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PLAYING POOL WITH π (THE NUMBER π FROM A BILLIARD POINT OF VIEW)

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Counting collisions in a simple dynamical system with two billiard balls can be used to estimate π to any accuracy.

(π)
It's irrational but well-rounded.
– found on a printed T-shirt

1. Introduction

The remarkable book of V. I. Arnold on differential equations [1] starts with the following sentence: "*The notion of the configuration space alone let us solve a very difficult mathematical problem.*" Then the problem is formulated and solved. The result of this article confirms Arnold's idea to use configuration space for another problem, the problem of calculating the number π with any precision.

There are many ways to calculate π with a good precision; some of them are known from ancient times, some are pretty recent. The methods use various elegant ideas [2]: *geometric* (inscribing and circumscribing regular polygons around a circle gives, in particular, the ancient values $3\frac{1}{7}$ and $3\frac{10}{71}$ for π); *number theory* (continued fractions allow us to find the regular fraction 355/113 as the simplest approximation for π accurate to the one millionth place); *analytical* (that use series, integrals, and infinite products); and many others (e.g., the *Monte Carlo Method*) which require modern electronic devices – powerful calculators and computers.

There is also an interesting experimental method for finding π discovered by a French mathematician Georges Louis Leclerc Comte de Buffon (1707–1788) in his article "Sur le jeu de franc-carreau" [3] published in 1777. Buffon suggested dropping a needle of length $L = D/2$ at random on a grid of parallel lines of spacing D. One drops the needle N times and counts the number of intersections, R, with the grid lines (note that since the needle is shorter than the distance between two consecutive grid lines, it intersects each time either exactly one line or none of the lines). The frequency of intersection with a line is R/N; on the other hand, one can show that the probability for the needle to intersect a grid line is $1/\pi$ (for an arbitrary needle of length, L, the probability equals $2L/\pi D$). Equating the

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4. Calculate the total number of hits in the system: the number of collisions between the balls plus the number of reflections of the small ball from the wall.

5. Write down the number $\mathbf{\Pi}$ of hits obtained from item #4 on a sheet of paper.

   (Note that we do not know a'priori if the number of hits is finite or infinite; we will prove it is finite.)

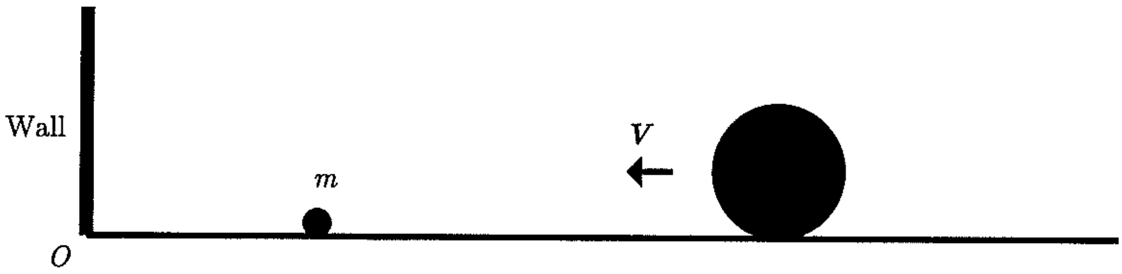

Fig. 1. Pushing the ball M towards the ball m

3. Investigation of a particular case

For different values of N the preceding Procedure gives us different values for the number $\mathbf{\Pi}$ (some of which could perhaps be infinity). Thus $\mathbf{\Pi} = \mathbf{\Pi}(N)$ is a function of the exponent N of the number 100^N.

Let us investigate the simplest case $N = 0$, which corresponds to the equality of the masses: $M = m$. The laws of conservation yield the following description of the system's behavior: if one ball is in the static position and the other one collides with it, then after the collision, the stationary ball starts moving with exactly the same velocity in the same direction as the second ball moved previously, while the second ball stops. It looks like the moving ball penetrates through the still ball without changing its velocity or affecting the still ball[3].

Then the moving ball hits the wall and reflects from it. The ball's velocity changes to the opposite one and, after that, it passes through the "transparent" ball and goes to infinity.

As you can see, the total number of hits in the system with $M = m$ is 3: two collisions and one reflection. Thus, $\mathbf{\Pi}(0) = 3$.

Note that 3 is the first digit of π. In what follows, the number of hits, $\mathbf{\Pi}$, is 31 (two first digits of π) and 314 (three digits of π) for, respectively, $M = 100m$ and $M = 100^2m$, i.e., $\mathbf{\Pi}(1) = 31$ and $\mathbf{\Pi}(2) = 314$.

4. The main result

Theorem. *The number of hits, $\mathbf{\Pi} = \mathbf{\Pi}(N)$, in the system described in the Procedure is always finite and equal to a number with $N+1$ digits,*

$$\mathbf{\Pi}(N) = 314159265358979323846264338327950288419716939937510\ldots,$$

whose first N digits coincide with the first N decimal digits of the number π (starting with 3).

The rest of the article is devoted to the proof of the theorem.

[3]Actually, the same behavior occurs when both identical balls move: after the collision, they just exchange their velocities; or, equivalently, they penetrate through each other without any effect upon each other. This reasoning solves easily the following fairly hard (at the first glance) problem: n identical balls move with the same speed along a line from left to right, and m other balls of the same kind move with the same speed along that line from right to left; how many collisions could occur in the system? The answer is now evident: mn, because each left ball penetrates each right ball.

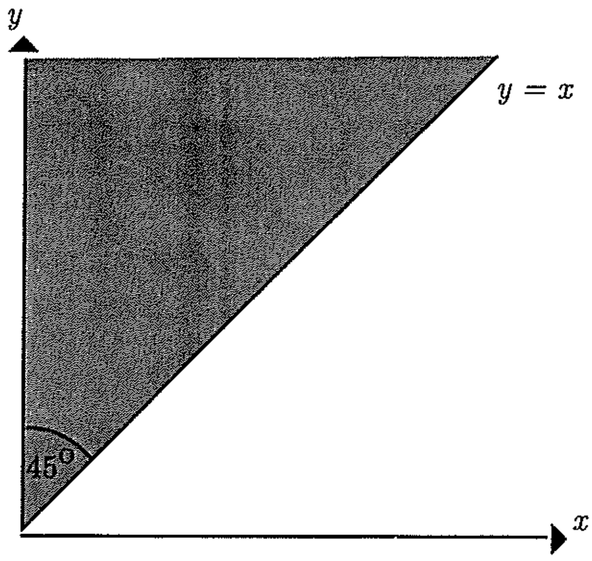

Fig. 3. The configuration space of the system

Step 2 (*between the first collision and the first reflection*). At the moment t_1, both balls begin to move along the horizontal line ℓ. The small ball moves with some velocity u and the big ball with velocity v, so that the laws of conservation of momentum and energy hold:

$$\begin{cases} mu + Mv = MV, \\ mu^2/2 + Mv^2/2 = MV^2/2, \end{cases}$$

where V is the initial (huge!) velocity of the ball M. We will not solve this system of equations quantitatively with respect to the variables u and v. Our aim is just to describe the behavior of the configuration point *after* the first collision.

Coming back to the initial dynamical system and using the system of equations (1), we can conclude that, after the first collision, the ball m will move very fast towards the wall (since the big ball gives it a big momentum) and the ball M also continues to move, a little bit slower than before, towards the wall. Both coordinates $x(t)$ and $y(t)$ are decreasing on the time interval after the first collision but before the reflection of the ball m from the wall. Therefore the configuration point $P(t)$ moves along a straight line segment inside the angle AOB, where O is the origin, OA is the positive y-axis, and OB is the ray $y = x$ outgoing from the origin in the first quadrant. Point $P(t)$ travels from the side OB to the side OA, approaching the origin O.

Step 3 (*reflection from the wall*). It is not hard to check that the small ball m moves faster than the big ball after the first collision, i.e. $u > v$.

Indeed, first note that $v < V$. This happens because the big ball gives some momentum to the small ball and accelerates it, so its new velocity v becomes smaller.

Multiplying the first equation of the system (1) by V and subtracting the double of the second equation yield

$$\begin{array}{rcrcl} & Mu \cdot V & + & Mv \cdot V & = MV^2 \\ - & mu^2 & + & Mv^2 & = MV^2 \\ \hline & mu(V-u)^2 & + & Mv(V-v) & = 0 \end{array}$$

$$\Downarrow$$

$$Mv(V-v) = mu(u-V).$$

As $V - v > 0$, $Mv > 0$, and $mu > 0$, we conclude that $u - V > 0$, so $u > V$. Thus $u > V > v$.[6]

[6]For simplicity, we consider the velocity of a ball moving from right to left to be positive, and from left to right to be negative. One can consider just speeds (the absolute values of velocities) instead of the velocities.

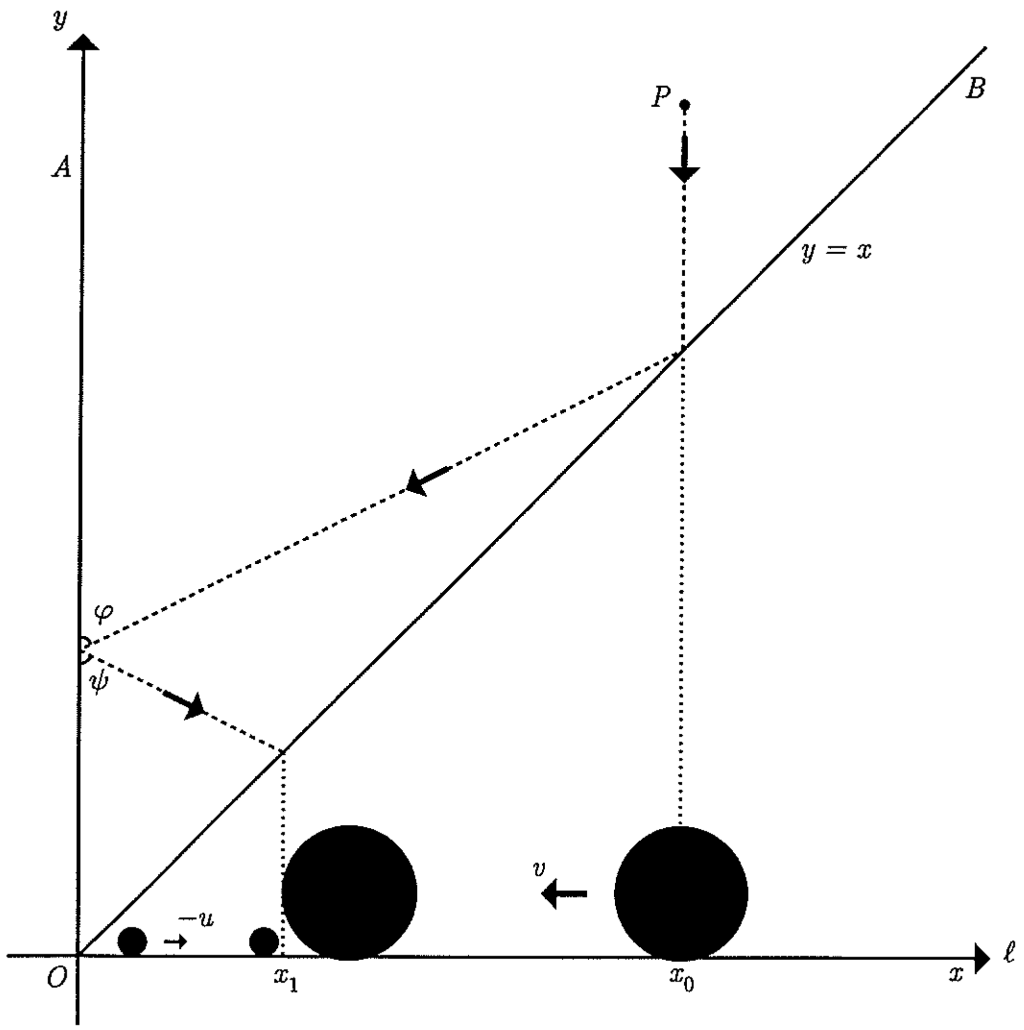

Fig. 5. $\varphi = \psi$, and the second collision

After that, the process repeats once again. This behavior of the system of the balls is reflected in the motion of the configuration point P (Figure 6).

It is absolutely unclear what kind behavior P exhibits:

(i) either P approaches the vertex O of the configuration angle AOB forever;

(ii) or P approaches O for a finite period of time, then moves away from O and reflects off the sides of the angle AOB infinitely many times; [7]

(iii) or P makes only finitely many reflections off the angle's sides, and, from some moment T_0 moves freely and rectilinearly.

Cases (i) and (ii) correspond to infinitely many hits in the system, i.e., collisions between the balls and reflections from the wall, and case (iii) to finitely many collisions and reflections. The theorem states that only the third case can occur, where the total number of hits in the system is $\Pi = 314159265\dots$.

[7] By the way, although P approaches vertex O during a finite period of time, P could make infinitely many reflections before it will start to move away from vertex O.

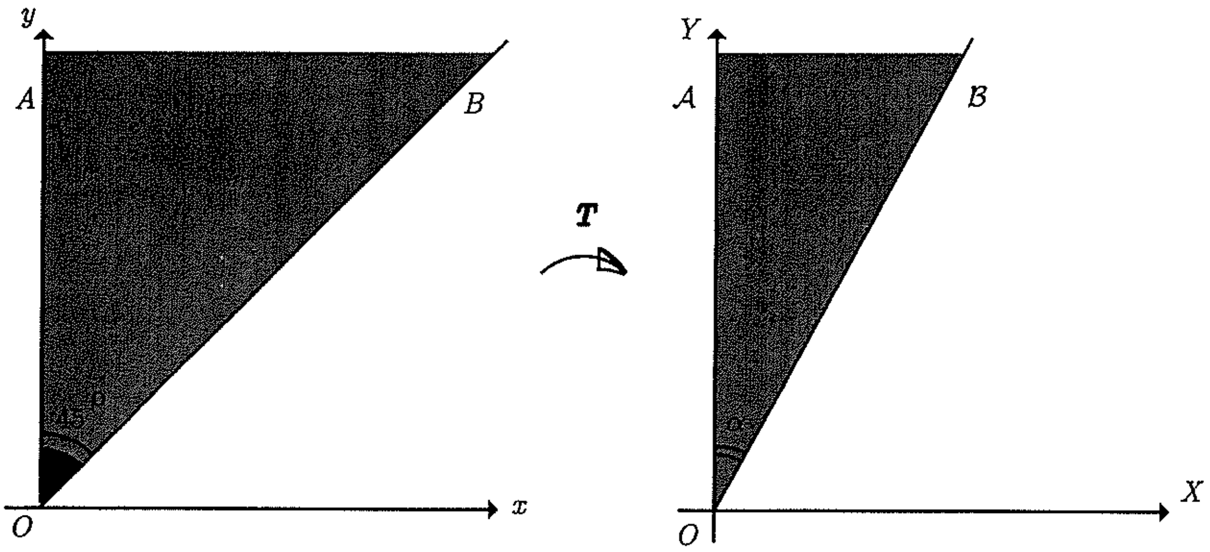

Fig. 7. Linear transformation T

Proof. Note that if point $P(x, y)$ has the velocity vector $\overrightarrow{\mathbf{w}} = (u, v) = (\dot{x}(t), \dot{y}(t))$ at moment t (different from a reflection), then point $\mathcal{P}(t)$ has the velocity vector

$$\overrightarrow{\mathbf{v}} = \left(\sqrt{m}\ \dot{x}(t), \sqrt{M}\ \dot{y}(t)\right) = \left(\sqrt{m}\ u, \sqrt{M}\ v\right)$$

at that moment. Thus the linear transformation T of the configuration space $\{(x, y)\}$ induces the same linear transformation in the velocity space $\{(\dot{x}, \dot{y})\}$.

Consider the following two cases: Case 1, in which point $\mathcal{P}(X, Y)$ reflects from the vertical side $X = 0$ (the Y-axis) of the angle α; and Case 2, in which $\mathcal{P}(X, Y)$ reflects off the side $Y = \sqrt{M/m}\ X$.

Case 1: Reflection from the Y-axis.

When the small ball reflects from the wall, its velocity u changes to $(-u)$. Then vector $\overrightarrow{\mathbf{v}}$ converts into vector

$$\overrightarrow{\mathbf{v}'} = \left(\sqrt{m}(-u), \sqrt{M}v\right).$$

which means $\varphi = \psi$ — the billiard reflection law (Figure 8).

Case 2: Reflection from the side $Y = \sqrt{M/m}\ X$.

This reflection corresponds to the ball collision. We will consider an interval of time in which only this collision occurs (i.e., the interval between two successive reflections of the ball m from the wall).

The system of moving balls has unchanging momentum during this interval of time, and the collision of the balls does not change it. The energy also doesn't change (it is always constant during the whole process).

For the sake of convenience, denote the momentum, p, by const_1 and twice the energy, $2E$, by const_2. Suppose the small ball has velocity u and the big ball has velocity v. The system (1) can be written as follows:

$$(1') \qquad \begin{cases} mu + Mv = \text{const}_1\ , \\ mu^2 + Mv^2 = \text{const}_2\ . \end{cases}$$

Some interesting geometry is hidden in the system $(1')$. Namely, consider the constant vector

$$\overrightarrow{\mathbf{m}} = (\sqrt{m}, \sqrt{M})$$

we obtain

$$\cos\varphi = \frac{\text{const}_1}{\text{const}_2}\cdot(m+M)^{-1/2} = \text{const}_3.$$

After reflection, point $\mathcal{P}$ moves with a new velocity, $\overrightarrow{\mathbf{v}'}$, satisfying the same system (1″). Therefore, the same reasoning for the angle ψ of $\mathcal{P}$'s reflection from the α's side $Y = \sqrt{M/m}\,X$ show that

$$\cos\psi = \text{const}_3 \qquad \text{(see Figure 9.)}$$

Consequently,

$$\psi = \varphi,$$

and the billiard law is proven for that reflection. The reduction to the billiard system in the angle α is finished. We shall call the angle $\mathcal{AOB}$ "the billiard configuration space" for the initial system.

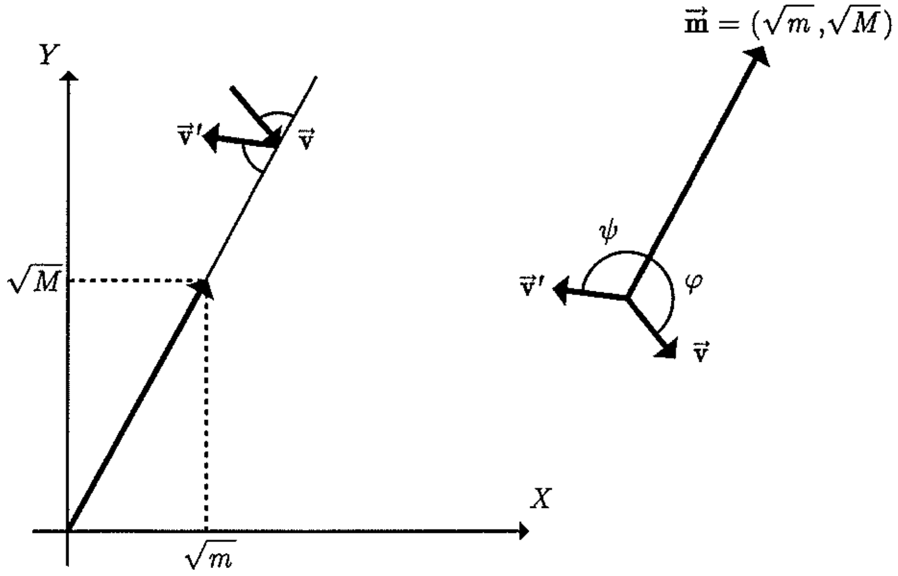

Fig. 9. *Reflection from the side* $Y = \sqrt{M/m}\,X$

7. The number of billiard reflections inside angle α

Lemma 2. **(a)** *The maximal number of reflections of a billiard point inside an angle* α*, over all possible billiard trajectories, is finite.*

(b) *This number equals* π/α *if* π/α *is an integer, and equals* $[\pi/\alpha]+1$ *if* π/α *is not an integer (where the "[ ]" is the greatest integer function).*

(c) *If the initial billiard ray is parallel to one side of the angle* α*, then the total number of reflections for this particular trajectory is one fewer than the maximum (i.e., equals* $\pi/\alpha - 1$ *if* $\pi/\alpha \in \mathbf{Z}$ *and* $[\pi/\alpha]$ *if* $\pi/\alpha \notin \mathbf{Z}$*)*.

Proof. Let us unfold the angle α together with the billiard trajectory γ in it. We just reflect angle α in its sides which the particle hits and consider the image of the trajectory γ under those reflections. The trajectory's image is a straight line, k, that lies in the corridor of reflected angles (see Figure 10).

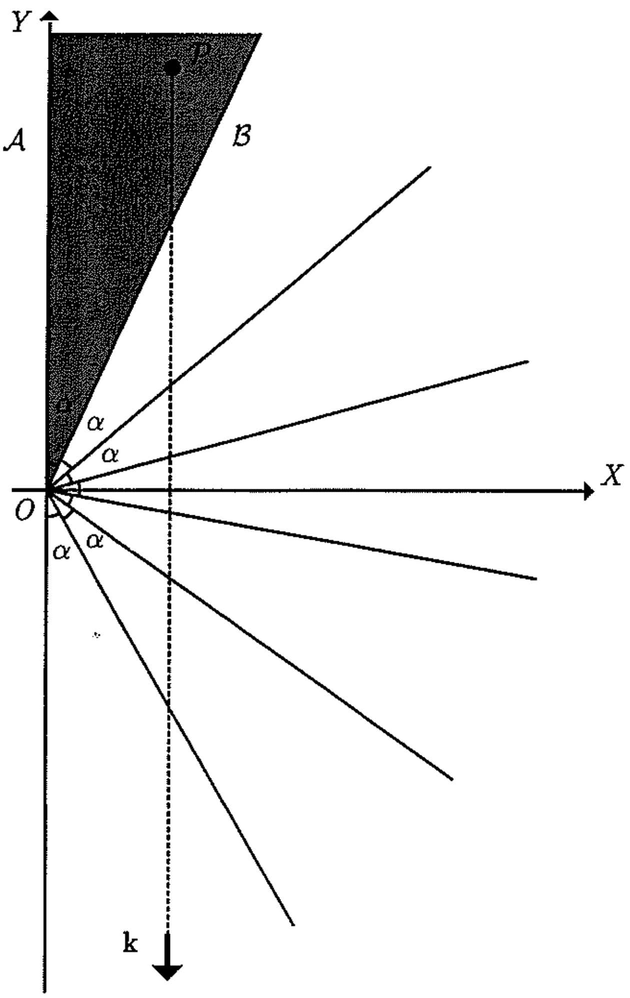

Fig. 11. *Ray* k *is parallel to the* Y*-axis*

is applicable in this case: the number of reflections equals

$$\mathbf{\Pi}(N) = \left[\frac{\pi}{\arctan(10^{-N})}\right]. \tag{8.1}$$

This is the precise formula; we will now use an approximation for $\arctan(10^{-N})$ to simplify (4).

The idea of the rest part of this section is as follows. If we replace the denominator of the fraction $\pi/\arctan(10^{-N})$ with the slightly larger number 10^{-N}, we get the slightly smaller fraction $\pi/10^{-N} = \pi \cdot 10^{N}$; if the integer part of the initial fraction and the perturbed fraction are the same, we can substitute $[\pi/\arctan(10^{-N})]$ by $[\pi \cdot 10^{N}]$. This will give us N correct digits of π.

However, the situation is much more delicate than it seems at first glance: *the integer parts of the two fractions could be different*! Fortunately, if N is sufficiently large, then they will differ by at most 1. Let us explain what occurs in more detail.

Denote $x = 10^{-N}$ for a moment. Recall that

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty}(-t^2)^n dt = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots ,$$

so $\arctan x \approx x$ as $x \to 0$. We will substitute x for $\arctan x$ in the formula (3).

then formula (6) yields

$$\mathbf{\Pi}(N) = [\pi \cdot 10^N] + 1 = 31415\ldots a_k \underbrace{999\ldots 9}_{N-k \text{ nines}} +1$$
$$= 31415\ldots (a_k+1) \underbrace{000\ldots 0}_{N-k \text{ zeros}} .$$

So, $\mathbf{\Pi}(N)$ gives only $k < N$ correct digits of π, while the last $N-k$ digits in $\mathbf{\Pi}(N)$ are incorrect, compared to those in π.

Fortunately, this happens (if it happens!) very rarely and does not affect the general result concerning *the whole sequence of integers* $\mathbf{\Pi}(N)$ as $N \to \infty$: the sequence of first digits in $\mathbf{\Pi}(0)$, $\mathbf{\Pi}(1)$, $\mathbf{\Pi}(2)$, ..., $\mathbf{\Pi}(N)$, ... will stabilize from some number, $\mathbf{\Pi}(N)$, and eventually we will be able to know **all the digits** of number π.

But most likely formula (6) is false for all N. Then (5) would always work and $\mathbf{\Pi}(N)$ would always give the precise N digits of π. We discuss this in the next section.

10. Some arithmetical questions and a conjecture

It would be very nice if only formula (5), which gives the correct value for $\mathbf{\Pi}(N)$, always applied and formula (6) was always false.

Let us pose four related questions in this connection.

Question 1. Is the formula (5),

$$\left[\frac{\pi}{\arctan(10^{-N})}\right] = \left[\frac{\pi}{10^{-N}}\right],$$

true for any natural N big enough?

Question 2. Is it true that for any positive *irrational number* a and for any positive x small enough,

$$\left[\frac{a}{\arctan x}\right] = \left[\frac{a}{x}\right] \quad ?$$

Question 3. Is the equality

$$\left[\frac{\pi}{\arctan(1/N)}\right] = \left[\frac{\pi}{1/N}\right]$$

true for any natural N big enough?

Question 4. Is the equality

$$\left[\frac{\sqrt{2}}{\arctan(10^{-N})}\right] = \left[\frac{\sqrt{2}}{10^{-N}}\right]$$

true for any natural N big enough?

Question 2 comes as a broad generalization of Question 1, while Questions 3 and 4 arise from Question 1 if you substitute 10^{-N} for $1/N$ and, respectively, π for $\sqrt{2}$. I am confident that the answer to the Question 1 is "YES" but cannot prove this. I know, with rigorous proofs, the exact answers to the other three questions (see the answers together with their proofs in the next section).

As for Question 1, the modern mathematics is powerless to answer it; in any event, several leading specialists in number theory and related topics explained to the author of this article that the modern mathematics is far from a solution to this problem. Here is a quotation from an e-mail of a famous Australian mathematician, Alf van der Poorten, to the author: "[...] One state of knowledge is so

11. Solutions to Questions 2, 3, and 4

Solution to Question 2. The answer to Question 2 is negative.

Statement 1. The equality $\left[\dfrac{a}{\arctan x}\right] = \left[\dfrac{a}{x}\right]$ cannot hold for all values of x.

Proof.

Let a be a positive number (no matter, rational or irrational). Let us set $x = \tan(a/n)$. Then the left hand side of the equality equals n, while the right hand side is strictly less than n (since $a/n < \tan(a/n)$). (A small shift of the value $x = \tan a$ does not change the inequality $\left[\dfrac{a}{\arctan x}\right] > \left[\dfrac{a}{x}\right]$.) ∎

Solution to Question 3. Let us denote $L_N = \left[\dfrac{\pi}{\arctan(1/N)}\right]$.

Statement 2. For each $N \geqslant 23$, either $L_N = [\pi N]$ or $L_{N+1} = [\pi(N+1)]$.

Proof.

Assume the contrary: $L_N \neq [\pi N]$ and $L_{N+1} \neq [\pi(N+1)]$. Then $L_N > \pi N$ and $L_{N+1} > \pi(N+1)$. By Lemma 3, **(b)**, we have

$$\frac{\pi}{\arctan x} < \pi x + \frac{\pi}{x} \ ,$$

so, plugging $x = 1/N$ and $x = 1/(N+1)$ we get two inequalities:

$$\pi N < L_N < \pi N + \frac{\pi}{N} \ ,$$

and

$$\pi(N+1) < L_{N+1} < \pi(N+1) + \frac{\pi}{N+1} \ .$$

Subtracting of the inequalities yields

$$\pi - \frac{\pi}{N} < L_{N+1} - L_N < \pi + \frac{\pi}{N+1} \ .$$

If $N \geqslant 23$, then $\pi - \dfrac{\pi}{N} > 3$ and $\pi + \dfrac{\pi}{N+1} < 4$, so we obtain

$$3 < L_{N+1} - L_N < 4 \ ,$$

which is a contradiction because $L_{N+1} - L_N$ is an integer. Statement 2 is proven. ∎

Solution to Question 4. The answer to Question 4 is "YES":

Statement 3. For natural N big enough, $\left[\dfrac{\sqrt{2}}{\arctan(10^{-N})}\right] = \left[\dfrac{\sqrt{2}}{10^{-N}}\right]$.

Proof.

We write $g(x) = \overline{o}\,(f(x))$ as $x \to 0$ if $\lim\limits_{x\to 0} \dfrac{g(x)}{f(x)} = 0$. As $x \to 0$,

$$\frac{\sqrt{2}}{\arctan x} = \frac{\sqrt{2}}{x - \frac{x^3}{3} + \overline{o}(x^4)} = \frac{\frac{\sqrt{2}}{x}}{1 - \frac{x^2}{3} + \overline{o}(x^3)} = \frac{\sqrt{2}}{x}\left(1 + \frac{x^2}{3} + \overline{o}(x^3)\right).$$

Let us think of the obstacle as a plane mirror. Then the two balls have their mirror images on the other side of the mirror. When the ball m reflects off the mirror, its image also reflects off the mirror from the other side; when the two balls collide with each other, their images also collide.

Let us simply remove the mirror and substitute the images of the two balls by real balls with the same masses, m and M, as their preimages. When the two small balls move towards each other with the same velocities and then collide, they just exchange their velocities, or, in other words, penetrate each other without any effect. The pairs of balls $(m - -M)$ and $(M - -m)$ collide simultaneously and we fix this event as one, not two, collisions. Thus the dynamical system

$$M - - m - - m - - M$$

counts the same number, **Π**, of hits as the previous system " **wall** $- -\ m\ - -\ M$ " does.

The difference between the two systems consist of the following:

1. The wall in the first system plays the role of a ball of infinite mass, and there is no obstacle or infinite mass in the second system;

2. The configuration space of the first system is 2-dimensional whereas that of the second system is 4-dimensional: it is the direct sum of two identical copies of the space for the first system and has the natural symmetries;

3. In the first system, one should distinguish the hits off the wall and the collisions of the balls. After a collision with the wall, the momentum of the system (of balls alone) changes, which does not occur when the balls collide with each other. In the second system one counts only the collisions of balls and uses only formula (1) or (1′).

The commonality between the systems is that they carry out the same function:

count the same number Π of hits!

13. Closing remarks

The author created the billiard method for finding π when he was preparing a mathematical colloquium talk at Eastern Illinois University about balls' collisions (the so-called "Sinai's Problem"). When the procedure was presented to the audience, no one believed it at first, but then the author gave a proof, the ease of which convinced everyone.

Later, the author talked about his discovery in several other American universities, with the same reaction of the audience: first complete distrust and then complete acceptance, due to the obviousness of the proof.

From the experimental (physical) point of view, the central theorem of the presented article (see sections "Procedure" and "The main result") is completely proven: the ratio of two real masses, M/m, that is used in the procedure of calculating π, cannot exceed the number of atoms in our Universe, which is much smaller than $10^{100,000,000}$ (actually, it is even less than 10^{200}); but we know (from the Internet) that our method gives correct first 100,000,000 digits of π. However, from mathematical point of view, the Conjecture is a real challenge.

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