Proof that π is irrational

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The number π (pi) has been studied since ancient times, and so has the concept of irrational numbers. An irrational number is any real number that cannot be expressed as a fraction *a*/*b*, where *a* is an integer and *b* is a non-zero integer.

It was not until the 18th century that Johann Heinrich Lambert proved that π is irrational. In the 19th century, Charles Hermite found a proof that requires no prerequisite knowledge beyond basic calculus. A simplification of Hermite's proof is due to Mary Cartwright. Two other such proofs are due to Ivan Niven and to Miklós Laczkovich.

In 1882, Ferdinand von Lindemann proved that π is not just irrational, but transcendental as well.[1]

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Lambert's proof

In 1761, Lambert proved[2] that π is irrational by first showing that this continued fraction expansion holds:

$$\text{tang}\left(\frac{\Phi}{\omega}\right) = \cfrac{\Phi}{\omega - \cfrac{\Phi\Phi}{3\omega - \cfrac{\Phi\Phi}{5\omega - \cfrac{\Phi\Phi}{7\omega - \cfrac{\Phi\Phi}{9\omega - \&c.}}}}}$$

Scan of formula on page 288 of Lambert's "Mémoires sur quelques propriétés remarquables des quantités transcendantes, circulaires et logarithmiques", Mémoires de l'Académie royale des sciences de Berlin (1768), 265–322.

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}}.$$

Then Lambert proved that if *x* is non-zero and rational then this expression must be irrational. Since tan(π/4) = 1, it follows that π/4 is irrational and therefore that π is irrational. A simplification of Lambert's proof is given below. This result can also be proved using even more basic tools of calculus (integrals instead of series).[3][4]

Hermite's proof

This proof[5][6] uses the characterization of π as the smallest positive number whose half is a zero of the cosine function and it actually proves that π^2 is irrational. As in many proofs of irrationality, the argument proceeds by reductio ad absurdum.

Consider the sequences $(A_n)_{n \geq 0}$ and $(U_n)_{n \geq 0}$ of functions from **R** into **R** thus defined:

1. $A_0(x) = \sin(x);$
2. $(\forall n \in \mathbb{Z}_+) : A_{n+1}(x) = \int_0^x yA_n(y)\,dy;$
3. $U_0(x) = \frac{\sin(x)}{x};$
4. $(\forall n \in \mathbb{Z}_+) : U_{n+1}(x) = -\frac{U_n'(x)}{x}.$

It can be proven by induction that

$$(\forall n \in \mathbb{Z}_+) : A_n(x) = \frac{x^{2n+1}}{(2n+1)!!} - \frac{x^{2n+3}}{2 \times (2n+3)!!} + \frac{x^{2n+5}}{2 \times 4 \times (2n+5)!!} \mp \cdots$$

and that

$$(\forall n \in \mathbb{Z}_+) : U_n(x) = \frac{1}{(2n+1)!!} - \frac{x^2}{2 \times (2n+3)!!} + \frac{x^4}{2 \times 4 \times (2n+5)!!} \mp \cdots$$

and therefore that

$$U_n(x) = \frac{A_n(x)}{x^{2n+1}}.$$

So

$$\frac{A_{n+1}(x)}{x^{2n+3}} = U_{n+1}(x) = -\frac{U_n'(x)}{x} = -\frac{1}{x}\frac{d}{dx}\left(\frac{A_n(x)}{x^{2n+1}}\right),$$

which is equivalent to

$$A_{n+1}(x) = (2n+1)A_n(x) - xA_n'(x) = (2n+1)A_n(x) - x^2A_{n-1}(x).$$

It follows by induction from this, together with the fact that $A_0(x) = \sin(x)$ and that $A_1(x) = -x\cos(x) + \sin(x)$, that $A_n(x)$ can be written as $P_n(x^2)\sin(x) + xQ_n(x^2)\cos(x)$, where P_n and Q_n are polynomial functions with integer coefficients and where the degree of P_n is smaller than or equal to $\lfloor n/2 \rfloor$. In particular,

$$A_n\left(\frac{\pi}{2}\right) = P_n\left(\frac{\pi^2}{4}\right).$$

Hermite also gave a closed expression for the function A_n, namely

$$A_n(x) = \frac{x^{2n+1}}{2^n n!} \int_0^1 (1-z^2)^n \cos(xz)\, dz.$$

He did not justify this assertion, but it can be proved easily. First of all, this assertion is equivalent to

$$\frac{1}{2^n n!} \int_0^1 (1-z^2)^n \cos(xz)\, dz = \frac{A_n(x)}{x^{2n+1}} = U_n(x).$$

Proceeding by induction, take n=0.

$$\int_0^1 \cos(xz)\, dz = \frac{\sin(x)}{x} = U_0(x)$$

and, for the inductive step, consider any $n \in \mathbf{Z}_+$. If

$$\frac{1}{2^n n!} \int_0^1 (1-z^2)^n \cos(xz)\, dz = U_n(x),$$

then, using integration by parts and Leibniz's rule, one gets

$$\begin{aligned} &\frac{1}{2^{n+1}(n+1)!} \int_0^1 (1-z^2)^{n+1} \cos(xz)\, dz \\ &= \frac{1}{2^{n+1}(n+1)!} \left(\overbrace{\left.(1-z^2)^{n+1} \frac{\sin(xz)}{x}\right|_{z=0}^{z=1}}^{=0} + \int_0^1 2(n+1)(1-z^2)^n z \frac{\sin(xz)}{x}\, dz \right) \\ &= \frac{1}{x} \cdot \frac{1}{2^n n!} \int_0^1 (1-z^2)^n z \sin(xz)\, dz \\ &= -\frac{1}{x} \cdot \frac{d}{dx} \left(\frac{1}{2^n n!} \int_0^1 (1-z^2)^n \cos(xz)\, dz \right) \\ &= -\frac{U_n'(x)}{x} = U_{n+1}(x). \end{aligned}$$

If $\pi^2/4 = p/q$, with p and q in N, then, since the coefficients of P_n are integers and its degree is smaller than or equal to $\lfloor n/2 \rfloor$, $q^{\lfloor n/2 \rfloor} P_n(\pi^2/4)$ is some integer N. In other words,

$$\begin{aligned} N &= q^{\lfloor \frac{n}{2} \rfloor} A_n\left(\frac{\pi}{2}\right) \\ &= q^{\lfloor \frac{n}{2} \rfloor} \frac{\left(\frac{p}{q}\right)^{n+\frac{1}{2}}}{2^n n!} \int_0^1 (1-z^2) \cos\left(\frac{\pi}{2} z\right)\, dz. \end{aligned}$$

But this number is clearly greater than 0; therefore, $N \in$ N. On the other hand, the integral that appears here is not greater than 1 and

$$\lim_{n\in\mathbb{N}} q^{\lfloor \frac{n}{2} \rfloor} \frac{\left(\frac{p}{q}\right)^{n+\frac{1}{2}}}{2^n n!} = 0.$$

So, if n is large enough, $N < 1$. Thereby, a contradiction is reached.

Hermite did not present his proof as an end in itself but as an afterthought within his search for a proof of the transcendence of π. He discussed the differential-recurrent relations to motivate and to obtain the convenient integral representation. Once the integral is obtained, there are various ways to present a succinct and self-contained proof starting from the integral (as in Cartwright's or Niven's presentations), which Hermite could easily see (as he did in his proof of the transcendence of e[7]).

Moreover, Hermite's proof is closer to Lambert's proof than it seems. In fact, $A_n(x)$ is the "residue" (or "remainder") of Lambert's continued fraction for tan(x).[4]

Cartwright's proof

Harold Jeffreys[8] wrote:

> The following was set as an example in the Mathematics Preliminary Examination at Cambridge in 1945 by Dame Mary Cartwright, but she has not traced its origin.

Consider the integrals

$$I_n(x) = \int_{-1}^{1} (1 - z^2)^n \cos(xz)\, dz.$$

Two integrations by parts give the recurrence relation

$$(\forall n \in \mathbb{N} \setminus \{1\}) : x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x)$$

If

$$J_n(x) = x^{2n+1} I_n(x),$$

then this becomes

$$J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x).$$

Also

$$J_0(x) = 2\sin(x) \text{ and } J_1(x) = -4x\cos(x) + 4\sin(x).$$

Hence for all $n \in \mathbf{Z}_+$,

$$J_n(x) = x^{2n+1} I_n(x) = n!\big(P_n(x)\sin(x) + Q_n(x)\cos(x)\big),$$

where $P_n(x)$ and $Q_n(x)$ are polynomials of degree $\le 2n$, and with integer coefficients (depending on n).

Take $x = \pi/2$, and suppose if possible that $\pi/2 = b/a$, where a and b are natural numbers (i.e., assume that π is rational). Then

$$\frac{b^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) = P_n\left(\frac{\pi}{2}\right) a^{2n+1}.$$

The right side is an integer. But $0 < I_n(\pi/2) < 2$ since the interval [−1, 1] has length 2 and the function which is being integrated takes only values between 0 and 1. On the other hand,

$$\frac{b^{2n+1}}{n!} \to 0 \text{ as } n \to \infty.$$

Hence for sufficiently large n

$$0 < \frac{b^{2n+1} I_n\left(\frac{\pi}{2}\right)}{n!} < 1,$$

that is, we could find an integer between 0 and 1. That is the contradiction that follows from the assumption that π is rational.

This proof is similar to Hermite's proof. Indeed,

$$\begin{aligned} J_n(x) &= x^{2n+1} \int_{-1}^{1} (1-z^2)^n \cos(xz)\, dz \\ &= 2x^{2n+1} \int_{0}^{1} (1-z^2)^n \cos(xz)\, dz \\ &= 2^{n+1} n! A_n(x). \end{aligned}$$

However, it is clearly simpler. This is achieved bypassing the inductive definition of the functions A_n and taking as a starting point their expression as an integral.

Niven's proof

This proof[9] uses the characterization of π as the smallest positive zero of the sine function.

Preparation: Suppose that π is rational, i.e. $\pi = a/b$ for some integers a and $b \neq 0$, which may be taken without loss of generality to be positive. Given any positive integer n, we define the polynomial function

$$f(x) = \frac{x^n (a - bx)^n}{n!}, \quad x \in \mathbb{R},$$

and denote by

$$F(x) = f(x) + \cdots + (-1)^j f^{(2j)}(x) + \cdots + (-1)^n f^{(2n)}(x), \quad x \in \mathbb{R},$$

the alternating sum of f and its first n even derivatives.

Claim 1: $F(0) + F(\pi)$ is an integer.

Proof: Expanding f as a sum of monomials, the coefficient of x^k is a number of the form $c_k/n!$ where c_k is an integer, which is 0 if $k < n$. Therefore, $f^{(k)}(0)$ is 0 when $k < n$ and it is equal to $(k!/n!)\, c_k$ if $n \le k \le 2n$; in each case, $f^{(k)}(0)$ is an integer and therefore $F(0)$ is an integer.

On the other hand, $f(\pi - x) = f(x)$ and so $(-1)^k f^{(k)}(\pi - x) = f^{(k)}(x)$ for each non-negative integer k.

In particular, $(-1)^k f^{(k)}(\pi) = f^{(k)}(0)$. Therefore, $f^{(k)}(\pi)$ is also an integer and so $F(\pi)$ is an integer (in fact, it is easy to see that $F(\pi) = F(0)$, but that is not relevant to the proof). Since $F(0)$ and $F(\pi)$ are integers, so is their sum.

Claim 2:

$$\int_0^\pi f(x)\sin(x)\,dx = F(0) + F(\pi)$$

Proof: Since $f^{(2n+2)}$ is the zero polynomial, we have

$$F'' + F = f.$$

The derivatives of the sine and cosine function are given by sin' = cos and cos' = −sin. Hence the product rule implies

$$(F' \cdot \sin - F \cdot \cos)' = f \cdot \sin$$

By the fundamental theorem of calculus

$$\int_0^\pi f(x)\sin(x)\,dx = \bigl(F'(x)\sin x - F(x)\cos x\bigr)\Big|_0^\pi.$$

Since $\sin 0 = \sin \pi = 0$ and $\cos 0 = -\cos \pi = 1$ (here we use the above mentioned characterization of π as a zero of the sine function), Claim 2 follows.

Conclusion: Since $f(x) > 0$ and $\sin x > 0$ for $0 < x < \pi$ (because π is the *smallest* positive zero of the sine function), Claims 1 and 2 show that $F(0) + F(\pi)$ is a *positive* integer. Since $0 \le x(a - bx) \le \pi a$ and $0 \le \sin x \le 1$ for $0 \le x \le \pi$, we have, by the original definition of f,

$$\int_0^\pi f(x)\sin(x)\,dx \le \pi\frac{(\pi a)^n}{n!}$$

which is smaller than 1 for large n, hence $F(0) + F(\pi) < 1$ for these n, by Claim 2. This is impossible for the positive integer $F(0) + F(\pi)$.

The above proof is a polished version, which is kept as simple as possible concerning the prerequisites, of an analysis of the formula

$$\int_0^\pi f(x)\sin(x)\,dx = \sum_{j=0}^{n}(-1)^j\bigl(f^{(2j)}(\pi) + f^{(2j)}(0)\bigr) + (-1)^{n+1}\int_0^\pi f^{(2n+2)}(x)\sin(x)\,dx,$$

which is obtained by $2n + 2$ integrations by parts. Claim 2 essentially establishes this formula, where the use of F hides the iterated integration by parts. The last integral vanishes because $f^{(2n+2)}$ is the zero polynomial. Claim 1 shows that the remaining sum is an integer.

Niven's proof is closer to Cartwright's (and therefore Hermite's) proof than what might look at first sight.[4] In fact,

$$\begin{aligned} J_n(x) &= x^{2n+1}\int_{-1}^{1}(1-z^2)^n\cos(xz)\,dz \\ &= \int_{-1}^{1}\left(x^2-(xz)^2\right)^n x\cos(xz)\,dz. \end{aligned}$$

Therefore, the substitution $xz = y$ turns this integral into

$$\int_{-x}^{x}(x^2-y^2)^n\cos(y)\,dy.$$

In particular,

$$\begin{aligned} J_n\left(\frac{\pi}{2}\right) &= \int_{-\pi/2}^{\pi/2}\left(\frac{\pi^2}{4}-y^2\right)^n\cos(y)\,dy \\ &= \int_0^{\pi}\left(\frac{\pi^2}{4}-\left(y-\frac{\pi}{2}\right)^2\right)^n\cos\left(y-\frac{\pi}{2}\right)\,dy \\ &= \int_0^{\pi} y^n(\pi-y)^n\sin(y)\,dy \\ &= \frac{n!}{b^n}\int_0^{\pi} f(x)\sin(x)\,dx. \end{aligned}$$

Another connection between the proofs lies in the fact that Hermite already mentions[5] that if f is a polynomial function and

$$F = f - f^{(2)} + f^{(4)} \mp \cdots,$$

then

$$\int f(x)\sin(x)\,dx = F'(x)\sin(x) - F(x)\cos(x),$$

from which it follows that

$$\int_0^{\pi} f(x)\sin(x)\,dx = F(\pi) + F(0).$$

Laczkovich's proof

Laczkovich's proof[10] is a simplification of Lambert's original proof. He considers the functions

$$\begin{aligned} f_k(x) = 1 - \frac{x^2}{k} + \frac{x^4}{2!k(k+1)} - \frac{x^6}{3!k(k+1)(k+2)} + \cdots \\ (k \notin \{0,-1,-2,\ldots\}). \end{aligned}$$

These functions are clearly defined for all $x \in \mathbf{R}$. Besides

$$f_{1/2}(x) = \cos(2x) \text{ and } f_{3/2}(x) = \frac{\sin(2x)}{2x}.$$

Claim 1: The following recurrence relation holds:

$$(\forall x\in\mathbb{R}):\frac{x^2}{k(k+1)}f_{k+2}(x)=f_{k+1}(x)-f_k(x).$$

Proof: This can be proved by comparing the coefficients of the powers of x.

Claim 2: For each $x \in \mathbf{R}$, $\lim_{k\to+\infty} f_k(x)=1.$

Proof: In fact, the sequence $x^{2n}/n!$ is bounded (since it converges to 0) and if C is an upper bound and if $k > 1$, then

$$\left|f_k(x)-1\right|\leqslant\sum_{n=1}^{\infty}\frac{C}{k^n}=C\frac{1/k}{1-1/k}=\frac{C}{k-1}.$$

Claim 3: If $x \neq 0$ and if x^2 is rational, then

$$(\forall k\in\mathbb{Q}\setminus\{0,-1,-2,\ldots\}):f_k(x)\neq 0\text{ and }\frac{f_{k+1}(x)}{f_k(x)}\notin\mathbb{Q}.$$

Proof: Otherwise, there would be a number $y \neq 0$ and integers a and b such that $f_k(x)=ay$ and $f_{k+1}(x)=by$. In order to see why, take $y = f_{k+1}(x)$, $a = 0$ and $b = 1$ if $f_k(x) = 0$; otherwise, choose integers a and b such that $f_{k+1}(x)/f_k(x) = b/a$ and define $y = f_k(x)/a = f_{k+1}(x)/b$. In each case, y cannot be 0, because otherwise it would follow from claim 1 that each $f_{k+n}(x)$ ($n \in \mathbf{N}$) would be 0, which would contradict claim 2. Now, take a natural number c such that all three numbers bc/k, ck/x^2 and c/x^2 are integers and consider the sequence

$$g_n=\begin{cases}f_k(x) & \text{if } n=0\\ \frac{c^n}{k(k+1)\cdots(k+n-1)}f_{k+n}(x) & \text{otherwise.}\end{cases}$$

Then

$$g_0=f_k(x)=ay\in\mathbb{Z}y\text{ and }g_1=\frac{c}{k}f_{k+1}(x)=\frac{bc}{k}y\in\mathbb{Z}y.$$

On the other hand, it follows from claim 1 that

$$\begin{aligned}g_{n+2}&=\frac{c^{n+2}}{x^2k(k+1)\cdots(k+n-1)}\cdot\frac{x^2}{(k+n)(k+n+1)}f_{k+n+2}(x)\\&=\frac{c^{n+2}}{x^2k(k+1)\cdots(k+n-1)}f_{k+n+1}(x)-\frac{c^{n+2}}{x^2k(k+1)\cdots(k+n-1)}f_{k+n}(x)\\&=\frac{c(k+n)}{x^2}g_{n+1}-\frac{c^2}{x^2}g_n\\&=\left(\frac{ck}{x^2}+\frac{c}{x^2}n\right)g_{n+1}-\frac{c^2}{x^2}g_n,\end{aligned}$$

which is a linear combination of g_{n+1} and g_n with integer coefficients. Therefore, each g_n is an integer multiple of y. Besides, it follows from claim 2 that each $g_n>0$ (and therefore that $g_n \geq |y|$) if n is large enough and that the sequence of all g_n's converges to 0. But, of course, a sequence of numbers greater than or equal to $|y|$ cannot converge to 0.

Since $f_{1/2}(\pi/4) = \cos(\pi/2) = 0$, it follows from claim 3 that $\pi^2/16$ is irrational and therefore π is irrational.

On the other hand, since

$$\tan x = \frac{\sin x}{\cos x} = x\frac{f_{3/2}(x/2)}{f_{1/2}(x/2)},$$

another consequence of claim 3 is that, if $x \in \mathbf{Q}\setminus\{0\}$, then tan *x* is irrational.

As Laczkovich himself acknowledges, his proof is really about the hypergeometric function. In fact, $f_k(x) = {}_0F_1(k; -x^2)$ and Gauss found a continued fraction expansion of the hypergeometric function using its functional equation.[11] This allowed Laczkovich to find a new (and simpler) proof of the fact that the tangent function has the continued fraction expansion that Lambert had discovered.

Laczkovich's result can also be expressed in Bessel functions of the first kind $J_\nu(x)$. In fact, $\Gamma(k)J_{k-1}(2x) = x^{k-1}f_k(x)$. So Laczkovich's result is equivalent to: If $x \neq 0$ and if x^2 is rational, then

$$(\forall k \in \mathbb{Q} \setminus \{0, -1, -2, \ldots\}) : \frac{xJ_k(x)}{J_{k-1}(x)} \notin \mathbb{Q}.$$

See also

- Proof that e is irrational
- Proof that π is transcendental

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