PARABOLIC OPERS AND DIFFERENTIAL OPERATORS

INDRANIL BISWAS, NIELS BORNE, SORIN DUMITRESCU, SEBASTIAN HELLER, AND CHRISTIAN PAULY

Abstract. Parabolic $\mathrm{SL}(r,\mathbb{C})$–opers were defined and investigated in [BDP] in the set-up of vector bundles on curves with a parabolic structure over a divisor. Here we introduce and study holomorphic differential operators between parabolic vector bundles over curves. We consider the parabolic $\mathrm{SL}(r,\mathbb{C})$–opers on a Riemann surface X with given singular divisor $S \subset X$ and with fixed parabolic weights satisfying the condition that all parabolic weights at any $x_i \in S$ are integral multiples of $\frac{1}{2N_i+1}$, where $N_i > 1$ are fixed integers. We prove that this space of opers is canonically identified with the affine space of holomorphic differential operators of order r between two natural parabolic line bundles on X (depending only on the divisor S and the weights N_i) satisfying the conditions that the principal symbol of the differential operators is the constant function 1 and the sub-principal symbol vanishes identically.

Contents

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1. Introduction

After the seminal work of Drinfeld and Sokolov [DS1], [DS1], the notion of opers was introduced by Beilinson and Drinfeld [BD1, BD2] as geometric structures on Riemann surfaces that formalize the notion of ordinary differential equations in a coordinate-free way. This broad formalism encapsulates the classical notion of a Riccati equation, or equivalently that of a complex projective structure on a Riemann surface, as being an $\mathrm{SL}(2,\mathbb{C})$–oper. Since then the notion of oper turned out to be very important, not only in the study of differential equations, but also in very diverse topics, as for example, geometric Langlands correspondence, nonabelian Hodge theory and also some branches of mathematics physics; see, for example, [BF], [DFK+], [FT], [FG1], [FG2], [CS], [Fr1], [Fr2], [BSY] and references therein. In contemporary research in mathematics and mathematical physics, the study of opers and their applications have been firmly established as an important topic, testified by the works of many. In particular, important progress in the understanding of opers was carried out in [BD1, BD2, FG1, FG2, AB, Wa, ABF, In, IIS1, IIS2].

In [BDP], three of the authors introduced and studied parabolic $\mathrm{SL}(r,\mathbb{C})$–opers on curves in the set-up of parabolic vector bundles as defined by Mehta and Seshadri, [MS], and also by Maruyama and Yokogawa [MY].

Later on, being inspired by the works [AB, Sa], in [BDHP] the infinitesimal deformations of parabolic $\mathrm{SL}(r,\mathbb{C})$–opers and also the monodromy map for parabolic $\mathrm{SL}(r,\mathbb{C})$–opers were studied. It may be mentioned that the appendix of [BDHP] provides an alternative definition of a parabolic $\mathrm{SL}(r,\mathbb{C})$–oper in terms of $\mathbb{R}$-filtered sheaves as introduced and studied by Maruyama and Yokogawa in [MY]. This definition is conceptually closer to the definition of an ordinary $\mathrm{SL}(r,\mathbb{C})$–oper and clarifies the one given in [BDP].

The aim of this article is to further investigate parabolic $\mathrm{SL}(r,\mathbb{C})$–opers and to characterize them as a special class of holomorphic differential operators on parabolic bundles. It should be recalled that the relation between opers and differential operators is established and well-known in the context of ordinary opers [BD1]. Here we introduce and study holomorphic differential operators on parabolic vector bundles over Riemann surfaces under the condition that at each point x_i on the singular divisor S all the parabolic weights are integral multiples of $1/2N_i+1$, with $N_i > 1$ being an integer. Under this assumption, the main result of the article, Theorem 6.2, proves that the space of all parabolic $\mathrm{SL}(r,\mathbb{C})$–opers on X with given singular set $S := \{x_1, \cdots, x_n\} \subset X$ and fixed parabolic weights integral multiples of $\frac{1}{2N_i+1}$ at each $x_i \in S$, is canonically identified with the affine space of r-order holomorphic differential operators between two natural parabolic line bundles on X (depending only on S and the weights N_i) having as principal symbol the constant function 1 and with vanishing sub-principal symbol.

The article is organized in the following way. Section 2 deals with parabolic $\mathrm{SL}(2,\mathbb{C})$–opers. In particular we introduce a rank two parabolic bundle which is a parabolic version of the indigenous bundle (also called Gunning bundle or uniformization bundle) introduced in [Gu] (see also [De]); recall that this indigenous bundle introduced by Gunning is the rank two holomorphic vector bundle associated to any ordinary $\mathrm{SL}(2,\mathbb{C})$–oper (e.g. a complex projective structure) on a given Riemann surface. It should be clarified that this parabolic analog of Gunning bundle depends only on the divisor S and the integers N_i. All parabolic $\text{SL}(2,\mathbb{C})$–opers with given singular set S and fixed weights are parabolic connections on the same parabolic Gunning bundle.

Section 3 starts with an explicit description of several (parabolic) symmetric powers of the rank two parabolic Gunning bundle constructed in Section 2; then $\text{SL}(r,\mathbb{C})$–opers on a Riemann surface X, singular over $S \subset X$, are defined (see Definition (3.3)). In this context Proposition 3.5 proves that parabolic $\text{SL}(r,\mathbb{C})$–opers on X with weights equal to integral multiples of $\frac{1}{2N_i+1}$ at each $x_i \in S$ are in natural bijection with invariant $\text{SL}(r,\mathbb{C})$–opers on a ramified Galois covering Y over X equipped with an action of the Galois group. This Proposition 3.5 is a generalization of Theorem 6.3 in [BDP] where a similar result was proved under the extra assumption that r is odd. The proof of Proposition 3.5 uses in an essential way the correspondence studied in [Bi1], [Bo1], [Bo2], and also a result (Corollary 2.5(3)) of Section 2 proving that, at each point of S, the monodromy of any parabolic connection on the parabolic Gunning bundle is semisimple.

Section 4 constructs the canonical parabolic filtration associated to any parabolic $\text{SL}(r,\mathbb{C})$–oper. This parabolic filtration depends only on S and the integers N_i. It is proved then that any parabolic connection on the associated parabolic bundle satisfies the Griffith transversality condition with respect to the above filtration (all corresponding second fundamental forms are actually isomorphisms).

Section 5 defines and study several equivalent definitions for holomorphic differential operators between parabolic vector bundles. Under the above rationality assumption on the parabolic weights, Proposition 5.2 proves that holomorphic differential operators between parabolic vector bundles are canonically identified with the invariant holomorphic differential operators between corresponding orbifold vector bundles on a ramified Galois covering Y over X equipped with an action of the Galois group. We deduce the construction of the principal symbol map defined on the space of differential operators in the parabolic set-up (see Lemma 5.3).

The last Section focuses on the class of holomorphic differential operators associated to $\text{SL}(r,\mathbb{C})$–opers. These are holomorphic differential operators between two parabolic line bundles over X naturally associated to the Gunning parabolic bundle (those line bundles only depend on the divisor S and the parabolic weights N_i). In this case the sub-principal symbol map (constructed in Lemma 6.1) defined on the affine space of parabolic differential operators between the appropriate parabolic line bundles vanish, and the principal symbol is the constant function 1. Then the main Theorem 6.2 stated above is proved.

2. A rank two parabolic bundle

Let X be a compact connected Riemann surface. Its canonical line bundle will be denoted by K_X. Fix a finite subset of n distinct points

$$S := \{x_1, \cdots, x_n\} \subset X. \tag{2.1}$$

The reduced effective divisor $x_1 + \ldots + x_n$ on X will also be denoted by S.

If $\text{genus}(X) = 0$, we assume that $n \geq 3$.

For any holomorphic vector bundle E on X, and any $k \in \mathbb{Z}$, the holomorphic vector bundle $E \otimes \mathcal{O}_X(kS)$ on X will be denoted by $E(kS)$.

Let us first start with the definition of a parabolic structure on a holomorphic vector bundle over X having S as the parabolic divisor.

2.1. **Parabolic bundles and parabolic connections.** A quasiparabolic structure on a holomorphic vector bundle E on X, associated to the divisor S, is a filtration of subspaces of the fiber E_{x_i} of E over x_i

$$E_{x_i} \,=\, E_{i,1} \,\supset\, E_{i,2} \,\supset\, \cdots \,\supset\, E_{i,l_i} \,\supset\, E_{i,l_i+1} \,=\, 0 \tag{2.2}$$

for every $1 \,\leq\, i \,\leq\, n$. A parabolic structure on E is a quasiparabolic structure as above together with a finite sequence of positive real numbers

$$0 \,\leq\, \alpha_{i,1} \,<\, \alpha_{i,2} \,<\, \cdots \,<\, \alpha_{i,l_i} \,<\, 1 \tag{2.3}$$

for every $1 \,\leq\, i \,\leq\, n$. The number $\alpha_{i,j}$ is called the parabolic weight of the corresponding subspace $E_{i,j}$ in (2.2) (see [MS], [MY]).

A parabolic vector bundle is a holomorphic vector bundle E with a parabolic structure $(\{E_{i,j}\}, \{\alpha_{i,j}\})$. It will be denoted by E_* for convenience.

A *logarithmic connection* on the holomorphic vector bundle E, singular over S, is a holomorphic differential operator of order one

$$D \,:\, E \,\longrightarrow\, E \otimes K_X \otimes \mathcal{O}_X(S)$$

satisfying the Leibniz rule, meaning

$$D(fs) \,=\, fD(s) + s \otimes df \tag{2.4}$$

for any locally defined holomorphic function f on X and any locally defined holomorphic section s of E.

Recall that any logarithmic connection on E over the Riemann surface is necessarily flat. Indeed, the curvature (2-form) vanishes identically because $\Omega^{2,0}_X \,=\, 0$.

Take a point $x_i \,\in\, S$. The fiber of $K_X \otimes \mathcal{O}_X(S)$ over x_i is identified with $\mathbb{C}$ by the Poincaré adjunction formula [GH, p. 146] which gives an isomorphism

$$\mathcal{O}_X(-x_i)_{x_i} \,\xrightarrow{\sim}\, (K_X)_{x_i}. \tag{2.5}$$

To describe this isomorphism, let z be a holomorphic coordinate function on X defined on an analytic open neighborhood of x_i such that $z(x_i) \,=\, 0$. We have an isomorphism $\mathcal{O}_X(-x_i)_{x_i} \,\longrightarrow\, (K_X)_{x_i}$ that sends z to $dz(x_i)$. It is straightforward to check that this map is actually independent of the choice of the holomorphic local coordinate z at x_i.

Let $D \,:\, E \,\longrightarrow\, E \otimes K_X \otimes \mathcal{O}_X(S)$ be a logarithmic connection on E. From (2.4) it follows that the composition of homomorphisms

$$E \,\xrightarrow{D}\, E \otimes K_X \otimes \mathcal{O}_X(S) \,\longrightarrow\, (E \otimes K_X \otimes \mathcal{O}_X(S))_{x_i} \,\xrightarrow{\sim}\, E_{x_i} \tag{2.6}$$

is $\mathcal{O}_X$–linear; the above isomorphism $(E \otimes K_X \otimes \mathcal{O}_X(S))_{x_i} \,\xrightarrow{\sim}\, E_{x_i}$ is given by the isomorphism in (2.5). Therefore, the composition of homomorphisms in (2.6) produces a $\mathbb{C}$–linear homomorphism

$$\mathrm{Res}(D\,, x_i) \,:\, E_{x_i} \longrightarrow E_{x_i}\,, \tag{2.7}$$

which is called the *residue* of the logarithmic connection D at x_i (see [De] for more details).

Remark 2.1. The local monodromy of D around x_i is conjugated to

$$\exp(-2\pi\sqrt{-1}\cdot \mathrm{Res}(D,\, x_i)) \,\in\, \mathrm{GL}(E_{x_i})$$

[De].

Consider now E with its parabolic structure $E_* \,=\, (E,\, (\{E_{i,j}\},\, \{\alpha_{i,j}\}))$; see (2.2), (2.3).

A *parabolic connection* on E_* is a logarithmic connection D on E, singular over S, such that

1. $\mathrm{Res}(D, x_i)(E_{i,j}) \,\subset\, E_{i,j}$ for all $1 \,\leq\, j \,\leq\, l_i$, $1 \,\leq\, i \,\leq\, n$ (see (2.2)), and
2. the endomorphism of $E_{i,j}/E_{i,j+1}$ induced by $\mathrm{Res}(D, x_i)$ coincides with multiplication by the parabolic weight $\alpha_{i,j}$ for all $1 \,\leq\, j \,\leq\, l_i$, $1 \,\leq\, i \,\leq\, n$ (see (2.3)).

Remark 2.2. The following necessary and sufficient condition for E_* to admit a connection was given in [BL]:

A parabolic vector bundle E_* admits a connection if and only if the parabolic degree of every direct summand of E_* is zero [BL, p. 594, Theorem 1.1].

2.2. **The parabolic Gunning bundle.** Choose a holomorphic line bundle $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes 2}$ is holomorphically isomorphic to K_X; also fix a holomorphic isomorphism between $\mathcal{L}^{\otimes 2}$ and K_X.

We have $H^1(X,\, \mathrm{Hom}(\mathcal{L}^*,\, \mathcal{L})) \,=\, H^1(X,\, K_X) \,=\, H^0(X,\, \mathcal{O}_X)^* \,=\, \mathbb{C}$ (Serre duality); note that here the chosen isomorphism between $\mathcal{L}^{\otimes 2}$ and K_X is being used. Consequently, there is a natural nontrivial extension $\widetilde{E}$ of $\mathcal{L}^*$ by $\mathcal{L}$ that corresponds to

$$1 \,\in\, H^1(X,\, \mathrm{Hom}(\mathcal{L}^*,\, \mathcal{L})).$$

So $\widetilde{E}$ fits in a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{L} \longrightarrow \widetilde{E} \xrightarrow{p_0} \mathcal{L}^* \longrightarrow 0\,; \tag{2.8}$$

this short exact sequence does not split holomorphically. Consider the subsheaf $\mathcal{L}^*(-S) \,\subset\, \mathcal{L}^*$. Define

$$E \,:=\, p_0^{-1}(\mathcal{L}^*(-S)) \,\subset\, \widetilde{E}\,,$$

where p_0 is the projection in (2.8). From (2.8) we know that this E fits in a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} E \xrightarrow{p} \mathcal{L}^*(-S) \longrightarrow 0\,; \tag{2.9}$$

the projection p in (2.9) is the restriction, to the subsheaf E, of p_0 in (2.8).

Lemma 2.3. *Take any point $x \,\in\, S$. The fiber E_x of E (see (2.9)) over x canonically decomposes as*

$$E_x \,=\, \mathcal{L}_x \oplus \mathcal{L}^*(-S)_x \,=\, \mathcal{L}_x \oplus \mathcal{L}_x\,.$$

Proof. Take $x \in S$. First we have the homomorphism

$$\iota(x) \,:\, \mathcal{L}_x \longrightarrow E_x\,, \tag{2.10}$$

where ι is the homomorphism in (2.9), which is evidently injective. On the other hand, tensoring (2.8) with $\mathcal{O}_X(-S)$ and using the natural map of it to (2.9) we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(-S) & \xrightarrow{\iota'} & \widetilde{E}(-S) & \xrightarrow{p'} & \mathcal{L}^*(-S) & \longrightarrow & 0 \\ & & \Big\downarrow \psi' & & \Big\downarrow \psi & & \Big\downarrow \mathrm{Id} & & \\ 0 & \longrightarrow & \mathcal{L} & \xrightarrow{\iota} & E & \xrightarrow{p} & \mathcal{L}^*(-S) & \longrightarrow & 0, \end{array} \tag{2.11}$$

where ι' and p' are the restrictions of ι and p respectively. Note that the composition of maps

$$\psi(x) \circ \iota'(x) \,:\, \mathcal{L}(-S)_x \longrightarrow E_x$$

in (2.11) is the zero homomorphism, because $\psi'(x) \,:\, \mathcal{L}(-S)_x \longrightarrow \mathcal{L}_x$ is the zero homomorphism and $\psi \circ \iota' = \iota \circ \psi'$ by the commutativity of (2.11). Since $\psi(x) \circ \iota'(x) = 0$, the homomorphism $\psi(x)$ is given by a homomorphism

$$q_x \,:\, \widetilde{E}(-S)_x/(\iota'(x)(\mathcal{L}(-S)_x)) \,=\, \mathcal{L}^*(-S)_x \longrightarrow E_x\,. \tag{2.12}$$

The homomorphism q_x in (2.12) is injective, because $\psi(x) \neq 0$. From (2.10) and (2.12) we have

$$\iota(x) \oplus q_x \,:\, \mathcal{L}_x \oplus \mathcal{L}^*(-S)_x \longrightarrow E_x \tag{2.13}$$

which is clearly an isomorphism.

Using (2.5) and the given isomorphism between $\mathcal{L}^{\otimes 2}$ and K_X we have

$$\mathcal{L}^*(-S)_x \,=\, ((K_X)_x \otimes \mathcal{L}^*_x)^* \otimes \mathcal{O}_X(-S)_x \,=\, (\mathcal{L}^*_x)^* \,=\, \mathcal{L}_x\,.$$

Hence the isomorphism in (2.13) gives that $E_x = \mathcal{L}_x \oplus \mathcal{L}^*(-S)_x = \mathcal{L}_x \oplus \mathcal{L}_x$. $\square$

For each $x_i \in S$ (see (2.1)), fix

$$c_i \in \mathbb{R} \tag{2.14}$$

such that $c_i > 1$. Using $\{c_i\}_{i=1}^n$ we will construct a parabolic structure on the holomorphic vector bundle E in (2.9).

For any $x_i \in S$, the quasiparabolic filtration of E_{x_i} is the following:

$$0 \subset \mathcal{L}^*(-S)_{x_i} \subset E_{x_i} \tag{2.15}$$

(see Lemma 2.3). The parabolic weight of $\mathcal{L}^*(-S)_{x_i}$ is $\frac{c_i+1}{2c_i+1}$; the parabolic weight of E_{x_i} is $\frac{c_i}{2c_i+1}$. The parabolic vector bundle defined by this parabolic structure on E will be denoted by E_*. Note that

$$\text{par-deg}(E_*) \,=\, \text{degree}(E) + \sum_{i=1}^n \left(\frac{c_i+1}{2c_i+1} + \frac{c_i}{2c_i+1}\right) \,=\, -n+n \,=\, 0\,; \tag{2.16}$$

in fact the parabolic second exterior product is

$$\det E_* \,=\, \bigwedge^2 E_* \,=\, (\bigwedge^2 E) \otimes \mathcal{O}_X(S) \,=\, \mathcal{O}_X\,, \tag{2.17}$$

where $\mathcal{O}_X$ is equipped with the trivial parabolic structure (no nonzero parabolic weights).

Proposition 2.4.

(1) *The holomorphic vector bundle* E *in* (2.9) *is isomorphic to a direct sum of holomorphic line bundles.*
(2) *The parabolic vector bundle* E_* *in* (2.16) *is not isomorphic to a direct sum of parabolic line bundles.*

Proof. Consider the short exact sequence in (2.9). Note that

$$H^1(X,\, \mathrm{Hom}(\mathcal{L}^*(-S),\, \mathcal{L})) \,=\, H^1(X,\, K_X(S)) \,=\, H^0(X,\, \mathcal{O}_X(-S))^* \,=\, 0\,.$$

Hence the short exact sequence in (2.9) splits holomorphically, and $E \,=\, \mathcal{L} \oplus \mathcal{L}^*(-S)$. This proves the first statement.

To prove the second statement by contradiction, assume that

$$E_* \,=\, A_* \oplus B_*\,, \tag{2.18}$$

where A_* and B_* are parabolic line bundles on X. Since

$$\text{par-deg}(A_*) + \text{par-deg}(B_*) \,=\, \text{par-deg}(E_*) \,=\, 0$$

(see (2.16)), at least one of A_* and B_* has nonnegative parabolic degree. Assume that $\text{par-deg}(A_*) \,\geq\, 0$. Since the parabolic degree of the quotient $\mathcal{L}^*(-S)$ in (2.9), equipped with the parabolic structure induced by E_*, is negative (recall that $n \,\geq\, 3$ if $\text{genus}(X) \,=\, 0$), there is no nonzero homomorphism from A_* to it (recall that $\text{par-deg}(A_*) \,\geq\, 0$). Consequently, the parabolic subbundle $A_* \,\subset\, E_*$ in (2.18) coincides with the subbundle $\mathcal{L}$ in (2.9) equipped with the parabolic structure induced by E_*. This implies that the following composition of homomorphisms

$$B \,\hookrightarrow\, E \,\longrightarrow\, E/\mathcal{L} \,=\, \mathcal{L}^*(-S)$$

is an isomorphism, where B denotes the holomorphic line bundle underlying B_* in (2.18). Therefore, the inclusion map $B \,\hookrightarrow\, E$ in (2.18) produces a holomorphic splitting

$$\rho\,:\, \mathcal{L}^*(-S) \,\longrightarrow\, E \tag{2.19}$$

of (2.9). Since ρ in (2.19) is given by (2.18), and the parabolic subbundle $A_* \,\subset\, E_*$ in (2.18) coincides with the subbundle $\mathcal{L}$ in (2.9) equipped with the parabolic structure induced by E_*, it follows that for all $x \,\in\, S$,

$$\rho(\mathcal{L}^*(-S)_x) \,=\, \mathcal{L}^*(-S)_x \,\subset\, E_x\,. \tag{2.20}$$

Recall that the quasiparabolic structure of E_* at x is given by the subspace $\mathcal{L}^*(-S)_x \,\subset\, E_x$ in Lemma 2.3, and therefore $\mathcal{L}^*(-S)_x$ must lie in the image, in E_*, of either A_* or B_*.

From (2.20) it follows that ρ in (2.19) satisfies the condition

$$\rho(\mathcal{L}^*(-S)) \,\subset\, \psi(\widetilde{E}(-S)) \,\subset\, E\,,$$

where ψ is the homomorphism in (2.11). Consequently, ρ produces a unique holomorphic homomorphism

$$\alpha\,:\, \mathcal{L}^*(-S) \,\longrightarrow\, \widetilde{E}(-S)$$

such that $\rho \,=\, \psi\circ\alpha$ on $\mathcal{L}^*(-S)$. This homomorphism α evidently gives a holomorphic splitting of the top exact sequence in (2.11), meaning $p'\circ\iota \,=\, \mathrm{Id}_{\mathcal{L}^*(-S)}$, where p' is the projection in (2.11). After tensoring the above homomorphism α with $\mathrm{Id}_{\mathcal{O}_X(S)}$ we get a homomorphism

$$\mathcal{L}^* = \mathcal{L}^*(-S) \otimes \mathcal{O}_X(S) \xrightarrow{\alpha \otimes \mathrm{Id}_{\mathcal{O}_X(S)}} \widetilde{E}(-S) \otimes \mathcal{O}_X(S) = \widetilde{E}$$

that splits holomorphically the short exact sequence in (2.8). But, as noted earlier, the short exact sequence in (2.8) does not split holomorphically. In view of this contradiction we conclude that there is no decomposition as in (2.18). $\square$

We recall that a parabolic connection on the parabolic vector bundle E_* in (2.16) is a logarithmic connection $D_0 : E \longrightarrow E \otimes K_X(S)$ on E singular over S such that the following conditions hold:

(1) for any $x_i \in S$ the eigenvalues of the residue $\mathrm{Res}(D_0, x_i)$ of D_0 at x_i are $\frac{c_i+1}{2c_i+1}$ and $\frac{c_i}{2c_i+1}$ (see (2.14)).
(2) The eigenspace in E_{x_i} for the eigenvalue $\frac{c_i+1}{2c_i+1}$ of $\mathrm{Res}(D_0, x_i)$ is the line
$$\mathcal{L}^*(-S)_x \subset E_{x_i}$$
in Lemma 2.3.

Let $D_0 : E \longrightarrow E \otimes K_X(S)$ be a logarithmic connection on E. Take the holomorphic line subbundle $\mathcal{L} \subset E$ in (2.9), and consider the composition of homomorphisms

$$\mathcal{L} \hookrightarrow E \xrightarrow{D_0} E \otimes K_X(S) \xrightarrow{p \otimes \mathrm{Id}_{K_X(S)}} \mathcal{L}^*(-S) \otimes K_X(S) = \mathcal{L},$$

where p is the projection in (2.9); this composition of homomorphisms will be denoted by $\mathcal{S}(D_0, \mathcal{L})$. This homomorphism

$$\mathcal{S}(D_0, \mathcal{L}) : \mathcal{L} \longrightarrow \mathcal{L} \tag{2.21}$$

is called the second fundamental form of the subbundle $\mathcal{L} \subset E$ for the logarithmic connection D_0. We note that $\mathcal{S}(D_0, \mathcal{L})$ is a constant scalar multiplication.

A parabolic connection on E_* induces a holomorphic connection on $\det E_* = \mathcal{O}_X$ (see (2.17)). Note that any holomorphic connection on $\mathcal{O}_X$ is of the form $d + \omega$, where d denotes the de Rham differential and $\omega \in H^0(X, K_X)$. A parabolic connection D_0 on E_* is called a parabolic $\mathrm{SL}(2, \mathbb{C})$–connection if the connection on $\det E_* = \mathcal{O}_X$ induced by D_0 coincides with the trivial connection d.

Corollary 2.5.

(1) *The parabolic vector bundle E_* in* (2.16) *admits a parabolic* $\mathrm{SL}(2, \mathbb{C})$*–connection.*
(2) *For any parabolic connection D_0 on E_*, the second fundamental form $\mathcal{S}(D_0, \mathcal{L})$ in* (2.21) *is an isomorphism of $\mathcal{L}$.*
(3) *For any parabolic connection D_0 on E_* the local monodromy of D_0 around any point of S is semisimple.*

Proof. In view of Remark 2.2, from (2.16) and the second statement in Proposition 2.4 it follows immediately that E_* admits a parabolic connection. Take a parabolic connection D_0 on E_*. Let $d + \omega$ be the connection on $\det E_* = \mathcal{O}_X$ induced by D_0, where $\omega \in H^0(X, K_X)$

and d is the de Rham differential. Then $D_0 - \frac{1}{2}\omega \otimes \mathrm{Id}_E$ is a parabolic $\mathrm{SL}(2,\mathbb{C})$–connection on E_*.

For any parabolic connection D_0 on E_*, consider the second fundamental form $\mathcal{S}(D_0,\, \mathcal{L})$ in the second statement. If $\mathcal{S}(D_0,\, \mathcal{L}) \,=\, 0$, then D_0 produces a parabolic connection on the line subbundle $\mathcal{L} \,\subset\, E$ in (2.9) equipped with the parabolic structure induced by E_*. But the parabolic degree of this parabolic line bundle is

$$g - 1 + \sum_{i=1}^{n} \frac{c_i}{2c_i + 1} \,>\, 0.$$

This implies that this parabolic line bundle does not admit any parabolic connection. Hence we conclude that $\mathcal{S}(D_0,\, \mathcal{L}) \,\neq\, 0$. This implies that $\mathcal{S}(D_0,\, \mathcal{L})$ is an isomorphism of $\mathcal{L}$.

The local monodromy of D_0 around any $x \,\in\, S$ is conjugate to $\exp(-2\pi\sqrt{-1}\cdot \mathrm{Res}(D_0,\, x))$ (see Remark 2.1). Hence the eigenvalues of the local monodromy for D_0 around each $x_i \,\in\, S$ are $\exp(-2\pi\sqrt{-1}\frac{c_i+1}{2c_i+1})$ and $\exp(-2\pi\sqrt{-1}\frac{c_i}{2c_i+1})$. This proves the third statement. $\square$

We will see in Corollary 4.2 that the endomorphism $\mathcal{S}(D_0,\, \mathcal{L})$ in Corollary 2.5(2) is actually independent of the parabolic connection D_0 on E_*.

Corollary 2.6. *Take any parabolic connection D_0 on E_*. There is no holomorphic line subbundle of E preserved by D_0.*

Proof. Let $L \,\subset\, E$ be a holomorphic line subbundle preserved by D_0. Denoted by L_* the parabolic line bundle defined by the parabolic structure on L induced by E_*. Since D_0 is a parabolic connection on E_*, its restriction to L is a parabolic connection on L_*. Therefore, we have

$$\text{par-deg}(L_*) \,=\, 0. \tag{2.22}$$

Consider the parabolic structure on the quotient $\mathcal{L}^*(-S)$ in (2.9) induced by E_*. Its parabolic degree is negative, and hence from (2.22) we conclude that there is no nonzero parabolic homomorphism from L_* to it. Consequently, the subbundle $L \,\subset\, E$ coincides with the subbundle $\mathcal{L}$ in (2.9). Since $L \,=\, \mathcal{L}$ is preserved by D_0, the second fundamental form $\mathcal{S}(D_0,\, \mathcal{L})$ in (2.21) vanishes identically. But this contradicts Corollary 2.5(2). Hence D_0 does not preserve any holomorphic line subbundle of E. $\square$

Given a parabolic connection D on E_*, consider its monodromy representation

$$\mathrm{Mon}_D \,:\, \pi_1(X \setminus S,\, y) \,\longrightarrow\, \mathrm{GL}(2,\mathbb{C})\,,$$

where $y \,\in\, X \setminus D$ is a base point. Corollary 2.6 implies that Mon_D is irreducible.

2.3. **Orbifold structure.**

In this subsection we assume that $\{c_i\}_{i=1}^n$ in (2.14) are all integers; recall that $c_i \,>\, 1$ for all $1 \,\leq\, i \,\leq\, n$,

There is a ramified Galois covering

$$\varphi \,:\, Y \,\longrightarrow\, X \tag{2.23}$$

satisfying the following two conditions:

- φ is unramified over the complement $X \setminus S$, and

- for every $x_i \in S$ and one (hence every) point $y \in \varphi^{-1}(x_i)$, the order of the ramification of φ at y is $2c_i + 1$.

Such a ramified Galois covering φ exists; see [Na, p. 26, Proposition 1.2.12]. Let

$$\Gamma := \mathrm{Gal}(\varphi) = \mathrm{Aut}(Y/X) \subset \mathrm{Aut}(Y) \tag{2.24}$$

be the Galois group for the Galois covering φ. A holomorphic vector bundle $V \xrightarrow{q_0} Y$ is called an *orbifold bundle* if Γ acts on the total space of V such that following three conditions hold:

1. The map $V \longrightarrow V$ given by the action of any element of Γ on V is holomorphic,
2. the projection q_0 is Γ–equivariant, and
3. the action of any $\gamma \in \Gamma$ on V is a holomorphic automorphism of the vector bundle V over the automorphism γ of Y.

Recall that the parabolic weights of E_* at any $x_i \in S$ are integral multiples of $\frac{1}{2c_i+1}$. Therefore, there is a unique, up to an isomorphism, orbifold vector bundle $\mathcal{V}$ of rank two on Y which corresponds to the parabolic vector bundle E_* [Bi], [Bo1], [Bo2]. The action of Γ on this $\mathcal{V}$ produces an action of Γ on the direct image $\varphi_*\mathcal{V}$. We have

$$(\varphi_*\mathcal{V})^\Gamma = E\,. \tag{2.25}$$

From (2.17) it follows that

$$\det \mathcal{V} = \bigwedge^2 \mathcal{V} = \mathcal{O}_Y\,, \tag{2.26}$$

and the action of Γ on the orbifold bundle $\det \mathcal{V}$ coincides with the action of Γ on $\mathcal{O}_Y$ given by the action of Γ on Y. Consider the subbundle $\mathcal{L} \subset E$ in (2.9). Let

$$\mathbf{L} \subset \mathcal{V} \tag{2.27}$$

be the orbifold line subbundle corresponding to it. So the action of Γ on $\mathcal{V}$ preserves the subbundle $\mathbf{L}$, and the subbundle

$$(\varphi_*\mathbf{L})^\Gamma \subset (\varphi_*\mathcal{V})^\Gamma = E\,.$$

coincides with $\mathcal{L}$.

The action of Γ on Y produces an action of Γ on the canonical bundle K_Y. For any automorphism $\gamma \in \Gamma$ consider its differential $d\gamma : TY \longrightarrow \gamma^*TY$. The action of γ on K_Y is given by $((d\gamma)^*)^{-1} = (d\gamma^{-1})^*$. Therefore, K_Y is an orbifold line bundle.

Lemma 2.7. *The orbifold line bundle $\mathbf{L}^{\otimes 2}$ (see (2.27)) is isomorphic to the orbifold line bundle K_Y.*

Proof. Let $\mathcal{L}_*$ denote the holomorphic line subbundle $\mathcal{L}$ in (2.9) equipped with the parabolic structure on it induced by E_*. So the underlying holomorphic line bundle for the parabolic bundle $\mathcal{L}_* \otimes \mathcal{L}_*$ is K_X, and the parabolic weight at any $x_i \in S$ is $\frac{2c_i}{2c_i+1}$. Hence the orbifold line bundle on Y corresponding to $\mathcal{L}_* \otimes \mathcal{L}_*$ is

$$(\varphi^* K_X) \otimes \mathcal{O}_Y\left(\sum_{i=1}^n 2c_i\varphi^{-1}(x_i)_{\mathrm{red}}\right) = K_Y$$

equipped with the action of Γ given by the action Γ on Y, where $\varphi^{-1}(x_i)_{\rm red}$ is the reduced inverse image of x_i. Since the orbifold line bundle $\mathbf{L}^{\otimes 2}$ corresponds to the parabolic line bundle $\mathcal{L}_*\otimes\mathcal{L}_*$, the lemma follows. $\square$

From Lemma 2.7 it follows that $\mathbf{L}$ is an orbifold theta characteristic on Y, and from (2.26) we have a short exact sequence of orbifold bundles

$$0 \longrightarrow \mathbf{L} \longrightarrow \mathcal{V} \longrightarrow \mathbf{L}^* \longrightarrow 0\,. \tag{2.28}$$

Corollary 2.8. *The short exact sequence in* (2.28) *does not admit any* Γ*–equivariant holomorphic splitting.*

Proof. If (2.28) has a Γ–equivariant holomorphic splitting, then $\mathcal{V}$ is a direct sum of orbifold line bundles. This would imply that the parabolic vector bundle E_* — that corresponds to $\mathcal{V}$ — is a direct sum of parabolic line bundles. Therefore, from Proposition 2.4(2) it follows that (2.28) does not admit any Γ–equivariant holomorphic splitting. $\square$

Actually a stronger form of Corollary 2.8 can be proved using it.

Proposition 2.9. *The short exact sequence of holomorphic vector bundles in* (2.28) *does not admit any holomorphic splitting.*

Proof. Assume that there is a holomorphic splitting

$$\rho\ :\ \mathbf{L}^* \longrightarrow \mathcal{V}$$

of the short exact sequence of holomorphic vector bundles in (2.28). Although ρ itself may not be Γ–equivariant, using it we will construct a Γ–equivariant splitting. For any $\gamma\in\Gamma$, the composition of homomorphisms

$$\mathbf{L}^* \xrightarrow{\ \gamma\ } \mathbf{L}^* \xrightarrow{\ \rho\ } \mathcal{V} \xrightarrow{\ \gamma^{-1}\ } \mathcal{V}\,,$$

which will be denoted by $\rho[\gamma]$, is also a holomorphic splitting of the short exact sequence of holomorphic vector bundles in (2.28). Now the average

$$\widetilde{\rho} := \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma}\rho[\gamma]\ :\ \mathbf{L}^* \longrightarrow \mathcal{V},$$

where $\#\Gamma$ is the order of Γ, is a Γ–equivariant holomorphic splitting of the short exact sequence of holomorphic vector bundles in (2.28). But this contradicts Corollary 2.8. Therefore, the short exact sequence of holomorphic vector bundles in (2.28) does not admit any holomorphic splitting. $\square$

The Γ–invariant holomorphic connections on $\mathcal{V}$ correspond to the parabolic connections on E_*. Moreover, the parabolic $\mathrm{SL}(2,\mathbb{C})$–connections on E_* correspond to the Γ–invariant holomorphic connections D_V on $\mathcal{V}$ that satisfy the condition that the holomorphic connection on $\det\mathcal{V} = \mathcal{O}_Y$ (see (2.26)) induced by D_V is the trivial connection on $\mathcal{O}_Y$ given by the de Rham differential.

Lemma 2.10. *The orbifold vector bundle $\mathcal{V}$ admits $\mathrm{SL}(2,\mathbb{C})$–oper connections. The parabolic $\mathrm{SL}(2,\mathbb{C})$–connections on the parabolic bundle E_* are precisely the Γ–invariant $\mathrm{SL}(2,\mathbb{C})$–oper structures on the orbifold bundle $\mathcal{V}$.*

Proof. From Proposition 2.9 it follows immediately that $\mathcal{V}$ admits $\mathrm{SL}(2,\mathbb{C})$–oper connections. Now the second statement of the lemma is deduced from the above observation that the parabolic $\mathrm{SL}(2,\mathbb{C})$–connections on E_* correspond to the Γ–invariant holomorphic connections D_V on $\mathcal{V}$ that satisfy the condition that the holomorphic connection on $\det \mathcal{V} = \mathcal{O}_Y$ induced by D_V is the trivial connection on $\mathcal{O}_Y$. $\square$

3. Symmetric powers of parabolic bundle

3.1. **Explicit description of some symmetric powers.** First we will explicitly describe the second symmetric power $\mathrm{Sym}^2(E_*)$ of the parabolic vector bundle E_* in (2.16). Consider the rank three holomorphic vector bundle $\mathrm{Sym}^2(E)$, where E is the vector bundle in (2.9). Since $\mathrm{Sym}^2(E)$ is a quotient of $E^{\otimes 2}$, any subspace of $E_x^{\otimes 2}$ produces a subspace of $\mathrm{Sym}^2(E)_x$. For each $x_i \in S$, let

$$B_i \subset \mathrm{Sym}^2(E)_{x_i} = \mathrm{Sym}^2(E_{x_i})$$

be the subspace given by the image of

$$E_{x_i} \otimes \mathcal{L}^*(-S)_{x_i} \subset E_{x_i}^{\otimes 2}$$

in $\mathrm{Sym}^2(E_{x_i})$, where $\mathcal{L}^*(-S)_{x_i} \subset E_{x_i}$ is the subspace in Lemma 2.3. Consider the unique holomorphic vector bundle E^2 of rank three on X that fits in the following short exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow E^2 \longrightarrow \mathrm{Sym}^2(E)(S) := \mathrm{Sym}^2(E) \otimes \mathcal{O}_X(S) \\ \longrightarrow \bigoplus_{i=1}^{n} \left(\mathrm{Sym}^2(E)_{x_i}/B_i\right) \otimes \mathcal{O}_X(S)_{x_i} \longrightarrow 0\,. \end{aligned} \tag{3.1}$$

The holomorphic vector bundle underlying the parabolic vector bundle $\mathrm{Sym}^2(E_*)$ is E^2.

Lemma 3.1. *For every $x_i \in S$, the fiber E_x^2 fits in a natural exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathcal{L}_{x_i}^{\otimes 2} \longrightarrow E_{x_i}^2 \longrightarrow B_i \otimes \mathcal{O}_X(S)_{x_i} \\ = (E_{x_i} \otimes \mathcal{L}^*(-S)_{x_i}) \otimes \mathcal{O}_X(S)_{x_i} = (E \otimes \mathcal{L}^*)_{x_i} \longrightarrow 0\,. \end{aligned}$$

Proof. Consider the commutative digram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Sym}^2(E) & \longrightarrow & \mathrm{Sym}^2(E)(S) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Sym}^2(E)(S)_{x_i} & \longrightarrow & 0 \\ & & \Big\downarrow \mathbf{f} & & \Big\downarrow \mathrm{Id} & & \Big\downarrow & & \\ 0 & \longrightarrow & E^2 & \longrightarrow & \mathrm{Sym}^2(E)(S) & \longrightarrow & \bigoplus_{i=1}^n \frac{\mathrm{Sym}^2(E)(S)_{x_i}}{B_i \otimes \mathcal{O}_X(S)_{x_i}} & \longrightarrow & 0. \end{array}$$

For any $x \in S$, the map $\mathbf{f}(x) : \mathrm{Sym}^2(E)_x \longrightarrow E_x^2$ is injective on the subspace $\mathcal{L}_{x_i}^{\otimes 2} \hookrightarrow \mathrm{Sym}^2(E)_{x_i}$, and moreover $\mathbf{f}(x_i)(\mathcal{L}_{x_i}^{\otimes 2}) \subset E_{x_i}^2$ coincides with $\mathbf{f}(x_i)(\mathrm{Sym}^2(E)_{x_i})$. Therefore, the subspace $\mathcal{L}_{x_i}^{\otimes 2} \hookrightarrow E_{x_i}^2$ in the lemma is the image of the homomorphism $\mathbf{f}(x_i)$.

For the map $E^2 \longrightarrow \mathrm{Sym}^2(E)(S) := \mathrm{Sym}^2(E)\otimes \mathcal{O}_X(S)$ in (3.1), the image of $E^2_{x_i}$ is

$$B_i\otimes \mathcal{O}_X(S)_{x_i} \,=\, (E_{x_i}\otimes \mathcal{L}^*(-S)_{x_i})\otimes \mathcal{O}_X(S)_{x_i} \,=\, (E\otimes \mathcal{L}^*)_{x_i} \,\subset\, \mathrm{Sym}^2(E)(S)_{x_i}\,.$$

This proves the lemma. □

For any $x_i \in S$, consider the subspace

$$\mathcal{L}^*(-S)^{\otimes 2}_{x_i} \,\subset\, B_i \,=\, (\mathcal{L}_{x_i}\otimes \mathcal{L}^*(-S)_{x_i})\oplus \mathcal{L}^*(-S)^{\otimes 2}_{x_i}\,.$$

Let

$$\mathcal{F}_i \,\subset\, E^2_{x_i} \tag{3.2}$$

be the inverse image of $\mathcal{L}^*(-S)^{\otimes 2}_{x_i}\otimes \mathcal{O}_X(S)_{x_i} \,\subset\, B_i\otimes \mathcal{O}_X(S)_{x_i}$ for the quotient map $E^2_{x_i} \longrightarrow B_i\otimes \mathcal{O}_X(S)_{x_i}$ in Lemma 3.1.

As mentioned before, the holomorphic vector bundle underlying the parabolic vector bundle $\mathrm{Sym}^2(E_*)$ is E^2. The quasiparabolic filtration of $E^2_{x_i}$, where $x_i \in S$, is the following:

$$\mathcal{L}^{\otimes 2}_{x_i} \,\subset\, \mathcal{F}_i \,\subset\, E^2_{x_i}\,, \tag{3.3}$$

where $\mathcal{L}^{\otimes 2}_{x_i}$ and $\mathcal{F}_i$ are the subspaces in Lemma 3.1 and (3.2) respectively. The parabolic weight of $\mathcal{L}^{\otimes 2}_{x_i}$ is $\frac{2c_i}{2c_i+1}$ and the parabolic weight of $\mathcal{F}_i$ is $\frac{1}{2c_i+1}$; the parabolic weight of $E^2_{x_i}$ is 0.

The parabolic symmetric product $\mathrm{Sym}^3(E_*)$ is actually a little easier to describe. The holomorphic vector bundle underlying the parabolic vector bundle $\mathrm{Sym}^3(E_*)$ is the rank three vector bundle

$$E^3 \,:=\, (\mathrm{Sym}^3(E))\otimes \mathcal{O}_X(S). \tag{3.4}$$

For each $x_i \in S$, the decomposition of E_{x_i} in Lemma 2.3 gives the following decomposition of the fiber $E^3_{x_i}$:

$$\big((\mathcal{L}^*(-S)^{\otimes 3}_{x_i})\oplus(\mathcal{L}^*(-S)^{\otimes 2}_{x_i}\otimes \mathcal{L}_{x_i})\oplus(\mathcal{L}^*(-S)_{x_i}\otimes \mathcal{L}^{\otimes 2}_{x_i})\oplus(\mathcal{L}^{\otimes 3}_{x_i})\big)\otimes \mathcal{O}_X(S)_{x_i} \,=\, E^3_{x_i}\,. \tag{3.5}$$

The quasiparabolic filtration of $E^3_{x_i}$ is

$$\begin{aligned}(\mathcal{L}^*(-S)^{\otimes 3}_{x_i})\otimes \mathcal{O}_X(S)_{x_i} \,&\subset\, \big((\mathcal{L}^*(-S)^{\otimes 3}_{x_i})\oplus(\mathcal{L}^*(-S)^{\otimes 2}_{x_i}\otimes \mathcal{L}_{x_i})\big)\otimes \mathcal{O}_X(S)_{x_i} \\ \subset\, \big((\mathcal{L}^*(-S)^{\otimes 3}_{x_i})&\oplus(\mathcal{L}^*(-S)^{\otimes 2}_{x_i}\otimes \mathcal{L}_{x_i})\oplus(\mathcal{L}^*(-S)_{x_i}\otimes \mathcal{L}^{\otimes 2}_{x_i})\big)\otimes \mathcal{O}_X(S)_{x_i} \,\subset\, E^3_{x_i}.\end{aligned} \tag{3.6}$$

The parabolic weight of $\mathcal{L}^*(-S)^{\otimes 3}_{x_i}\otimes \mathcal{O}_X(S)_{x_i}$ is $\frac{c_i+2}{2c_i+1}$, The parabolic weight of

$$\big((\mathcal{L}^*(-S)^{\otimes 3}_{x_i})\oplus(\mathcal{L}^*(-S)^{\otimes 2}_{x_i}\otimes \mathcal{L}_{x_i})\big)\otimes \mathcal{O}_X(S)_{x_i}$$

is $\frac{c_i+1}{2c_i+1}$, the parabolic weight of $\big((\mathcal{L}^*(-S)^{\otimes 3}_{x_i})\oplus(\mathcal{L}^*(-S)^{\otimes 2}_{x_i}\otimes \mathcal{L}_{x_i})\oplus(\mathcal{L}^*(-S)_{x_i}\otimes \mathcal{L}^{\otimes 2}_{x_i})\big)\otimes \mathcal{O}_X(S)_{x_i}$ is $\frac{c_i}{2c_i+1}$, and the parabolic weight of $E^3_{x_i}$ is $\frac{c_i-1}{2c_i+1}$.

Finally, we will describe the parabolic symmetric product $\mathrm{Sym}^4(E_*)$. Consider the rank five vector bundle

$$\mathrm{Sym}^4(E)(2S) \,=\, (\mathrm{Sym}^4(E))\otimes \mathcal{O}_X(2S)\,.$$

Using Lemma 2.3, the fiber $\mathrm{Sym}^4(E)(2S)_{x_i}$, where $x_i \in S$, decomposes into a direct sum of lines. More precisely, as in (3.5),

$$\begin{aligned}\mathrm{Sym}^4(E)(2S)_{x_i} \,=\, &((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i}\oplus((\mathcal{L}^*)^{\otimes 3}\otimes \mathcal{L}(-S))_{x_i} \\ &\oplus((\mathcal{L}^*)^{\otimes 2}\otimes \mathcal{L}^{\otimes 2})_{x_i}\oplus(\mathcal{L}^*\otimes \mathcal{L}^{\otimes 3}(S))_{x_i}\oplus(\mathcal{L}^{\otimes 4}(2S))_{x_i}.\end{aligned} \tag{3.7}$$

Let E^4 denote the vector bundle of rank five defined by the following short exact sequence of sheaves:

$$0 \longrightarrow E^4 \xrightarrow{\mathbf{h}} \mathrm{Sym}^4(E)(2S) \longrightarrow \tag{3.8}$$
$$\bigoplus_{i=1}^n \mathcal{Q}_i = \bigoplus_{i=1}^n \frac{\mathrm{Sym}^4(E)(2S)_{x_i}}{((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2})_{x_i}} \longrightarrow 0,$$

where

$$\mathcal{Q}_i := \frac{\mathrm{Sym}^4(E)(2S)_{x_i}}{((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2})_{x_i}}. \tag{3.9}$$

The holomorphic vector bundle underlying the parabolic vector bundle $\mathrm{Sym}^4(E_*)$ is E^4 defined in (3.8).

Lemma 3.2. *For every $x_i \in S$, the fiber $E^4_{x_i}$ fits in the following short exact sequence of vector spaces:*

$$0 \longrightarrow (\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i} \longrightarrow E^4_{x_i}$$
$$\xrightarrow{\rho_i} ((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2})_{x_i} \longrightarrow 0\,.$$

Proof. The projection

$$\rho_i : E^4_{x_i} \longrightarrow ((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2})_{x_i}$$

in the lemma is given by the homomorphism $\mathbf{h}(x_i)$ in (3.8). To describe the homomorphism

$$(\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i} \longrightarrow E^4_{x_i}$$

in the lemma, we consider the commutative diagram of homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sym}^4(E)(S) \arrow[r] \arrow[d, "\mathbf{f}"] & \mathrm{Sym}^4(E)(2S) \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathrm{Sym}^4(E)(2S)_{x_i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^4 \arrow[r] & \mathrm{Sym}^4(E)(2S) \arrow[r] & \bigoplus_{i=1}^n \mathcal{Q}_i \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{Q}_i$ is defined in (3.9). Let

$$\mathbf{f}(x_i) : \mathrm{Sym}^4(E)(S)_{x_i} \longrightarrow E^4_{x_i} \tag{3.10}$$

be the restriction of it to $x_i \in S$. As in (3.7), we have the decomposition

$$\mathrm{Sym}^4(E)(S)_{x_i} = ((\mathcal{L}^*)^{\otimes 4}(-3S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-2S))_{x_i}$$
$$\oplus((\mathcal{L}^*)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}(-S))_{x_i} \oplus (\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i}.$$

The subspace

$$((\mathcal{L}^*)^{\otimes 4}(-3S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-2S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}(-S))_{x_i} \subset \mathrm{Sym}^4(E)(S)_{x_i}$$

is the kernel of the homomorphism $\mathbf{f}(x_i)$ in (3.10). The restriction of $\mathbf{f}(x_i)$ to the subspace

$$(\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i} \subset \mathrm{Sym}^4(E)(S)_{x_i}$$

is injective. Therefore, $\mathbf{f}(x_i)$ gives the homomorphism

$$(\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i} \longrightarrow E^4_{x_i}$$

in the lemma. It is evident that the quotient map $E^4_{x_i} \longrightarrow E^4_{x_i}/((\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i})$ coincides with ρ_i. $\square$

Define the subspaces

$$\mathcal{F}_3^i := \rho_i^{-1}(((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i}) \subset \mathcal{F}_4^i := \rho_i^{-1}(((\mathcal{L}^*)^{\otimes 4}(-2S))_{x_i} \oplus ((\mathcal{L}^*)^{\otimes 3} \otimes \mathcal{L}(-S))_{x_i}) \subset E^4_{x_i} \tag{3.11}$$

where ρ_i is the homomorphism in Lemma 3.2.

As mentioned before, the holomorphic vector bundle underlying the parabolic vector bundle $\mathrm{Sym}^4(E_*)$ is E^4. The quasiparabolic filtration of $E^4_{x_i}$ is

$$(\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \subset (\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i} \subset \mathcal{F}_3^i \subset \mathcal{F}_4^i \subset E^4_{x_i}$$

(see Lemma 3.2 and (3.11)). The parabolic weight of $(\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i}$ is $\frac{2c_i}{2c_i+1}$, the parabolic weight of $(\mathcal{L}^* \otimes \mathcal{L}^{\otimes 3})_{x_i} \oplus (\mathcal{L}^{\otimes 4}(S))_{x_i}$ is $\frac{2c_i-1}{2c_i+1}$, the parabolic weight of $\mathcal{F}_3^i$ is $\frac{2}{2c_i+1}$, the parabolic weight of $\mathcal{F}_4^i$ is $\frac{1}{2c_i+1}$ and the parabolic weight of $E^4_{x_i}$ is 0.

3.2. **Higher rank parabolic opers.** For any $r \geq 2$, consider the parabolic vector bundle of rank r defined by the symmetric product $\mathrm{Sym}^{r-1}(E_*)$ of the parabolic vector bundle E_* in (2.16). Since $\det E_* = \mathcal{O}_X$ (see (2.17)), it follows that

$$\det \mathrm{Sym}^{r-1}(E_*) = \bigwedge^r \mathrm{Sym}^{r-1}(E_*) = \mathcal{O}_X, \tag{3.12}$$

where $\mathcal{O}_X$ is equipped with the trivial parabolic structure (no nonzero parabolic weights).

A parabolic $\mathrm{SL}(r,\mathbb{C})$–connection on $\mathrm{Sym}^{r-1}(E_*)$ is a parabolic connection on $\mathrm{Sym}^{r-1}(E_*)$ satisfying the condition that the induced parabolic connection on $\det \mathrm{Sym}^{r-1}(E_*) = \mathcal{O}_X$ is the trivial connection.

Two parabolic $\mathrm{SL}(r,\mathbb{C})$–connections on $\mathrm{Sym}^{r-1}(E_*)$ are called equivalent if they differ by a holomorphic automorphism of the parabolic bundle $\mathrm{Sym}^{r-1}(E_*)$. If D_1 is a parabolic $\mathrm{SL}(r,\mathbb{C})$–connection on $\mathrm{Sym}^{r-1}(E_*)$, and D_2 is another parabolic connection on $\mathrm{Sym}^{r-1}(E_*)$ equivalent to D_1, then D_2 is clearly a parabolic $\mathrm{SL}(r,\mathbb{C})$–connection. Indeed, this follows immediately from the fact that the holomorphic automorphisms of a holomorphic line bundle $\mathbb{L}$ on X act trivially on the space of all logarithmic connections on $\mathbb{L}$.

Definition 3.3. A parabolic $\mathrm{SL}(r,\mathbb{C})$–*oper* on X is an equivalence class of parabolic $\mathrm{SL}(r,\mathbb{C})$–connections on $\mathrm{Sym}^{r-1}(E_*)$.

Proposition 3.4.

(1) *The parabolic vector bundle* $\mathrm{Sym}^{r-1}(E_*)$ *admits a parabolic* $\mathrm{SL}(r,\mathbb{C})$*–connection.*
(2) *For any parabolic connection* D_r *on* $\mathrm{Sym}^{r-1}(E_*)$*, the local monodromy of* D_r *around any* $x_i \in S$ *is semisimple.*

Proof. Any parabolic connection on E_* induces a parabolic connection on $\mathrm{Sym}^{r-1}(E_*)$. Moreover, a parabolic $\mathrm{SL}(2,\mathbb{C})$–connection on E_* induces a parabolic $\mathrm{SL}(r,\mathbb{C})$–connection on $\mathrm{Sym}^{r-1}(E_*)$. Therefore, from Corollary 2.5(1) it follows that $\mathrm{Sym}^{r-1}(E_*)$ admits a parabolic connection on E_*.

Let D_2 be a parabolic $\mathrm{SL}(2,\mathbb{C})$–connection on E_*. Denote by D_r the parabolic connection on $\mathrm{Sym}^{r-1}(E_*)$ induced by D_2. From Corollary 2.5(3) we know that the local monodromy of D_2 around any $x_i \in S$ is semisimple. Since the local monodromy of D_r around any

$x_i \in S$ is simply the $(r-1)$-th symmetric product of the local monodromy of D_2 around $x_i \in S$, and the local monodromy of D_2 around $x_i \in S$ is semisimple, it follows that the local monodromy of D_r around $x_i \in S$ is semisimple.

We have shown that $\mathrm{Sym}^{r-1}(E_*)$ admits a parabolic connection for which the local monodromy around any $x_i \in S$ is semisimple. On the other hand, the space of parabolic connections on $\mathrm{Sym}^{r-1}(E_*)$ is an affine space for the vector space

$$H^0(X,\, \mathrm{End}^n(\mathrm{Sym}^{r-1}(E_*)) \otimes K_X(S)),$$

where

$$\mathrm{End}^n(\mathrm{Sym}^{r-1}(E_*)) \subset \mathrm{End}(\mathrm{Sym}^{r-1}(E_*)) \tag{3.13}$$

is the subsheaf defined by the sheaf of endomorphisms nilpotent with respect to the quasi-parabolic filtrations of $\mathrm{Sym}^{r-1}(E_*)$ over S. Consequently, using Remark 2.1 it follows that for every parabolic connection D'_r on $\mathrm{Sym}^{r-1}(E_*)$ the local monodromy of D'_r around any $x_i \in S$ is semisimple. ☐

In the rest of this section we assume that c_i, $1 \le i \le n$, in (2.14) are integers. Take a ramified Galois covering $\varphi : Y \longrightarrow X$ as in (2.23). As in Section 2, let $\mathcal{V}$ denote the orbifold bundle on Y corresponding to the parabolic bundle E_* on X. The action of the Galois group $\Gamma = \mathrm{Gal}(\varphi)$ on $\mathcal{V}$ produces an action of Γ on $\mathrm{Sym}^{r-1}(\mathcal{V})$. A holomorphic connection on $\mathrm{Sym}^{r-1}(\mathcal{V})$ is called *equivariant* if it is preserved by the action of Γ on $\mathrm{Sym}^{r-1}(\mathcal{V})$.

From (3.12) it follows immediately that

$$\det \mathrm{Sym}^{r-1}(\mathcal{V}) \,=\, \bigwedge^r \mathrm{Sym}^{r-1}(\mathcal{V}) \,=\, \mathcal{O}_Y.$$

An $\mathrm{SL}(r,\mathbb{C})$–connection on $\mathrm{Sym}^{r-1}(\mathcal{V})$ is a holomorphic connection D'_r on $\mathrm{Sym}^{r-1}(\mathcal{V})$ such that the connection on $\det \mathrm{Sym}^{r-1}(\mathcal{V}) = \mathcal{O}_Y$ induced by D'_r coincides with the trivial connection on $\mathcal{O}_Y$. Two equivariant $\mathrm{SL}(r,\mathbb{C})$–connections on $\mathrm{Sym}^{r-1}(\mathcal{V})$ are called equivalent if they differ by a holomorphic Γ–equivariant automorphism of $\mathrm{Sym}^{r-1}(\mathcal{V})$.

Proposition 3.5. *There is a natural bijection between that parabolic* $\mathrm{SL}(r,\mathbb{C})$*–opers on* X *and the equivalence classes of equivariant* $\mathrm{SL}(r,\mathbb{C})$*–connections on* $\mathrm{Sym}^{r-1}(\mathcal{V})$.

Proof. Let D_2 be a parabolic connection on E_*. Since the local monodromy of D_2 around any $x_i \in S$ is semisimple, it corresponds to an equivariant holomorphic connection $\widehat{D}_2$ on $\mathcal{V}$. Let $\widehat{D}_r$ be the equivariant connection on $\mathrm{Sym}^{r-1}(\mathcal{V})$ induced by $\widehat{D}_2$. As before, D_r denotes the parabolic connection on $\mathrm{Sym}^{r-1}(E_*)$ induced by D_2. Therefore, $\widehat{D}_r$ corresponds to D_r.

The holomorphic vector bundle underlying the parabolic bundle $\mathrm{Sym}^{r-1}(E_*)$ is denoted by $\mathrm{Sym}^{r-1}(E_*)_0$ [MY]. As in (3.13), let

$$\mathrm{End}^n(\mathrm{Sym}^{r-1}(E_*)) \subset \mathrm{End}(\mathrm{Sym}^{r-1}(E_*)_0)$$

be the coherent analytic subsheaf consisting of all locally defined sections s of the endomorphism bundle $\mathrm{End}(\mathrm{Sym}^{r-1}(E_*)_0)$ satisfying the condition that $s(x)$ is nilpotent with respect to the quasi-parabolic filtration of $\mathrm{Sym}^{r-1}(E_*)_x$ for all $x \in S$ lying in the domain of s. Recall that any parabolic connection on $\mathrm{Sym}^{r-1}(E_*)$ is of the form $D_r + \theta$ for some

$$\theta \in H^0(X,\, \mathrm{End}^n(\mathrm{Sym}^{r-1}(E_*)) \otimes K_X(S)).$$

We have

$$H^0(X,\, \text{End}^n(\text{Sym}^{r-1}(E_*)) \otimes K_X(S)) \;=\; H^0(Y,\, \text{End}(\text{Sym}^{r-1}(\mathcal{V})))^\Gamma. \tag{3.14}$$

Also the space of all equivariant holomorphic connections on $\text{Sym}^{r-1}(\mathcal{V})$ is an affine space for $H^0(Y,\, \text{End}(\text{Sym}^{r-1}(\mathcal{V})))^\Gamma$.

The parabolic connection $D_r + \theta$, where $\theta \in H^0(X,\, \text{End}^n(\text{Sym}^{r-1}(E_*)) \otimes K_X(S))$, corresponds to the equivariant connection $\widehat{D}_r + \widehat{\theta}$ on $\text{Sym}^{r-1}(\mathcal{V})$, where $\widehat{\theta} \in H^0(Y,\, \text{End}(\text{Sym}^{r-1}(\mathcal{V})))^\Gamma$ corresponds to θ by the isomorphism in (3.14). Also, parabolic automorphisms of $\text{Sym}^{r-1}(E_*)$ are identified with the Γ–equivariant automorphisms of $\mathcal{V}$. Now the proposition follows from (3.14), Proposition 3.4 and Definition 3.3. $\square$

The above Proposition 3.5 is a generalization of Theorem 6.3 in [BDP] where a similar statement was proved under the extra assumption that r is odd.

4. Some properties of parabolic opers

Consider the vector bundle E in (2.9). Let

$$\text{End}^n(E_*) \subset \text{End}(E) \tag{4.1}$$

be the coherent analytic subsheaf defined by the conditions that $s(E_x) \subset \mathcal{L}^*(-S)_x$ and $s(\mathcal{L}^*(-S)_x) = 0$ for all $x \in S$ lying in the domain of the local section s of $\text{End}(E)$ (see Lemma 2.3). Take any

$$\phi \in H^0(X,\, \text{End}^n(E_*) \otimes K_X(S)).$$

Let

$$\widehat{\phi} : \mathcal{L} \longrightarrow \mathcal{L}^*(-S) \otimes K_X(S) = \mathcal{L} \tag{4.2}$$

be the homomorphism given by the following composition of homomorphisms:

$$\mathcal{L} \xrightarrow{\iota} E \xrightarrow{\phi} E \otimes K_X(S) \xrightarrow{p \otimes \text{Id}_{K_X(S)}} \mathcal{L}^*(-S) \otimes K_X(S) = \mathcal{L},$$

where ι and p are the homomorphisms in (2.9); recall that $\mathcal{L}^{\otimes 2} = K_X$.

Proposition 4.1. *For every $\phi \in H^0(X,\, \text{End}^n(E_*) \otimes K_X(S))$ the homomorphism $\widehat{\phi}$ constructed from it in* (4.2) *vanishes identically.*

Proof. Tensoring the diagram in (2.11) with $K_X(S)$ we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L} \otimes K_X & \longrightarrow & \widetilde{E} \otimes K_X & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow q & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} \otimes K_X(S) & \longrightarrow & E \otimes K_X(S) & \longrightarrow & \mathcal{L} & \longrightarrow & 0.
\end{array} \tag{4.3}$$

Take any $\phi \in H^0(X,\, \text{End}^n(E_*) \otimes K_X(S))$. Consider the composition of homomorphisms

$$\widetilde{E}(-S) \xrightarrow{\psi} E \xrightarrow{\phi} E \otimes K_X(S),$$

where ψ is the homomorphism in (2.11), and denote this composition by $\widetilde{\phi}$. From (4.3), (4.1) and the construction of the decomposition in Lemma 2.3 it follows that the image of this homomorphism $\widetilde{\phi} : \widetilde{E}(-S) \longrightarrow E \otimes K_X(S)$ is contained in the image of the homomorphism

q in (4.3); in other words, the subsheaf $\phi\circ\psi(\widetilde{E}(-S)) \subset E\otimes K_X(S)$ lies in the image of the homomorphism

$$\psi\otimes \mathrm{Id}_{K_X(S)} \,:\, \widetilde{E}(-S)\otimes K_X(S) \,=\, \widetilde{E}\otimes K_X \,\longrightarrow\, E\otimes K_X(S)\,.$$

Consequently, ϕ produces a homomorphism

$$\phi' \,:\, \widetilde{E}(-S) \,\longrightarrow\, \widetilde{E}\otimes K_X\,. \tag{4.4}$$

More precisely, ϕ' is determined uniquely by the condition

$$\widetilde{\phi} \,=\, (\psi\otimes \mathrm{Id}_{K_X(S)})\circ\phi'.$$

Let

$$\phi'' \,:\, \mathcal{L}(-S) \,\longrightarrow\, \mathcal{L} \tag{4.5}$$

denote the following composition of homomorphisms

$$\mathcal{L}(-S) \xrightarrow{\iota'} \widetilde{E}(-S) \xrightarrow{\phi'} \widetilde{E}\otimes K_X \xrightarrow{p_0\otimes \mathrm{Id}_{K_X}} \mathcal{L}^*\otimes K_X \,=\, \mathcal{L}\,,$$

where ι' and p_0 are the homomorphisms in (2.11) and (2.8) respectively. To prove the proposition it suffices to show that ϕ'' in (4.5) vanishes identically.

Take any $x_i \in S$. Since

$$q(\phi'(x_i)(\widetilde{E}(-S)_{x_i})) \,=\, \phi(\psi(x_i)(\widetilde{E}(-S)_{x_i})) \,=\, \phi(\mathcal{L}^*(-S)_{x_i}) \,=\, 0\,,$$

where ψ, ϕ' and q are the homomorphisms in (2.11), (4.4) and (4.3) respectively, we conclude that

$$\phi'(x_i)(\widetilde{E}(-S)_{x_i}) \,\subset\, (\mathcal{L}\otimes K_X)_{x_i} \,\subset\, (\widetilde{E}\otimes K_X)_{x_i}\,, \tag{4.6}$$

where ϕ' is the homomorphism in (4.4) and $\mathcal{L}\,\subset\,\widetilde{E}$ is the subbundle in (2.8).

Furthermore, it can be shown that

$$\phi'(x_i)(\mathcal{L}(-S)_{x_i}) \,=\, 0\,; \tag{4.7}$$

see (2.11) for the subspace $\mathcal{L}(-S)_{x_i} \,\subset\, \widetilde{E}(-S)_{x_i}$. Indeed, this again follows from (2.11), (4.3), (4.1) and the construction of the decomposition in Lemma 2.3.

In view of (4.6) and (4.7), the homomorphism ϕ'' in (4.5) vanishes at each x_i. Therefore, ϕ'' produces a homomorphism

$$\phi''' \,:\, \mathcal{L}(-S) \,\longrightarrow\, \mathcal{L}(-S)\,. \tag{4.8}$$

Consider the image $\phi'(\mathcal{L}(-S)) \,\subset\, \widetilde{E}\otimes K_X$, where ϕ' is the homomorphism in (4.4). If the homomorphism ϕ''' in (4.8) in nonzero, then this subsheaf $\phi'(\mathcal{L}(-S))$ produces a holomorphic splitting of the top short exact sequence in (2.11) tensored with K_X. Indeed, in that case the homomorphism $p'\otimes\mathrm{Id}_{K_X}$ (see (2.11) for p') maps $\phi'(\mathcal{L}(-S))$ surjectively to $\mathcal{L}^*(-S)\otimes K_X \,=\, \mathcal{L}(-S)$ and hence $\phi'(\mathcal{L}(-S))$ gives a holomorphic splitting of the short exact sequence

$$0 \longrightarrow \mathcal{L}(-S)\otimes K_X \longrightarrow \widetilde{E}(-S)\otimes K_X \longrightarrow \mathcal{L}^*(-S)\otimes K_X \longrightarrow 0$$

obtained from the top exact sequence in (2.11) by tensoring it with K_X. A holomorphic splitting of the above exact sequence produces a holomorphic splitting of the top short exact sequence in (2.11). But the exact sequence in (2.8) does not split holomorphically, which implies that the top short exact sequence in (2.11) does not split holomorphically. This implies that $\phi''' = 0$ (see (4.8)), and hence $\phi'' = 0$ (see (4.5)). As noted before, to prove the proposition it is enough to show that ϕ'' vanishes identically. This completes the proof. $\square$

Corollary 4.2. *The endomorphism $\mathcal{S}(D_0, \mathcal{L}) : \mathcal{L} \longrightarrow \mathcal{L}$ in Corollary 2.5(2) does not depend on the parabolic connection D_0.*

Proof. The space of parabolic connections on E_* is an affine space for the vector space $H^0(X, \mathrm{End}^n(E_*) \otimes K_X(S))$. Note that for any parabolic connection D on E_* and any $\phi \in H^0(X, \mathrm{End}^n(E_*) \otimes K_X(S))$, we have

$$\mathcal{S}(D + \phi, \mathcal{L}) = \mathcal{S}(D, \mathcal{L}) + \widehat{\phi},$$

where $\widehat{\phi}$ is constructed in (4.2) from ϕ. Therefore, from Proposition 4.1 it follows immediately that $\mathcal{S}(D + \phi, \mathcal{L}) = \mathcal{S}(D, \mathcal{L})$. $\square$

As before, let $\mathcal{L}_*$ denote the holomorphic line subbundle $\mathcal{L}$ in (2.9) equipped with the parabolic structure on it induced by E_*. We denote by $E_*/\mathcal{L}_*$ the quotient line bundle $E/\mathcal{L}$ in (2.9) equipped with the parabolic structure on it induced by E_*. So from (2.9) we have a short exact sequence of parabolic bundles

$$0 \longrightarrow \mathcal{L}_* \longrightarrow E_* \longrightarrow E_*/\mathcal{L}_* \longrightarrow 0\,. \tag{4.9}$$

For notational convenience, both $\mathrm{Sym}^0(E_*)$ and $(\mathcal{L}_*)^0$ will denote the trivial holomorphic line bundle $\mathcal{O}_X$ equipped with the trivial parabolic structure (no nonzero parabolic weights). Since $\mathrm{Sym}^{r-1}(E_*)$ is a quotient of $(E_*)^{\otimes(r-1)}$, we have a natural homomorphism of parabolic bundles

$$\tau_j : \mathrm{Sym}^{j-1}(E_*) \otimes (\mathcal{L}_*)^{r-j} \longrightarrow \mathrm{Sym}^{r-1}(E_*)$$

for every $1 \leq j \leq r$ (see (4.9)). This τ_j is an injective homomorphism, and its image is a parabolic subbundle of $\mathrm{Sym}^{r-1}(E_*)$. Let

$$\mathcal{F}^j_* := \mathrm{Image}(\tau_j) \subset \mathrm{Sym}^{r-1}(E_*)$$

be the parabolic subbundle; its rank is j. So we have a filtration of parabolic subbundles

$$0 = \mathcal{F}^0_* \subset \mathcal{F}^1_* \subset \mathcal{F}^2_* \subset \cdots \subset \mathcal{F}^{r-1}_* \subset \mathcal{F}^r_* = \mathrm{Sym}^{r-1}(E_*). \tag{4.10}$$

The holomorphic vector bundle underlying any $\mathcal{F}^i_*$ will be denoted by $\mathcal{F}^i_0$.

For any $1 \leq j \leq r$, the quotient parabolic line bundle $\mathcal{F}^j_*/\mathcal{F}^{j-1}_*$ in (4.10) actually has the following description:

$$\mathcal{F}^j_*/\mathcal{F}^{j-1}_* = (\mathcal{L}_*)^{r-j} \otimes (E_*/\mathcal{L}_*)^{j-1}\,. \tag{4.11}$$

Indeed, this follows immediately from (4.9); by convention, $(E_*/\mathcal{L}_*)^0$ is the trivial line bundle $\mathcal{O}_X$ with the trivial parabolic structure. It can be shown that

$$(\mathcal{L}_*)^* = E_*/\mathcal{L}_*. \tag{4.12}$$

Indeed, from (2.17) it follows that $\mathcal{L}_* \otimes (E_*/\mathcal{L}_*) = \det E_*$ is the trivial line bundle $\mathcal{O}_X$ with the trivial parabolic structure, and hence (4.12) holds. Therefore, from (4.11) it follows that

$$\text{par-deg}(\mathcal{F}^j_*/\mathcal{F}^{j-1}_*) = (2j-r-1)\cdot\text{par-deg}(E_*/\mathcal{L}_*) = (2j-r-1)\cdot\left(1-g-n+\sum_{i=1}^{n}\frac{c_i+1}{2c_i+1}\right), \tag{4.13}$$

where $g = \text{genus}(X)$. Now from (4.10) and (4.13) it is deduced that

$$\text{par-deg}(\mathcal{F}^j_*) = \sum_{i=1}^{j}\text{par-deg}(\mathcal{F}^i_*/\mathcal{F}^{i-1}_*) = j(r-j)\cdot\left(g-1+\sum_{i=1}^{n}\frac{c_i}{2c_i+1}\right). \tag{4.14}$$

Lemma 4.3. *Let D be any parabolic connection on the parabolic bundle* $\text{Sym}^{r-1}(E_*)$*. Then the following two hold:*

1. *For any $1 \leq j \leq r-1$, the parabolic subbundle $\mathcal{F}^j_*$ in* (4.10) *is not preserved by D.*
2. *$D(\mathcal{F}^j_0) \subset \mathcal{F}^{j+1}_0 \otimes K_X(S)$, where $\mathcal{F}^i_0$ is the holomorphic vector bundle underlying $\mathcal{F}^i$, for all* $1 \leq j \leq r-1$.

Proof. From (4.14) it follows that $\text{par-deg}(\mathcal{F}^j_*) \neq 0$ (in fact, $\text{par-deg}(\mathcal{F}^j_*) > 0$) for all $1 \leq j \leq r-1$. Consequently, D does not preserve $\mathcal{F}^j_*$.

For any $1 \leq j \leq r-2$, and any $2 \leq k \leq r-j$, consider the parabolic line bundle

$$(\mathcal{F}^j_*/\mathcal{F}^{j-1}_*)^* \otimes (\mathcal{F}^{j+k}_*/\mathcal{F}^{j+k-1}_*) = ((\mathcal{L}_*)^{r-j}\otimes(E_*/\mathcal{L}_*)^{j-1})^* \otimes ((\mathcal{L}_*)^{r-j-k}\otimes(E_*/\mathcal{L}_*)^{j+k-1})$$
$$= (\mathcal{L}_*)^{r-j-k-(r-j)} \otimes (E_*/\mathcal{L}_*)^{j+k-1-(j-1)} = (\mathcal{L}_*)^{-k}\otimes(E_*/\mathcal{L}_*)^{k} = (E_*/\mathcal{L}_*)^{2k};$$

see (4.11) and (4.12) for the above isomorphisms. The holomorphic line bundle underlying the parabolic line bundle $(\mathcal{F}^j_*/\mathcal{F}^{j-1}_*)^* \otimes (\mathcal{F}^{j+k}_*/\mathcal{F}^{j+k-1}_*) = (E_*/\mathcal{L}_*)^{2k}$ will be denoted by $\xi_{r,k}$. We have

$$\text{degree}(\xi_{r,k}) = 2k\cdot\text{degree}(E/\mathcal{L}) + \sum_{i=1}^{n}\left[\frac{2k(c_i+1)}{2c_i+1}\right]$$
$$= 2k(1-g-n)+kn+\sum_{i=1}^{n}\left[\frac{k}{2c_i+1}\right] = k(2-2g-n)++\sum_{i=1}^{n}\left[\frac{k}{2c_i+1}\right],$$

where $[t] \in \mathbb{Z}$ denotes the integral part of t, meaning $0 \leq t-[t] < 1$. This implies that

$$\text{degree}(\xi_{r,k}) < 2-2g-n = -\text{degree}(K_X(S))$$

(recall that $n \geq 3$ if $g = 0$), and hence $\text{degree}(\xi_{r,k}\otimes K_X(S)) < 0$. Consequently, we have

$$H^0(X,\, \xi_{r,k}\otimes K_X(S)) = 0.$$

This implies that

$$H^0(X,\, (\mathcal{F}^j_*/\mathcal{F}^{j-1}_*)^* \otimes (\mathcal{F}^{j+k}_*/\mathcal{F}^{j+k-1}_*)\otimes K_X(S)) = 0. \tag{4.15}$$

From (4.15) it is deduced that the following composition of homomorphisms

$$\mathcal{F}^j_0 \xrightarrow{D} \mathcal{F}^r_0 \otimes K_X(S) \longrightarrow (\mathcal{F}^r_0/\mathcal{F}^{j+1}_0)\otimes K_X(S) \tag{4.16}$$

vanishes identically, where $\mathcal{F}^\ell_0$ is the holomorphic vector bundle underlying the parabolic bundle $\mathcal{F}^\ell_*$. To see this, observe that the parabolic vector bundle

$$\mathrm{Hom}(\mathcal{F}^j_*,\, (\mathcal{F}^r_*/\mathcal{F}^{j+1}_*)\otimes K_X(S)) \,=\, (\mathcal{F}^r_*/\mathcal{F}^{j+1}_*)\otimes K_X(S)\otimes(\mathcal{F}^j_*)^* \,=\, (\mathcal{F}^r_*/\mathcal{F}^{j+1}_*)\otimes(\mathcal{F}^j_*)^*\otimes K_X(S)$$

has a filtration of parabolic subbundles such that the successive quotients are

$$(\mathcal{F}^j_*/\mathcal{F}^{j-1}_*)^*\otimes(\mathcal{F}^{j+k}_*/\mathcal{F}^{j+k-1}_*)\otimes K_X(S)\,, \quad 2\,\leq\, k\,\leq\, r-j.$$

So (4.15) implies that the composition of homomorphisms in (4.16) vanishes identically. Since the composition of homomorphisms in (4.16) vanishes identically we have

$$D(\mathcal{F}^j_*)\,\subset\,\mathcal{F}^{j+1}_*$$

for all $1\,\leq\, j\,\leq\, r-1$. □

From (4.11) it follows that for any $1\,\leq\, j\,\leq\, r-1$, the parabolic line bundle

$$(\mathcal{F}^j_*/\mathcal{F}^{j-1}_*)^*\otimes(\mathcal{F}^{j+1}_*/\mathcal{F}^j_*)\,=\,(E_*/\mathcal{L}_*)\otimes\mathcal{L}^*_*\,=\,(E_*/\mathcal{L}_*)^{\otimes 2}$$

is $TX(-S)\,=\,K_X(S)^*$ equipped with the parabolic weight $\frac{1}{2c_i+1}$ at each $x_i\,\in\, S$ (see (4.12) for the above isomorphism). Therefore, from Lemma 4.3(2) we conclude that for any parabolic connection D on the parabolic bundle $\mathrm{Sym}^{r-1}(E_*)$, the second fundamental forms for the parabolic subbundles in (4.10) are given by a collection of holomorphic homomorphisms

$$\psi(D,j)\,\in\, H^0(X,\, \mathrm{Hom}(\mathcal{F}^j_*/\mathcal{F}^{j-1}_*,\, \mathcal{F}^{j+1}_*/\mathcal{F}^j_*)\otimes K_X(S))\,=\,H^0(X,\,\mathcal{O}_X) \tag{4.17}$$

with $1\,\leq\, j\,\leq\, r-1$.

Corollary 4.4. *For each $1\,\leq\, j\,\leq\, r-1$, the section $\psi(D,j)$ in (4.17) is a nonzero constant.*

Proof. From Lemma 4.3(1) it follows immediately that $\psi(D,j)\,\neq\,0$. □

5. Differential operators on parabolic bundles

In this section we will describe differential operators between parabolic vector bundles. As before, fix a compact Riemann surface X and a reduced effective divisor $S\,=\,\sum_{i=1}^n x_i$ on it; if genus$(X)\,=\,0$, then assume that $n\,\geq\,3$. For each point $x_i\,\in\, S$ fix an integer $N_i\,\geq\,2$. We will consider parabolic bundles on X with parabolic structure on S such that all the parabolic weights at each $x_i\,\in\, S$ are integral multiples of $1/N_i$.

There is a ramified Galois covering

$$\varphi\,:\, Y\,\longrightarrow\, X \tag{5.1}$$

satisfying the following two conditions:

- φ is unramified over the complement $X\setminus S$, and
- for every $x_i\,\in\, S$ and one (hence every) point $y\,\in\,\varphi^{-1}(x_i)$, the order of the ramification of φ at y is N_i.

Such a ramified Galois covering φ exists; see [Na, p. 26, Proposition 1.2.12]. Let

$$\Gamma := \mathrm{Gal}(\varphi) := \mathrm{Aut}(Y/X) \subset \mathrm{Aut}(Y) \tag{5.2}$$

be the Galois group for φ. So the restriction

$$\varphi' := \varphi\big|_{Y'} : Y' := Y \setminus \varphi^{-1}(S) \longrightarrow X' := X \setminus S \tag{5.3}$$

is an étale Galois covering with Galois group Γ.

As before, a holomorphic vector bundle V on Y is called an *orbifold bundle* if Γ acts on V as holomorphic bundle automorphisms over the action of Γ on Y.

Consider the trivial vector bundle

$$\mathbb{C}[\Gamma]_Y := Y \times \mathbb{C}[\Gamma] \longrightarrow Y\,, \tag{5.4}$$

where $\mathbb{C}[\Gamma]$ is the group algebra for Γ with coefficients in $\mathbb{C}$. The usual action of Γ on $\mathbb{C}[\Gamma]$ and the Galois action of Γ on Y together produce an action of Γ on $Y \times \mathbb{C}[\Gamma]$. This action makes $Y \times \mathbb{C}[\Gamma] = \mathbb{C}[\Gamma]_Y$ an orbifold bundle on Y. Let

$$\mathcal{E}_* \longrightarrow X \tag{5.5}$$

be the corresponding parabolic vector bundle on X with parabolic structure on S [Bi], [Bo1], [Bo2]. The action of Γ on the vector bundle $\mathbb{C}[\Gamma]_Y$ in (5.4) produces an action of Γ on its direct image $\varphi_*\mathbb{C}[\Gamma]_Y$ over the trivial action of Γ on X. We have

$$\mathcal{E}_0 = (\varphi_*\mathbb{C}[\Gamma]_Y)^\Gamma \subset \varphi_*\mathbb{C}[\Gamma]_Y\,, \tag{5.6}$$

where $(\varphi_*\mathbb{C}[\Gamma]_Y)^\Gamma$ is the Γ–invariant part, and $\mathcal{E}_0$ is the holomorphic vector bundle underlying the parabolic bundle $\mathcal{E}_*$ in (5.5).

It can be shown that the holomorphic vector bundle $\mathcal{E}_0 = (\varphi_*\mathbb{C}[\Gamma]_Y)^\Gamma$ is identified with $\varphi_*\mathcal{O}_Y$. Indeed, there is a natural Γ–equivariant isomorphism

$$\varphi_*\mathbb{C}[\Gamma]_Y \xrightarrow{\sim} (\varphi_*\mathcal{O}_Y) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]\,;$$

it is in fact given by the projection formula. Therefore, the natural isomorphism

$$\varphi_*\mathcal{O}_Y \xrightarrow{\sim} ((\varphi_*\mathcal{O}_Y) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma])^\Gamma$$

(any complex Γ–module M is naturally identified with $(M \otimes_{\mathbb{C}} \mathbb{C}[\Gamma])^\Gamma$) produces an isomorphism

$$\varphi_*\mathcal{O}_Y \xrightarrow{\sim} (\varphi_*\mathbb{C}[\Gamma]_Y)^\Gamma. \tag{5.7}$$

The direct image $\varphi_*\mathcal{O}_Y$ has a natural parabolic structure which we will now describe.

Take any $x_i \in S$. Fix an analytic open neighborhood $U \subset X$ of x_i such that $U \bigcap S = x_i$. Let $\mathcal{U} := \varphi^{-1}(U) \subset Y$ be the inverse image. The restriction of φ to $\mathcal{U}$ will be denoted by $\widetilde{\varphi}$. Let $\widetilde{D}_i := \varphi^{-1}(x_i)_{\mathrm{red}} \subset Y$ be the reduced inverse image. For all $k \in [1,\, N_i]$, define the vector bundle

$$V_k := \widetilde{\varphi}_*\mathcal{O}_{\mathcal{U}}(-(N_i - k))\widetilde{D}_i) \longrightarrow U\,.$$

So we have a filtration of subsheaves of $V_{N_i} = (\varphi_*\mathcal{O}_Y)\big|_U$:

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{N_i - 1} \subset V_{N_i} = (\varphi_*\mathcal{O}_Y)\big|_U\,.$$

The restriction of this filtration of subsheaves to x_i gives a filtration of subspaces

$$0 \subset (V_1)'_{x_i} \subset (V_2)'_{x_i} \subset \cdots \subset (V_{N_i-1})'_{x_i} \subset (V_{N_i})_{x_i} = (\varphi_*\mathcal{O}_Y)_{x_i} \tag{5.8}$$

of the fiber $(\varphi_*\mathcal{O}_Y)_{x_i}$. We note that $(V_k)'_{x_i}$ in (5.8) is the image, in the fiber $(\varphi_*\mathcal{O}_Y)_{x_i}$, of the fiber $(V_k)_{x_i}$ over x_i of the vector bundle V_k.

The parabolic structure on $\varphi_*\mathcal{O}_Y$ is defined as follows. The parabolic divisor is S. The quasiparabolic filtration over any $x_i \in S$ is the filtration of $(\varphi_*\mathcal{O}_Y)_{x_i}$ constructed in (5.8). The parabolic weight of the subspace $(V_k)_{x_i}$ in (5.8) is $\frac{N_i-k}{N_i}$. The resulting parabolic vector bundle is identified with $\mathcal{E}_*$ in (5.5); recall from (5.6) and (5.7) that $\mathcal{E}_0$ is identified with $\varphi_*\mathcal{O}_Y$.

The trivial connection on the trivial vector bundle $\mathbb{C}[\Gamma]_Y := Y \times \mathbb{C}[\Gamma]$ in (5.4) is preserved by the action of the Galois group Γ on $\mathbb{C}[\Gamma]_Y$. Therefore, this trivial connection produces a parabolic connection on the corresponding parabolic vector bundle $\mathcal{E}_*$ in (5.5). This parabolic connection on $\mathcal{E}_*$ will be denoted by $\nabla^{\mathcal{E}}$.

Using the isomorphism between $\mathcal{E}_0$ and $\varphi_*\mathcal{O}_Y$ (see (5.6) and (5.7)), the logarithmic connection on $\mathcal{E}_0$ defining the above parabolic connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}_*$ produces a logarithmic connection on $\varphi_*\mathcal{O}_Y$. This logarithmic connection on $\varphi_*\mathcal{O}_Y$ given by $\nabla^{\mathcal{E}}$ is easy to describe. To describe it, take the de Rham differential $d : \mathcal{O}_Y \longrightarrow K_Y$ on Y. Let

$$\varphi_* d : \varphi_*\mathcal{O}_Y \longrightarrow \varphi_* K_Y \tag{5.9}$$

be its direct image. On the other hand, using the projection formula, the natural homomorphism

$$K_Y \hookrightarrow K_Y \otimes \mathcal{O}_Y(\varphi^{-1}(S)_{\rm red}) = \varphi^*(K_X \otimes \mathcal{O}_X(S)).$$

produces a homomorphism

$$\varphi_* K_Y \longrightarrow \varphi_*(\varphi^*(K_X \otimes \mathcal{O}_X(S))) = (\varphi_*\mathcal{O}_Y) \otimes K_X \otimes \mathcal{O}_X(S).$$

Combining this with $\varphi_* d$ in (5.9) we obtain homomorphisms

$$\varphi_*\mathcal{O}_Y \longrightarrow \varphi_* K_Y \longrightarrow (\varphi_*\mathcal{O}_Y) \otimes K_X \otimes \mathcal{O}_X(S).$$

This composition of homomorphisms $\varphi_*\mathcal{O}_Y \longrightarrow (\varphi_*\mathcal{O}_Y) \otimes K_X \otimes \mathcal{O}_X(S)$ defines a logarithmic connection on $\varphi_*\mathcal{O}_Y$. This logarithmic connection coincides with the one that defines the above constructed parabolic connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}_*$.

The parabolic connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}_*$ defines a nonsingular holomorphic connection ∇' on

$$\mathcal{E}'_0 := \mathcal{E}_0\big|_{X'} = \varphi_{1*}\mathcal{O}_{Y'}$$

over X' (see (5.3)). For any holomorphic vector bundle V' on X', note that

$$J^k(V' \otimes \mathcal{E}'_0) = J^k(V') \otimes \mathcal{E}'_0 \tag{5.10}$$

for all $k \geq 0$. To see this isomorphism, for any $x \in X'$ and $u \in (\mathcal{E}'_0)_x$, let $\widetilde{u}$ denote the unique flat section of $\mathcal{E}'_0$ for the connection ∇', defined on any simply connected open neighborhood of x, such that $\widetilde{u}(x) = u$. Now the homomorphism

$$J^k(V') \otimes \mathcal{E}'_0 \longrightarrow J^k(V' \otimes \mathcal{E}'_0)$$

that sends any $v \otimes u$ to the image of $v \otimes \widetilde{u}$, where $v \in J^k(V')_x$ and $u \in (\mathcal{E}'_0)_x$ with $x \in X'$, is evidently an isomorphism.

Take holomorphic vector bundles V' and W' on a nonempty Zariski open subset $U \subset X'$. Recall that a holomorphic differential operator of order k from V' to W' is a holomorphic homomorphism $J^k(V') \longrightarrow W'$. Let

$$D' : J^k(V') \longrightarrow W'$$

be a holomorphic differential operator of order k from V' to W' on U.

We will show that D' extends to a holomorphic differential operator

$$\widetilde{D'} : J^k(V' \otimes \mathcal{E}'_0) \longrightarrow W' \otimes \mathcal{E}'_0 \tag{5.11}$$

from $V' \otimes \mathcal{E}'_0$ to $W' \otimes \mathcal{E}'_0$ over U. To construct $\widetilde{D'}$, using the isomorphism in (5.10) we have

$$J^k(V' \otimes \mathcal{E}'_0) = J^k(V') \otimes \mathcal{E}'_0 \xrightarrow{D' \otimes \mathrm{Id}_{\mathcal{E}'_0}} W' \otimes \mathcal{E}'_0 .$$

This homomorphism is the one in (5.11).

Let V_* and W_* be parabolic vector bundles on X. Denote the restrictions $V_0\big|_{X'}$ and $W_0\big|_{X'}$ by V' and W' respectively. The holomorphic vector bundle underlying the parabolic tensor product $V_* \otimes \mathcal{E}_*$ (respectively, $W_* \otimes \mathcal{E}_*$) will be denoted by $(V_* \otimes \mathcal{E}_*)_0$ (respectively, $(W_* \otimes \mathcal{E}_*)_0$), where $\mathcal{E}_*$ is the parabolic bundle in (5.5).

Definition 5.1. A *holomorphic differential operator* of order k from V_* to W_* over an open subset $\widetilde{U} \subset X$ is a holomorphic homomorphism

$$D' : J^k(V') \longrightarrow W'$$

over $U := \widetilde{U} \bigcap X'$ such that the homomorphism

$$\widetilde{D'} : J^k(V' \otimes \mathcal{E}'_0) \longrightarrow W' \otimes \mathcal{E}'_0$$

in (5.11) extends to a holomorphic homomorphism $J^k((V_* \otimes \mathcal{E}_*)_0) \longrightarrow (W_* \otimes \mathcal{E}_*)_0$ over entire $\widetilde{U}$.

It is straightforward to check that the above definition does not depend on the choice of the map φ.

We denote by $\mathrm{Diff}^k_X(V_*, W_*)$ the sheaf of holomorphic differential operators of order k from V_* to W_*. Define

$$\mathrm{DO}^k_P(V_*, W_*) := H^0(X, \mathrm{Diff}^k_X(V_*, W_*))$$

to be the space of all holomorphic differential operators of order k from V_* to W_* over X.

Let $\mathbb{V}$ and $\mathbb{W}$ denote the orbifold vector bundles on Y corresponding to the parabolic vector bundles V_* and W_* respectively. Consider the space

$$\mathrm{DO}^k(\mathbb{V}, \mathbb{W}) := H^0(Y, \mathrm{Hom}(J^k(\mathbb{V}), \mathbb{W}))$$

of holomorphic differential operators of order k from $\mathbb{V}$ to $\mathbb{W}$ over Y. Then the actions of Γ on $\mathbb{V}$ and $\mathbb{W}$ together produce an action of Γ on $\mathrm{DO}^k(\mathbb{V}, \mathbb{W})$. Let

$$H^0(Y, \mathrm{Hom}(J^k(\mathbb{V}), \mathbb{W}))^\Gamma = \mathrm{DO}^k(\mathbb{V}, \mathbb{W})^\Gamma \subset \mathrm{DO}^k(\mathbb{V}, \mathbb{W})$$

be the space of all Γ–invariant differential operators of order k from $\mathbb{V}$ to $\mathbb{W}$.

Proposition 5.2. *There is a natural isomorphism*

$$\mathrm{DO}^k(\mathbb{V},\, \mathbb{W})^\Gamma \ \xrightarrow{\sim}\ \mathrm{DO}^k_P(V_*,\, W_*)\,.$$

Proof. We will first prove that

$$\varphi_*\mathbb{V} \,=\, (V_* \otimes \mathcal{E}_*)_0\,, \tag{5.12}$$

where $\mathcal{E}_*$ is the parabolic bundle in (5.5) and $(V_* \otimes \mathcal{E}_*)_0$ is the vector bundle underlying the parabolic vector bundle $V_* \otimes \mathcal{E}_*$. To prove (5.12), first note that

$$\varphi_*\mathbb{V} \,=\, \left(\varphi_*(\mathbb{V} \otimes \mathbb{C}[\Gamma]_Y)\right)^\Gamma\,, \tag{5.13}$$

where $\mathbb{C}[\Gamma]_Y$ is the orbifold bundle in (5.4). Since $\mathcal{E}_*$ and V_* correspond to the orbifold bundles $\mathbb{C}[\Gamma]_Y$ and $\mathbb{V}$ respectively, the parabolic bundle corresponding to the orbifold bundle $\mathbb{V} \otimes \mathbb{C}[\Gamma]_Y$ is $V_* \otimes \mathcal{E}_*$. In particular, we have

$$\left(\varphi_*(\mathbb{V} \otimes \mathbb{C}[\Gamma]_Y)\right)^\Gamma \,=\, (V_* \otimes \mathcal{E}_*)_0\,.$$

This and (5.13) together give the isomorphism in (5.12).

Let $D \,:\, \mathbb{V} \,\longrightarrow\, \mathbb{W}$ be a holomorphic differential operator of order k on Y. Taking its direct image for the map φ, we have

$$\varphi_*D \,:\, \varphi_*\mathbb{V} \,\longrightarrow\, \varphi_*\mathbb{W}\,.$$

Now if $D \,\in\, \mathrm{DO}^k(\mathbb{V},\, \mathbb{W})^\Gamma$, then clearly

$$\varphi_*D((\varphi_*\mathbb{V})^\Gamma) \,\subset\, (\varphi_*\mathbb{W})^\Gamma\,.$$

Let

$$D_\varphi \,:=\, (\varphi_*D)\big|_{(\varphi_*\mathbb{V})^\Gamma} \,:\, (\varphi_*\mathbb{V})^\Gamma \,\longrightarrow\, (\varphi_*\mathbb{W})^\Gamma$$

be the restriction of φ_*D to $(\varphi_*\mathbb{V})^\Gamma \,\subset\, \varphi_*\mathbb{V}$.

Using (5.12) it is now straightforward to check that D_φ defines a holomorphic differential operator of order k from the parabolic bundle V_* to W_*. The corresponding homomorphism $J^k((V_* \otimes \mathcal{E}_*)_0) \,\longrightarrow\, (W_* \otimes \mathcal{E}_*)_0$ in Definition 3.3 is given by φ_*D using the isomorphism in (5.12).

The isomorphism in the proposition sends any $D \,\in\, \mathrm{DO}^k(\mathbb{V},\, \mathbb{W})^\Gamma$ to $D_\varphi \,\in\, \mathrm{DO}^k_P(V_*,\, W_*)$ constructed above from D.

For the inverse map, given any $\mathbf{D} \,\in\, \mathrm{DO}^k_P(V_*,\, W_*)$, consider the homomorphism

$$J^k((V_* \otimes \mathcal{E}_*)_0) \,\longrightarrow\, (W_* \otimes \mathcal{E}_*)_0$$

in Definition 3.3 given by the differential operator $\mathbf{D}$. Using the isomorphism in (5.12) it produces a holomorphic differential operator from $\mathbb{V}$ to $\mathbb{W}$. This differential operator is evidently fixed by the action of Γ on $\mathrm{DO}^k(\mathbb{V},\, \mathbb{W})$. $\square$

5.1. **Another description of differential operators on parabolic bundles.** We will give an alternative description of the holomorphic differential operators between two parabolic vector bundles. Let $\text{Diff}^k_Z(A,\, B)$ denote the sheaf of holomorphic differential operators of order k from a holomorphic vector bundle A on a complex manifold Z to another holomorphic vector bundle B on Z. The sheaf $\text{Diff}^k_Z(\mathcal{O}_Z,\, \mathcal{O}_Z) \,=\, J^k(\mathcal{O}_Z)^*$ has both left and right $\mathcal{O}_Z$–module structures, and

$$\text{Diff}^k_Z(A,\, B) \,=\, B \otimes_{\mathcal{O}_Z} \text{Diff}^k_Z(\mathcal{O}_Z,\, \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} A^*\,. \tag{5.14}$$

We have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \text{Diff}^k_Z(\mathcal{O}_Z,\, \mathcal{O}_Z) \xrightarrow{\ \alpha\ } \text{Diff}^{k+1}_Z(\mathcal{O}_Z,\, \mathcal{O}_Z) \xrightarrow{\ \eta\ } \text{Sym}^{k+1}(TZ) \longrightarrow 0\,, \tag{5.15}$$

where η is the symbol map. The homomorphism

$$\text{Id}_B \otimes \alpha \otimes \text{Id}_{A^*}\,:\, B \otimes_{\mathcal{O}_Z} \text{Diff}^k_Z(\mathcal{O}_Z,\, \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} A^* \,\longrightarrow\, B \otimes_{\mathcal{O}_Z} \text{Diff}^{k+1}_Z(\mathcal{O}_Z,\, \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} A^*\,,$$

where α is the homomorphism in (5.15), coincides with the natural inclusion map

$$\text{Diff}^k_Z(A,\, B) \,\hookrightarrow\, \text{Diff}^{k+1}_Z(A,\, B).$$

The holomorphic differential operators between two parabolic vector bundles will be described along the above line.

Consider the pair $(Y,\, \varphi)$ in (5.1). The action of $\Gamma \,=\, \text{Gal}(\varphi)$ on Y produces an action of Γ on $\mathcal{O}_Y$. This action of Γ on $\mathcal{O}_Y$ induces an action of Γ on $J^k(\mathcal{O}_Y)$, which in turn induces an action of Γ on the dual vector bundle $J^k(\mathcal{O}_Y)^* \,=\, \text{Diff}^k_Y(\mathcal{O}_Y,\, \mathcal{O}_Y)$. As mentioned before, $\text{Diff}^k_Y(\mathcal{O}_Y,\, \mathcal{O}_Y)$ is equipped with left and right $\mathcal{O}_Y$–module structures. These module structures are Γ–equivariant. Let $\mathcal{J}^k_*$ denote the parabolic vector bundle on X associated to the orbifold vector bundle $J^k(\mathcal{O}_Y)^* \,=\, \text{Diff}^k_Y(\mathcal{O}_Y,\, \mathcal{O}_Y)$ on Y.

Note that the rank of $\mathcal{J}^k_*$ is $k+1$. The parabolic line bundle $\mathcal{J}^0_*$ is the trivial line bundle $\mathcal{O}_X$ equipped with the trivial parabolic structure. The underlying holomorphic vector bundle for the parabolic bundle $\mathcal{J}^1_*$ is $\mathcal{O}_X \oplus TX(-S)$. The quasiparabolic filtration of $\mathcal{J}^1_*$ over any point $x_i \,\in\, S$ is

$$TX(-S)_{x_i} \,\subset\, (\mathcal{O}_X)_{x_i} \oplus TX(-S)_{x_i} \,=\, (\mathcal{J}^1_0)_{x_i}\,.$$

The parabolic weight of $TX(-S)_{x_i}$ is $\frac{1}{N_i}$ and the parabolic weight of $(\mathcal{J}^1_0)_{x_i}$ is 0. Let

$$TX(-S)_* \,\longrightarrow\, X \tag{5.16}$$

denote the parabolic line bundle defined by $TX(-S)$ equipped with the parabolic weight $\frac{1}{N_i}$ at each $x_i \,\in\, S$. So

$$\mathcal{J}^1_* \,=\, TX(-S)_* \oplus \mathcal{O}_X,$$

where $\mathcal{O}_X$ has the trivial parabolic structure.

Using the homomorphism α in (5.15) for Y and $j \,=\, k$ we see that $\mathcal{J}^j_*$ is a parabolic subbundle of $\mathcal{J}^{j+1}_*$ for all $j \,\geq\, 0$. Consequently, we have filtration of parabolic subbundles

$$\mathcal{J}^0_* \,\subset\, \mathcal{J}^1_* \,\subset\, \cdots \,\subset\, \mathcal{J}^{k-1}_* \,\subset\, \mathcal{J}^k_* \tag{5.17}$$

for all $k \,\geq\, 0$ such that each successive quotient is a parabolic line bundle.

We will describe the quotient parabolic line bundle $\mathcal{J}^j_*/\mathcal{J}^{j-1}_*$ in (5.17) for all $1 \,\leq\, j \,\leq\, k$.

The holomorphic line bundle underlying the parabolic bundle $\mathcal{J}^j_*/\mathcal{J}^{j-1}_*$ is

$$(TX)^{\otimes j}(-jS)\otimes\mathcal{O}_X\left(\sum_{i=1}^{n}\left[\frac{j}{N_i}\right]x_i\right),$$

where $\left[\frac{j}{N_i}\right]\in\mathbb{Z}$ is the integral part of $\frac{j}{N_i}$, and its parabolic weight at any $x_i\in S$ is $\frac{j}{N_i}-\left[\frac{j}{N_i}\right]$. Indeed, from (5.15) we know that the parabolic line bundle $\mathcal{J}^j_*/\mathcal{J}^{j-1}_*$ corresponds to the orbifold line bundle $(TY)^{\otimes j}$ on Y. On the other hand, the parabolic line bundle $TX(-S)_*$ defined in (5.16) corresponds to the orbifold line bundle TY. Therefore, we have

$$\mathcal{J}^j_*/\mathcal{J}^{j-1}_*\,=\,TX(-S)^{\otimes j}_*\,. \tag{5.18}$$

The above description of $\mathcal{J}^j_*/\mathcal{J}^{j-1}_*$ follows immediately from (5.18).

The Γ–equivariant left and right $\mathcal{O}_Y$–module structures on $\mathrm{Diff}^k_Y(\mathcal{O}_Y,\,\mathcal{O}_Y)$ produces left and right $\mathcal{O}_X$–module structures on $\mathcal{J}^k_*$.

Then, for any two parabolic bundles V_* and W_* over X, it follows from Proposition 5.2 and (5.14) that $\mathrm{Diff}^k_X(V_*,\,W_*)$ coincides with the holomorphic vector bundle underlying the parabolic tensor product

$$W_*\otimes_{\mathcal{O}_X}\mathcal{J}^k_*\otimes_{\mathcal{O}_X}V^*_*\,;$$

in other words, we have

$$\mathrm{Diff}^k_X(V_*,\,W_*)\,=\,(W_*\otimes_{\mathcal{O}_X}\mathcal{J}^k_*\otimes_{\mathcal{O}_X}V^*_*)_0\,.$$

5.2. **The symbol map.** Consider the quotient map

$$\gamma\,:\,\mathcal{J}^k_*\,\longrightarrow\,\mathcal{J}^k_*/\mathcal{J}^{k-1}_*\,=\,TX(-S)^{\otimes k}_*$$

(see (5.17), (5.18)). It produces a map

$$\begin{aligned}\sigma\,:=\,(\mathrm{Id}_{W_*}\otimes\gamma\otimes\mathrm{Id}_{V^*_*})_0\,:\,\mathrm{Diff}^k_X(V_*,\,W_*)\,&=\,(W_*\otimes_{\mathcal{O}_X}\mathcal{J}^k_*\otimes_{\mathcal{O}_X}\otimes V^*_*)_0\\ \longrightarrow\,(W_*\otimes TX(-S)^{\otimes k}_*\otimes V^*_*)_0\,&=\,(TX(-S)^{\otimes k}_*\otimes\mathrm{Hom}(V_*,\,W_*)_*)_0\,.\end{aligned} \tag{5.19}$$

The above homomorphism σ is the *symbol* map of differential operators between parabolic bundles.

Take any $\widehat{D}\,\in\,\mathrm{DO}^k_P(V_*,\,W_*)$. Denote by $\mathbb{V}$ (respectively, $\mathbb{W}$) the orbifold bundle on Y corresponding to V_* (respectively, W_*), and let

$$D\,\in\,\mathrm{DO}^k(\mathbb{V},\,\mathbb{W})^\Gamma$$

be the invariant differential operator given by $\widehat{D}$ using Proposition 5.2. Let

$$\sigma(\widehat{D})\,\in\,H^0(X,\,(TX(-S)^{\otimes k}_*\otimes\mathrm{Hom}(V_*,\,W_*)_*)_0)$$

be the symbol of $\widehat{D}$ (see (5.19)). Let

$$\sigma(D)\,\in\,H^0(Y,\,\mathrm{Hom}(\mathbb{V},\,\mathbb{W})\otimes(TY)^{\otimes k})$$

be the symbol of D. We have

$$\sigma(D)\,\in\,H^0(Y,\,\mathrm{Hom}(\mathbb{V},\,\mathbb{W})\otimes(TY)^{\otimes k})^\Gamma$$

because D is fixed by the action of Γ on $\mathrm{DO}^k(\mathbb{V},\, \mathbb{W})$. The proof of the following lemma is straightforward.

Lemma 5.3. *The parabolic vector bundle* $TX(-S)_*^{\otimes k} \otimes \mathrm{Hom}(V_*,\, W_*)_*$ *on* X *corresponds to the orbifold vector bundle* $Hom(\mathbb{V},\, \mathbb{W}) \otimes (TY)^{\otimes k}$ *on* Y*. The natural isomorphism*

$$H^0(X,\, (TX(-S)_*^{\otimes k} \otimes \mathrm{Hom}(V_*,\, W_*)_*)_0) \stackrel{\sim}{\longrightarrow} H^0(Y,\, \mathrm{Hom}(\mathbb{V},\, \mathbb{W}) \otimes (TY)^{\otimes k})^{\Gamma}$$

takes the symbol $\sigma(\widehat{D})$ *to the symbol* $\sigma(D)$*.*

6. Parabolic opers and differential operators

Recall the short exact sequence in (4.9) and the isomorphism in (4.12). For notational convenience, $(\mathcal{L}_*)^* = E_*/\mathcal{L}_*$ will be denoted by $\mathcal{L}_*^{-1}$. For any $j \leq 1$, the parabolic line bundle $(\mathcal{L}_*)^{\otimes j}$ (respectively, $(\mathcal{L}_*^*)^{\otimes j}$) will be denoted by $\mathcal{L}_*^j$ (respectively, $\mathcal{L}_*^{-j}$). Also, $\mathcal{L}_*^0$ will denote the trivial line bundle $\mathcal{O}_X$ with the trivial parabolic structure.

We note that

$$\mathcal{L}_*^{-2} \,=\, TX(-S)_*\,, \tag{6.1}$$

where $TX(-S)_*$ is the parabolic line bundle in (5.16). From (5.18) and (6.1) it follows that

$$\mathcal{J}_*^j/\mathcal{J}_*^{j-1} \,=\, \mathcal{L}_*^{-2j} \tag{6.2}$$

for all $j \geq 1$.

For any integer $r \geq 2$, consider the space of parabolic differential operators of order r

$$\mathrm{DO}_P^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}) \,:=\, H^0(X,\, \mathrm{Diff}_X^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}))$$

from $\mathcal{L}_*^{1-r}$ to $\mathcal{L}_*^{r+1}$. Let

$$\begin{aligned} \sigma \,:\, \mathrm{DO}_P^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}) \,&\longrightarrow\, (\mathcal{L}_*^{r+1} \otimes (TX(-S)_*)^{\otimes r} \otimes \mathcal{L}_*^{r-1})_0 \\ &= (\mathcal{L}_*^{r+1} \otimes \mathcal{L}_*^{-2r} \otimes \mathcal{L}_*^{r-1})_0 \,=\, (\mathcal{L}_*^0)_0 \,=\, \mathcal{O}_X \end{aligned} \tag{6.3}$$

be the symbol map constructed in (5.19) (see (6.2) for the isomorphism used in (6.3)).

Let

$$\widetilde{\mathrm{DO}}_P^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}) \,\subset\, \mathrm{DO}_P^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}) \tag{6.4}$$

be the affine subspace consisting of parabolic differential operators whose symbol is the constant function 1.

The following Lemma constructs the sub-principal symbol of the operator:

Lemma 6.1. *There is a natural map*

$$\Psi \,:\, \widetilde{\mathrm{DO}}_P^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}) \,\longrightarrow\, H^0(X,\, K_X)\,.$$

Proof. As in (2.27), let $\mathbf{L}$ denote the orbifold line bundle on Y corresponding to $\mathcal{L}$. So the parabolic bundle $\mathcal{L}_*^{1-r}$ (respectively, $\mathcal{L}_*^{r+1}$) corresponds to the orbifold line bundle $\mathbf{L}^{1-r}$ (respectively, $\mathbf{L}^{r+1}$). Take any

$$D \,\in\, \widetilde{\mathrm{DO}}_P^r(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}).$$

Now Proposition 5.2 says that D corresponds to a Γ–invariant holomorphic differential operator of order r from $\mathbf{L}^{1-r}$ to $\mathbf{L}^{r+1}$. Let

$$\mathcal{D} \in \mathrm{DO}^r(\mathbf{L}^{1-r},\, \mathbf{L}^{r+1})^\Gamma \tag{6.5}$$

be the Γ–invariant differential operator corresponding to D. As the orbifold bundle $\mathbf{L}^2$ is isomorphic to TY (see Lemma 2.7), the symbol of $\mathcal{D}$ is a section of $\mathcal{O}_Y$. Since the symbol of D is the constant function 1, from Lemma 5.3 it follows that the symbol of $\mathcal{D}$ is the constant function 1 on Y.

We will now show that a differential operator $\mathbf{D} \in \mathrm{DO}^r(\mathbf{L}^{1-r},\, \mathbf{L}^{r+1})$ of symbol 1 produces a section

$$\theta_{\mathbf{D}} \in H^0(Y,\, K_Y)\, . \tag{6.6}$$

Consider the short exact sequence of jet bundles

$$0 \longrightarrow \mathbf{L}^{1-r}\otimes K_Y^{\otimes r} = \mathbf{L}^{r+1} \stackrel{\mu}{\longrightarrow} J^r(\mathbf{L}^{1-r}) \stackrel{\nu}{\longrightarrow} J^{r-1}(\mathbf{L}^{1-r}) \longrightarrow 0 \tag{6.7}$$

(see Lemma 2.7 for the above isomorphism) together with the homomorphism

$$\mathbf{D}' \,:\, J^r(\mathbf{L}^{1-r}) \longrightarrow \mathbf{L}^{r+1}$$

defining the given differential operator $\mathbf{D}$. Since the symbol of $\mathbf{D}$ is 1, we have

$$\mathbf{D}'\circ\mu = \mathrm{Id}_{\mathbf{L}^{r+1}}\, ,$$

where μ is the homomorphism in (6.7). Therefore, $\mathbf{D}'$ produces a holomorphic splitting of the short exact sequence in (6.7). Let

$$\tau \,:\, J^{r-1}(\mathbf{L}^{1-r}) \longrightarrow J^r(\mathbf{L}^{1-r}) \tag{6.8}$$

be the holomorphic homomorphism given by this splitting of the short exact sequence in (6.7), so τ is uniquely determined by the following two conditions:

- $\nu\circ\tau = \mathrm{Id}_{J^{r-1}(\mathbf{L}^{1-r})}$, where ν is the projection in (6.7), and
- $\mathrm{image}(\tau) = \mathrm{kernel}(\mathbf{D}') \subset J^r(\mathbf{L}^{1-r})$.

Next consider the following natural commutative diagram of homomorphisms of jet bundles:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbf{L}^{1-r}\otimes K_Y^{\otimes r} = \mathbf{L}^{r+1} \arrow[r, "\mu"] \arrow[d] & J^r(\mathbf{L}^{1-r}) \arrow[r, "\nu"] \arrow[d, "\varpi"] & J^{r-1}(\mathbf{L}^{1-r}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^{r-1}(\mathbf{L}^{1-r})\otimes K_Y \arrow[r] \arrow[d] & J^1(J^{r-1}(\mathbf{L}^{1-r})) \arrow[r, "\alpha"] \arrow[d, "\zeta"] & J^{r-1}(\mathbf{L}^{1-r}) \arrow[r] & 0 \\
0 \arrow[r] & J^{r-2}(\mathbf{L}^{1-r})\otimes K_Y \arrow[r, "="] \arrow[d] & J^{r-2}(\mathbf{L}^{1-r})\otimes K_Y \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{6.9}$$

where the horizontal sequences are the natural jet sequences, and the vertical sequence in the left is the jet sequence tensored with K_Y; the homomorphism ϖ is the natural homomorphism of jet bundles. The homomorphism ζ in (6.9) is constructed as follows: We have the natural homomorphism

$$h_1 \,:\, J^1(J^{r-1}(\mathbf{L}^{1-r})) \longrightarrow J^{r-1}(\mathbf{L}^{1-r}).$$

On the other hand, we have the composition of homomorphisms

$$J^1(J^{r-1}(\mathbf{L}^{1-r})) \longrightarrow J^1(J^{r-2}(\mathbf{L}^{1-r})) \longrightarrow J^{r-1}(\mathbf{L}^{1-r}),$$

which will be denoted by h_2. Now, we have $\zeta \,=\, h_1 - h_2$; note that $J^{r-2}(\mathbf{L}^{1-r}) \otimes K_Y$ is a subbundle of $J^{r-1}(\mathbf{L}^{1-r})$.

Next consider the homomorphism

$$\varpi \circ \tau \,:\, J^{r-1}(\mathbf{L}^{1-r}) \longrightarrow J^1(J^{r-1}(\mathbf{L}^{1-r}))\,,$$

where τ and ϖ are the homomorphisms in (6.8) and (6.9) respectively. We have

$$\alpha \circ (\varpi \circ \tau) \,=\, \mathrm{Id}_{J^{r-1}(\mathbf{L}^{1-r})}\,, \tag{6.10}$$

where α is the projection in (6.9), because (6.9) is a commutative diagram.

From (6.10) it follows immediately that $\varpi \circ \tau$ gives a holomorphic splitting of the bottom exact sequence in (6.9). But a holomorphic splitting of the bottom exact sequence in (6.9) is a holomorphic connection on $J^{r-1}(\mathbf{L}^{1-r})$.

Let ∇ denote the holomorphic connection on $J^{r-1}(\mathbf{L}^{1-r})$ given by $\varpi \circ \tau$. The holomorphic connection on $\bigwedge^r J^{r-1}(\mathbf{L}^{1-r}) \,=\, \mathcal{O}_Y$ (see Lemma 2.7) induced by ∇ will be denoted by ∇^0. So the connection ∇^0 is of the form

$$\nabla^0 \,=\, d + \theta_{\mathbf{D}}\,,$$

where $\theta_{\mathbf{D}} \,\in\, H^0(Y,\, K_Y)$ and d is the de Rham differential on $\mathcal{O}_Y$. This $\theta_{\mathbf{D}}$ is the holomorphic 1-form in (6.6).

By the construction of it, the form $\theta_{\mathbf{D}}$ vanishes identically if and only if the above connection ∇ on $J^{r-1}(\mathbf{L}^{1-r})$ induces the trivial connection on $\bigwedge^r J^{r-1}(\mathbf{L}^{1-r}) \,=\, \mathcal{O}_Y$. Therefor $\theta_{\mathbf{D}}$ should be seen as a sub-principal symbol.

Consider $\theta_{\mathcal{D}} \,\in\, H^0(Y,\, K_Y)$ (as in (6.6)) for the differential operator $\mathcal{D}$ in (6.5). Since $\mathcal{D}$ is Γ–invariant, we know that $\theta_{\mathcal{D}}$ is also Γ–invariant. On the other hand,

$$H^0(Y,\, K_Y)^\Gamma \,=\, H^0(X,\, K_X)\,.$$

The element of $H^0(X,\, K_X)$ corresponding to $\theta_{\mathcal{D}}$ will be denoted by $\theta'_{\mathcal{D}}$.

Now we have a map

$$\Psi \,:\, \widetilde{\mathrm{DO}}^r_P(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}) \longrightarrow H^0(X,\, K_X)$$

that sends any D to $\theta'_{\mathcal{D}}$ constructed above from D. $\square$

The following main Theorem deals with the space of all parabolic $\mathrm{SL}(r,\mathbb{C})$–opers on X (see Definition 3.3) with given singular set $S \,:=\, \{x_1,\, \cdots,\, x_n\} \,\subset\, X$ and fixed integers $c_i \,=\, N_i$ (see (2.14)).

Theorem 6.2. *The space of all parabolic* $\mathrm{SL}(r,\mathbb{C})$*–opers on* X *is identified with the inverse image*

$$\Psi^{-1}(0) \subset \widetilde{\mathrm{DO}}^r_P(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}),$$

where Ψ *is the map in Lemma 6.1.*

Proof. In view of Proposition 3.5, Proposition 5.2, Lemma 5.3 and Lemma 6.1, the theorem is straightforward.

As before, fix a ramified Galois covering

$$\varphi : Y \longrightarrow X$$

satisfying the following two conditions:

- φ is unramified over the complement $X \setminus S$, and
- for every $x_i \in S$ and one (hence every) point $y \in \varphi^{-1}(x_i)$, the order of ramification of φ at y is $2N_i + 1$.

As before, Γ denotes $\mathrm{Aut}(Y/X)$. Parabolic $\mathrm{SL}(r,\mathbb{C})$–opers on X are in a natural bijective correspondence with the equivariant $\mathrm{SL}(r,\mathbb{C})$–opers on Y. Equivariant $\mathrm{SL}(r,\mathbb{C})$–opers on Y are in a natural bijective correspondence with the subspace of $\mathcal{D} \in \mathrm{DO}^r(\mathbf{L}^{1-r},\, \mathbf{L}^{r+1})^\Gamma$ (see (6.5)) defined by all invariant differential operators D satisfying the following two conditions:

- the symbol of D is the constant function 1, and
- the element in $H^0(Y,\, K_Y)$ corresponding to D (see (6.6)) vanishes (this is equivalent to the vanishing of the sub-principal symbol of D [BD1]).

This subspace of $\mathrm{DO}^r(\mathbf{L}^{1-r},\, \mathbf{L}^{r+1})^\Gamma$ is in a natural bijective correspondence with

$$\Psi^{-1}(0) \subset \widetilde{\mathrm{DO}}^r_P(\mathcal{L}_*^{1-r},\, \mathcal{L}_*^{r+1}),$$

where Ψ is the map in Lemma 6.1. ∎

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

Email address: indranil@math.tifr.res.in

Université Lille 1, Cité scientifique U.M.R. CNRS 8524, U.F.R. de Mathématiques 59 655 Villeneuve d'Ascq Cédex, France

Email address: Niels.Borne@math.univ-lille1.fr

Université Côte d'Azur, CNRS, LJAD, France

Email address: dumitres@unice.fr

Beijing Institute of Mathematical Sciences and Applications, Yanqi Island, Huairou District, Beijing 101408

Email address: sheller@bimsa.cn

Université Côte d'Azur, CNRS, LJAD, France

Email address: pauly@unice.fr