side is $\ln k - \ln(k-1)$; therefore if we sum both sides for $2 \leqslant k \leqslant n$ the left-hand sum telescopes and we get

$$\begin{aligned} \ln n - \ln 1 \;&=\; \sum_{k=2}^{n} \left(\frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \frac{1}{4k^4} + \cdots\right) \\ &=\; (H_n - 1) + \tfrac{1}{2}(H_n^{(2)} - 1) + \tfrac{1}{3}(H_n^{(3)} - 1) + \tfrac{1}{4}(H_n^{(4)} - 1) + \cdots\,. \end{aligned}$$

Rearranging, we have an expression for the difference between H_n and $\ln n$:

$$H_n - \ln n = 1 - \tfrac{1}{2}(H_n^{(2)}-1) - \tfrac{1}{3}(H_n^{(3)}-1) - \tfrac{1}{4}(H_n^{(4)}-1) - \cdots .$$

When $n \to \infty$, the right-hand side approaches the limiting value

$$1 - \tfrac{1}{2}(\zeta(2)-1) - \tfrac{1}{3}(\zeta(3)-1) - \tfrac{1}{4}(\zeta(4)-1) - \cdots,$$

which is now known as *Euler's constant* and conventionally denoted by the Greek letter γ. In fact, $\zeta(r) - 1$ is approximately $1/2^r$, so this infinite series converges rather rapidly and we can compute the decimal value

"Huius igitur quantitatis constantis C valorem deteximus, quippe est C = 0,577218." —L. Euler [103]

$$\gamma = 0.5772156649\ldots . \tag{6.64}$$

Euler's argument establishes the limiting relation

$$\lim_{n\to\infty} (H_n - \ln n) = \gamma; \tag{6.65}$$

thus H_n lies about 58% of the way between the two extremes in (6.60). We are gradually homing in on its value.

Further refinements are possible, as we will see in Chapter 9. We will prove, for example, that

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\epsilon_n}{120n^4}, \qquad 0 < \epsilon_n < 1. \tag{6.66}$$

This formula allows us to conclude that the millionth harmonic number is

$$H_{1000000} \approx 14.3927267228657236313811275,$$

without adding up a million fractions. Among other things, this implies that a stack of a million cards can overhang the edge of a table by more than seven cardlengths.

What does (6.66) tell us about the worm on the rubber band? Since H_n is unbounded, the worm will definitely reach the end, when H_n first exceeds 100. Our approximation to H_n says that this will happen when n is approximately

$$e^{100-\gamma} \approx e^{99.423}.$$

In fact, exercise 9.49 proves that the critical value of n is either $\lfloor e^{100-\gamma} \rfloor$ or $\lceil e^{100-\gamma} \rceil$. We can imagine W's triumph when he crosses the finish line at last, much to K's chagrin, some 287 decillion centuries after his long crawl began. (The rubber band will have stretched to more than 10^{27} light years long; its molecules will be pretty far apart.)

Well, they can't really go at it this long; the world will have ended much earlier, when the Tower of Brahma is fully transferred.

6.4 HARMONIC SUMMATION

Now let's look at some sums involving harmonic numbers, starting with a review of a few ideas we learned in Chapter 2. We proved in (2.36) and (2.57) that

$$\sum_{0\le k<n} H_k = nH_n - n; \qquad (6.67)$$

$$\sum_{0\le k<n} kH_k = \frac{n(n-1)}{2}H_n - \frac{n(n-1)}{4}. \qquad (6.68)$$

Let's be bold and take on a more general sum, which includes both of these as special cases: What is the value of

$$\sum_{0\le k<n} \binom{k}{m} H_k,$$

when m is a nonnegative integer?

The approach that worked best for (6.67) and (6.68) in Chapter 2 was called *summation by parts*. We wrote the summand in the form $u(k)\Delta v(k)$, and we applied the general identity

$$\textstyle\sum_a^b u(x)\Delta v(x)\,\delta x = u(x)v(x)\big|_a^b - \sum_a^b v(x+1)\Delta u(x)\,\delta x. \qquad (6.69)$$

Remember? The sum that faces us now, $\sum_{0\le k<n}\binom{k}{m}H_k$, is a natural for this method because we can let

$$u(k) = H_k, \qquad \Delta u(k) = H_{k+1} - H_k = \frac{1}{k+1};$$

$$v(k) = \binom{k}{m+1}, \qquad \Delta v(k) = \binom{k+1}{m+1} - \binom{k}{m+1} = \binom{k}{m}.$$

(In other words, harmonic numbers have a simple Δ and binomial coefficients have a simple Δ^{-1}, so we're in business.) Plugging into (6.69) yields

$$\sum_{0\le k<n}\binom{k}{m}H_k = \textstyle\sum_0^n \binom{x}{m}H_x\,\delta x = \binom{x}{m+1}H_x\Big|_0^n - \sum_0^n \binom{x+1}{m+1}\frac{\delta x}{x+1}$$

$$= \binom{n}{m+1}H_n - \sum_{0\le k<n}\binom{k+1}{m+1}\frac{1}{k+1}.$$

The remaining sum is easy, since we can absorb the $(k+1)^{-1}$ using our old standby, equation (5.5):

$$\sum_{0\le k<n}\binom{k+1}{m+1}\frac{1}{k+1} = \sum_{0\le k<n}\binom{k}{m}\frac{1}{m+1} = \binom{n}{m+1}\frac{1}{m+1}.$$

Thus we have the answer we seek:

$$\sum_{0\leqslant k<n}\binom{k}{m}H_k = \binom{n}{m+1}\Bigl(H_n-\frac{1}{m+1}\Bigr). \tag{6.70}$$

(This checks nicely with (6.67) and (6.68) when $m=0$ and $m=1$.)

The next example sum uses division instead of multiplication: Let us try to evaluate

$$S_n = \sum_{k=1}^{n}\frac{H_k}{k}.$$

If we expand H_k by its definition, we obtain a double sum,

$$S_n = \sum_{1\leqslant j\leqslant k\leqslant n}\frac{1}{j\cdot k}.$$

Now another method from Chapter 2 comes to our aid; equation (2.33) tells us that

$$S_n = \frac{1}{2}\biggl(\Bigl(\sum_{k=1}^{n}\frac{1}{k}\Bigr)^2+\sum_{k=1}^{n}\frac{1}{k^2}\biggr) = \frac{1}{2}\bigl(H_n^2+H_n^{(2)}\bigr). \tag{6.71}$$

It turns out that we could also have obtained this answer in another way if we had tried to sum by parts (see exercise 26).

Now let's try our hands at a more difficult problem [354], which doesn't submit to summation by parts:

$$U_n = \sum_{k\geqslant 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}(n-k)^n, \qquad \text{integer } n\geqslant 1.$$

(This sum doesn't explicitly mention harmonic numbers either; but who knows when they might turn up?)

(Not to give the answer away or anything.)

We will solve this problem in two ways, one by grinding out the answer and the other by being clever and/or lucky. First, the grinder's approach. We expand $(n-k)^n$ by the binomial theorem, so that the troublesome k in the denominator will combine with the numerator:

$$\begin{aligned} U_n &= \sum_{k\geqslant 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}\sum_j\binom{n}{j}(-k)^j n^{n-j} \\ &= \sum_j\binom{n}{j}(-1)^{j-1}n^{n-j}\sum_{k\geqslant 1}\binom{n}{k}(-1)^k k^{j-1}. \end{aligned}$$

This isn't quite the mess it seems, because the k^{j-1} in the inner sum is a polynomial in k, and identity (5.40) tells us that we are simply taking the

nth difference of this polynomial. Almost; first we must clean up a few things. For one, k^{j-1} isn't a polynomial if $j = 0$; so we will need to split off that term and handle it separately. For another, we're missing the term $k = 0$ from the formula for nth difference; that term is nonzero when $j = 1$, so we had better restore it (and subtract it out again). The result is

$$\begin{aligned} U_n &= \sum_{j\geq 1}\binom{n}{j}(-1)^{j-1}n^{n-j}\sum_{k\geq 0}\binom{n}{k}(-1)^k k^{j-1} \\ &\quad - \sum_{j\geq 1}\binom{n}{j}(-1)^{j-1}n^{n-j}\binom{n}{0}0^{j-1} \\ &\quad - \binom{n}{0}n^n\sum_{k\geq 1}\binom{n}{k}(-1)^k k^{-1}. \end{aligned}$$

OK, now the top line (the only remaining double sum) is zero: It's the sum of multiples of nth differences of polynomials of degree less than n, and such nth differences are zero. The second line is zero except when $j = 1$, when it equals $-n^n$. So the third line is the only residual difficulty; we have reduced the original problem to a much simpler sum:

$$U_n = n^n(T_n - 1), \qquad \text{where } T_n = \sum_{k\geq 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}. \tag{6.72}$$

For example, $U_3 = \binom{3}{1}\frac{8}{1} - \binom{3}{2}\frac{1}{2} = \frac{45}{2}$; $T_3 = \binom{3}{1}\frac{1}{1} - \binom{3}{2}\frac{1}{2} + \binom{3}{3}\frac{1}{3} = \frac{11}{6}$; hence $U_3 = 27(T_3 - 1)$ as claimed.

How can we evaluate T_n? One way is to replace $\binom{n}{k}$ by $\binom{n-1}{k} + \binom{n-1}{k-1}$, obtaining a simple recurrence for T_n in terms of T_{n-1}. But there's a more instructive way: We had a similar formula in (5.41), namely

$$\sum_k \binom{n}{k}\frac{(-1)^k}{x+k} = \frac{n!}{x(x+1)\ldots(x+n)}.$$

If we subtract out the term for $k = 0$ and set $x = 0$, we get $-T_n$. So let's do it:

$$\begin{aligned} T_n &= \left(\frac{1}{x} - \frac{n!}{x(x+1)\ldots(x+n)}\right)\bigg|_{x=0} \\ &= \left(\frac{(x+1)\ldots(x+n) - n!}{x(x+1)\ldots(x+n)}\right)\bigg|_{x=0} \\ &= \left(\frac{x^n{n+1 \brack n+1} + \cdots + x{n+1 \brack 2} + {n+1 \brack 1} - n!}{x(x+1)\ldots(x+n)}\right)\bigg|_{x=0} = \frac{1}{n!}\left[{n+1 \atop 2}\right]. \end{aligned}$$

(We have used the expansion (6.11) of $(x+1)\ldots(x+n) = x^{\overline{n+1}}/x$; we can divide x out of the numerator because $\left[{n+1 \atop 1}\right] = n!$.) But we know from (6.58) that $\left[{n+1 \atop 2}\right] = n!\,H_n$; hence $T_n = H_n$, and we have the answer:

$$U_n \;=\; n^n(H_n - 1)\,. \tag{6.73}$$

That's one approach. The other approach will be to try to evaluate a much more general sum,

$$U_n(x,y) \;=\; \sum_{k\geqslant 1}\binom{n}{k}\frac{(-1)^{k-1}}{k}(x+ky)^n\,, \qquad \text{integer } n \geqslant 0; \tag{6.74}$$

the value of the original U_n will drop out as the special case $U_n(n,-1)$. (We are encouraged to try for more generality because the previous derivation "threw away" most of the details of the given problem; somehow those details must be irrelevant, because the nth difference wiped them away.)

We could replay the previous derivation with small changes and discover the value of $U_n(x,y)$. Or we could replace $(x+ky)^n$ by $(x+ky)^{n-1}(x+ky)$ and then replace $\binom{n}{k}$ by $\binom{n-1}{k}+\binom{n-1}{k-1}$, leading to the recurrence

$$U_n(x,y) \;=\; xU_{n-1}(x,y) + x^n/n + yx^{n-1}\,; \tag{6.75}$$

this can readily be solved with a summation factor (exercise 5).

But it's easiest to use another trick that worked to our advantage in Chapter 2: differentiation. The derivative of $U_n(x,y)$ with respect to y brings out a k that cancels with the k in the denominator, and the resulting sum is trivial:

$$\begin{aligned}\frac{\partial}{\partial y}U_n(x,y) \;&=\; \sum_{k\geqslant 1}\binom{n}{k}(-1)^{k-1}n(x+ky)^{n-1}\\ &=\; \binom{n}{0}nx^{n-1} - \sum_{k\geqslant 0}\binom{n}{k}(-1)^k n(x+ky)^{n-1} \;=\; nx^{n-1}\,.\end{aligned}$$

(Once again, the nth difference of a polynomial of degree $< n$ has vanished.)

We've proved that the derivative of $U_n(x,y)$ with respect to y is nx^{n-1}, independent of y. In general, if $f'(y) = c$ then $f(y) = f(0) + cy$; therefore we must have $U_n(x,y) = U_n(x,0) + nx^{n-1}y$.

The remaining task is to determine $U_n(x,0)$. But $U_n(x,0)$ is just x^n times the sum $T_n = H_n$ we've already considered in (6.72); therefore the general sum in (6.74) has the closed form

$$U_n(x,y) \;=\; x^nH_n + nx^{n-1}y\,. \tag{6.76}$$

In particular, the solution to the original problem is $U_n(n,-1) = n^n(H_n - 1)$.