Geodesics and Shortest Paths Approach in Pedestrian Motions

B. Nkonga and M. Rascle[1] and F. Decoupigny and G. Maignant[2]

[1] Laboratoire Dieudonné, UMR CNRS 6621 and University of Nice, Parc Valrose, 06108 Nice Cedex 2, France `nkonga or rascle@unice.fr`

[2] Laboratoire Espace, UMR CNRS 6012 and University of Nice, BP 3209, 06204 Nice Cedex 3, France `decoupig or maignant@unice.fr`

0.1 Introduction

The first two Authors are from Applied Mathematics (M.R.) or Computational Fluid Mechanics (B.N.), whereas F.D. and G.M. are from Geography.

We present and revisit here in some details the modeling of pedestrian flows based on the shortest path and geodesics approach, i.e. on variants of the Hamilton Jacobi (HJ) equation, more precisely on the eikonal equation. For general mathematical references, among a huge literature, see e.g. [3], [10]. In the context of pedestrian flows [1], and many subsequent papers, e.g. [2, 14].

Our approach here was motivated for the last two Authors by previous works of Decoupigny on various problems, including discrete models of optimal paths for visitors of French national parks [12] ... and for Rascle by previous works on mesh generation [4], apparently far away from, but in fact closely related to this subject.

In fact, besides its historical origin on light or particle propagation, this fairly general view has been intensely used in the last decades on a huge variety of fields, e.g. by Osher [9], Sethian [5], ... and many others, e.g. in the group of P. Markowich at Cambridge, not forgetting a lot of clever works, e.g. [7], ... on the numerical approximation of the viscosity solution, see e.g. [10] ...

This approach is thus not new in the Pedestrian experts community, see again [1], many people know it and its limitations, in particular its numerical cost. Of course, there are so many aspects in pedestrian traffic analysis that it would be naive to pretend cover all of them with this single method. Nevertheless, its beauty and the relevant global information its contains make it quite elegant and appealing.

By the way, it is perhaps worth to emphasize that, here like in (car) traffic flow, there is no physics or continuum mechanics involved, no conservation of momentum etc ... The aim is simply to propose one optimization principle which might govern the choice of pedestrian trajectories in a crowded area.

It would be also quite interesting to compare it carefully with competing methods, in particular with the social forces (or acceleration) methods, see e.g. [8], [13] ...

In the next section, we briefly recall a few basic facts on this highly classical method, up to the notion of distance $d(x, C)$ from a curve C to any point x. Typically, say in a big hall or a plaza, each population k of pedestrians travels towards a door or a destination $C := D_k$ with a velocity determined by the gradient of $d_k(x) := d(x, D_k)$.

In section 3, we present the full corresponding macroscopic model and give some details on its (expensive) approximation, before studying in section 4 a (much cheaper) semi-discrete description, in which each population k of pedestrians is defined in a discrete way, but each corresponding distance d_k in a macroscopic way. We then show a few preliminary results and conclude.

0.2 Basic facts on Hamilton-Jacobi equation. Distance

We first briefly summarize the main ingredients.

Variational Problem. Distance

For any curve $\{X(t) = (X_1(t), X_2(t)), 0 \leq t \leq T\}$ from point $y = (y_1, y_2)$ (origin) to point $z = (z_1, z_2)$ (destination), define:

$$L_1[X] := \int_0^T L(X(t), \dot{X}(t))\, dt, \tag{1}$$

where the *cost* $L(x, v) = L(X(t), \dot{X}(t)) \geq 0$ is often called the *Lagrangian*. We set

$$d_L(y, z) = \inf\{L_1[X]\,,\, X(0) = y\,,\, X(T) = z\}. \tag{2}$$

The infimum, taken on all curves from $y = (y_1, y_2)$ to $z = (z_1, z_2)$, depends on y, z and (non locally) on the choice of L. Since $L \geq 0$, d_L is a distance, provided that $L(x, v) \equiv L(x, -v)$, which implies: $d_L(y, z) = d_L(z, y)$ for all pairs (y, z).

Euler-Lagrange equation

If a trajectory is optimal , then a Taylor expansion and an integration by parts shows that necessarily this celebrated relation:

$$\frac{d}{dt}(\, \partial_v L(X, \dot{X})) = \partial_x L(X, \dot{X}), \tag{3}$$

must be satisfied. Here $v = (v_1, v_2) = \dot{X}(.)$, $x = X(.)$ and the gradients $\partial_x L$, $\partial_v L$ are $2D$ vectors.
In particular, straight lines are optimal if we choose $L(x, v) \equiv L(v) := \frac{1}{2}|v|^2$, and broken lines are optimal if $L(x, v) = \frac{1}{2c(x)}|v|^2$, with $c(x) > 0$ is piecewise-constant, say $c(x) = c_\pm$ for $\pm\, x_2 > 0$: Descartes refraction law.

Legendre transform in *v*. Link with Hamilton-Jacobi equation.

For any position x and any 2D vector p, the *Hamiltonian* is defined as the Legendre transform of $L(x, v)$:

$$H(x, p) := \sup_v \{p.v - L(x, v)\}. \tag{4}$$

For convex *Lagrangian* with respect to v, extremality conditions give

$$H(x, p) = p.v - L(x, v), \quad for \quad v = \partial_p H(x, p) \tag{5}$$

In particular in the cases considered below, the *Lagrangian* is v-convex and of the following form

$$L(x, v) = \frac{1}{2c(x)}|v|^2, \tag{6}$$

where $c(x)$ is a given strictly positive function. In this case, we obtain

$$H(x, p) := \sup_v \{p.v - \frac{1}{2c(x)}|v|^2 - \frac{c(x)}{2}|p|^2\} + \frac{c(x)}{2}|p|^2 = \frac{c(x)}{2}|p|^2, \tag{7}$$

and (extremality relations), for any x, the supremum in v is reached for

$$v = c(x)p. \tag{8}$$

Link with Euler-Lagrange. Hamiltonian system

Using these extremality relations leads classically to the characterization of optimal trajectories by the celebrated Hamiltonian characteristic system:

$$\dot{X} = \frac{dX}{dt} = \partial_p H(X, p) = v, \tag{9}$$

$$\dot{p} = \frac{dp}{dt} = -\partial_x H(X, p) = \partial_x L(X, v), \tag{10}$$

which itself leads, in a highly nontrivial way, to the corresponding *eikonal* or more generally *Hamilton-Jacobi (HJ)* equation.

Hamilton-Jacobi equation:

Finally, written here in the time-dependent version:

$$\partial_t \Phi + H(x, \partial_x \Phi(x, t)) = 0, \tag{11}$$

or in its **stationary** version

$$H(x, \partial_x \varphi(x)) = f(x), \tag{12}$$

with in either case suitable *initial and/or boundary conditions*, this famous equation characterizes the optimal trajectories and therefore the solution of variational problems formulated as in equation (2), with the same cost L and Hamiltonian H defined in (4).

Application: distance from a given curve (C).

In view of its sense of propagation, we should perhaps call it : distance from (C) to x. Anyway, this distance, according to a cost function L, is defined as:

$$d(C,x) := d_L(C,x) := \inf_{y \in (C)} \{d_L(y,x)\}, \tag{13}$$

In particular, we choose the cost function L given by (6) and H in (7), with $f(x) \equiv \frac{1}{2}$. In the next sections two choices are considered

Choice 1 : $c(x) = \beta(\rho(x))$, Choice 2 : $c(x) \equiv 1$.

The function $\beta(\rho)$ will vanish at high local densities when $\rho(x) = \rho_\infty$, i.e. in **congested regions.**. We will use $\beta(\rho(x)) = 1.0 - \frac{\rho(x)}{\rho_\infty}$. It turns out that $\varphi(x) := d(C,x)$ satisfies (12), and finally satisfies:

$$H(x, \partial_x \varphi(x)) := \frac{c(x)}{2} |\partial_x \varphi(x)|^2 = f(x), \quad \text{with} \quad \varphi(x) \equiv 0 \text{ on } (C). \tag{14}$$

Indeed, using the extremality relations (8), with choice 1 and again $f \equiv 1/2$, we see that on each optimal trajectory, parameterized by some artificial time, say $x = X(t)$:

$$v = \dot{X}(t) = c(x)\partial_x \varphi(x) \text{ and } |v| = \sqrt{c(x)}. \tag{15}$$

Therefore, as expected, **the smaller $c(x)$, the smaller the speed**. For numerous applications, see again the Osher-Sethian literature.

Moreover, $\Phi(x,t) := \varphi(x) - \frac{t}{2}$ is a solution of the evolution equation (11). Of course, similar relations would arise if we used another variant of eikonal equation:

$$\sqrt{c(x)}\, |\partial_x \varphi(x)| = 1. \tag{16}$$

Finally, we note that, see e.g. [3], even with smooth data, the solutions of (11) or (12) are not uniquely defined: *shocks* can arise, at points x which are equidistant from at least two points of curve (C). Then the theory of *viscosity solutions*, [10], provides a uniqueness criterion by selecting the first arrival "time", i.e. the trajectory coming from the nearest point on (C), and this criterion is built in efficient numerical schemes.

For instance, in some figures in section below, where we plot the distance $d_k(x)$ from exit door k at any point x in a plaza, these shocks appear on the graph as "passes" near a "peak" separating two different paths from the door to x.

Two applications:

As we already said, most of the material here is classical. We have been motivated by two different previous works of ours:

- One concerns mesh generation or refinement, with the same ideas as here, e.g. refine a mesh by moving at constant Riemannian speed on the graph of a previous approximation of the solution. We refer to [4] for many illustration of the results.
- The other one concerns the optimal location of parking lots in a national park in order to restrict the number of paths. The idea (Decoupigny) is to identify a cost on each path, depending on the slope and of the interest of each site, find the geodesics, compare with existing paths, make recommendations. For related ideas, see [12]. A joint work in this direction is in progress.

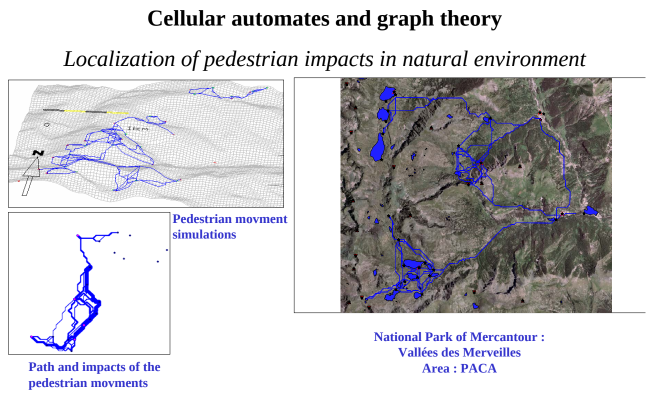

Fig. 1. Pedestrian paths in the Mercantour French National Park

0.3 Pedestrian Flow: Eulerian description

We are going to apply the above-mentioned notion of distance depending on the density. We refer again to the literature quoted in the Introduction and so many references therein, in particular we revisit here with minor variations the approach of [1].

Principle

Consider, see Figure 2 below, a plaza or a big hall with several doors $D_k, k = 1..K$, all being possibly both entrance and exit doors. At each time t and point x, let $\rho_k(x,t), k = 1..K$ be the local density of pedestrians moving towards door D_k. When time is discrete, set $\rho_k^n(x) := \rho_k(x, t_n)$ and let $\rho^n(x) := \sum_k \rho_k^n$ be the local **total density**. Finally, let $d_k^n(x, D_k)$ be the distance to door D_k, defined as in section 0.2 by:

$$H(x, \partial_x d_k^n) := \frac{\beta(\rho^n(x))}{2} |\partial_x d_k^n|^2 = \frac{1}{2} \text{ in the plaza,} \tag{17}$$
$$d_k^n(x) \equiv 0 \text{ on door } D_k, \tag{18}$$

where the associated Lagrangian depends on the (local) total density:

$$L_k^n(x, v) = \frac{1}{2c(x)} |v|^2, \text{ with } c(x) := c_k^n(x) = \beta(\rho^n(x)), \tag{19}$$

and $f(.)$ is a given non-increasing function of ρ, vanishing for large densities.

With the background of section 0.2 in mind, we now **postulate** that at each discrete time t_n, each population k responds to the gradient of the distance $d_k^n(x, D_k)$ by moving towards door D_k with a velocity given by equation (20) below.

Compare with extremality conditions (8) and note the difference of signs: here, the distance d_k^n is a nonlocal information which travels *from* door D_k, whereas the k-pedestrians travel *towards* this door. Consistently with this observation, we note that for each transport equation (21) to the unknown density ρ_k, see below, the other entrance doors $D_j, j \neq k$ only play a role in boundary data, as (given) *entering* fluxes into the plaza.

The algorithm :

Step 1: assuming $\rho := \rho^n$ is known at time t_n, compute each **distance** $d_k := d_k^n$ (we drop the index n), by solving the above problem (17).

Step 2: using the extremality condition, the corresponding velocity field v_k is then (note again the - sign):

$$v_k = -\frac{1}{2\,\beta(\rho(x))}\,\partial_x d_k \tag{20}$$

Step 3: for each k, refresh ρ_k from t_n to t_{n+1} by approximating the solution at time t_{n+1} to the Initial Value Problem (IVP):

$$\partial_t \rho_k + \nabla.(\rho_k\, v_k) = 0 \text{ in } \Omega \times (t_n, t_{n+1}), \tag{21}$$
$$\rho_k(x, t_n) = \rho_k^n(x) \text{ in } \Omega, \tag{22}$$

... with the classical **entering boundary conditions:** the flux $\rho\, v$ at each point x of the boundary $\partial\Omega$ is imposed whenever $v_k^n \cdot \nu < 0$, $\nu(x)$ being the exterior normal vector at point x. Practically, the flux $\rho_k \cdot v_k$ is only entering in the plaza, call it Ω through the other doors $D_j, j \neq k$.

Step 4: knowing the partial densities ρ_k^{n+1} at time t_{n+1}, refresh the total density ρ^{n+1}. Go to step 1.

0.4 Pedestrian Flow: Eulerian-Lagrangian description

Principle

Consider each discrete population k, going to door D_k, assume that their individual velocities are given from a continuous velocity field.
As above, at each time t, e.g. $t = t_n$, we assume that $v_k(x) := v_k(x,t) = -c(x,t)\partial_x d_k(x, D_k)$, where $d_k(x, D_k) = \varphi_k$ is the solution to (17), and again

$$c(x) := c(x,t) = \beta(\rho(x,t)) \tag{23}$$

is a function of a total *continuous* approximation of the total discrete density $\rho = \sum_k \rho_k$ at time t, as in steps 1 and 2 of our above fully Eulerian algorithm.

Now, for each individual j in population k, the resulting semi-discrete problem at time t is, for each population k:

$$\dot{X}_j(t) = v_j(t) \quad \text{with} \quad v_j(t) = -c(X_j(t),t)\,\partial_x d_k(X_j(t), D_k). \tag{24}$$

where $c(X_j(t),t) \simeq \displaystyle\int_\Omega \sum_k \sum_{\ell\in k} \delta_\ell(x) K(x - X_j(t))dx$ with $\delta_\ell(x)$ a Dirac function centered at the position of the particle ℓ and $K(\xi)$ is a smoothing kernel.

Comments

We obviously approximate this ODE system, say by the explicit Euler scheme. At each time step, the function $c(.)$ is defined by (23), and we have to decide either to *refresh* the density (either at each time step, or more rarely, e.g. at each time of visualization), or even to keep the (obsolete) initial density ...

When refreshing the density, the obvious problem is passing from the discrete positions to a continuous density. A simple choice is to count the number of particles in each numerical cell.

A crucial issue in designing pedestrian models is their ability of predicting and describing the appearance and subsequent evacuation of high concentration populations, where panic can be lethal. Of course, we do not claim to solve this challenging problem with such a simple model. Nevertheless, we see in Figures below that dense regions of space become repulsive, in a way which depends on the stiffness of function β in equation (20). This is related with a constrained model with a maximal ρ_{max} has been introduced and intensely studied several years ago by Maury, Venel and (on a more theoretical viewpoint) Santambrogio *et al*, see e.g. [11] ... An obvious idea is to approximate this limit model with the above model studied here, and using in (20) a function β stiff near the maximal density ρ_{max}. In the semi-discrete approach, that requires a careful calibration in the link between discrete and continuous densities, which requires more work.

1 Numerical tests:

We present here a few preliminary tests and visualizations, in which we try to combine and/or compare some of these approaches.

Test 1: We consider here a curve-shaped hall, with an (Nord-Est) entrance door and a (Sud-West) exit door. We consider that at a given time there is a dense region corresponding to $c(x) = c_1(x) = 1 - 0.99\exp(-(x - x_0)^2)$. Figure 2 shows on the left the level curves of the distance to the exit : see the shock (the "pass") in the NE corner and note that the trajectories, not shown here, are orthogonal to these level curves. On the right, we plot the sum of the two distances to (SW) and to (NE), computed on an unstructured mesh, via a numerical solver of the evolution HJ equation [6]. Even *without* any computation of the trajectories, the optimal paths from NE to SW or conversely (the purple areas) at a given time look pretty obvious, whereas running the full algorithm of section 3, much more expensive, provides the velocity field, see Figure 2, right.

Test 2: Here, a corridor or a platform with an exit door (right), with given incoming flux on the left door that is non zero for $0 < t < 120$, The geometry and positions of doors are taken from [7]. In figure 3 we plot the **density** at two different times. A high density front is formed near the obstacle and, by the distance update, is transversally (anisotropic) diffused. The local behavior around the obstacle is clearly different to results in [7] where high densities are avoided by adding an isotropic diffusion in the velocity.

Test 3: Using the semi-discrete approach of section 4 We have made a couple of comparisons between:
Choice 1: use the above eikonal equation with a variable coefficient $c(x) = \beta(\rho(x))$, in order to prevent the density from becoming too large. Here, we have only tested it with a smooth $\beta(\rho) = 1.0 - \frac{\rho}{\rho_\infty}$ and a given ρ_∞.
Choice 2: cf Maury, Venel (and Santambrogio *et al* on more theoretical side), eikonal equation with a constant $c(x)$ and a *constraint* on the maximal (total) density with corresponding Lagrange multipliers and a numerical solution based on Uzawa algorithm. Very nice mathematically and numerically, but expensive. Its "incompressible" feature in congested regions is perhaps a good cartoon for *lane formation*.

2 Final comments and conclusion

As we said, we have revisited existing ideas on the eikonal equation, combined them with a discrete Lagrangian description, and made some preliminary numerical tests on some of these ideas. Clearly, by far the most tractable method is the semi-discrete algorithm used in section 4.

We could also see, on numerical movies that we cannot show here, that this method can nicely describe counterintuitive motions, like moving **away** from the target in order to avoid crowded areas, which is a neat advantage on fully discrete methods. Going further requires a more refined calibration, with an *ad hoc* function β, in order to see how far one can go in this direction. We also mention again the importance of this notion of geodesics in geography and more generally in any field related to space organization.

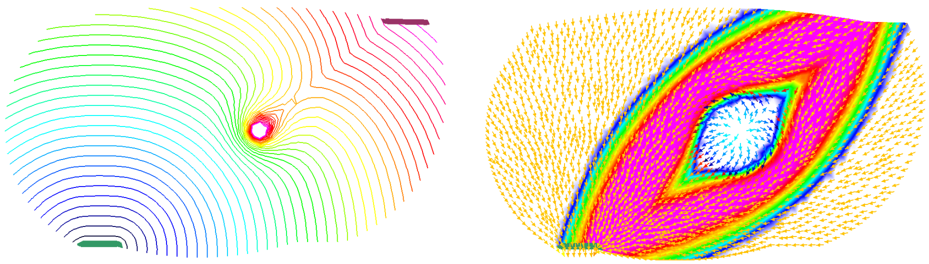

Fig. 2. Level curves of distance (left) from the exit door, computed for $c(x) = c_1(x)$, estimation of optimal paths (right) as sum of distances and local optimal trajectory direction for the exit.

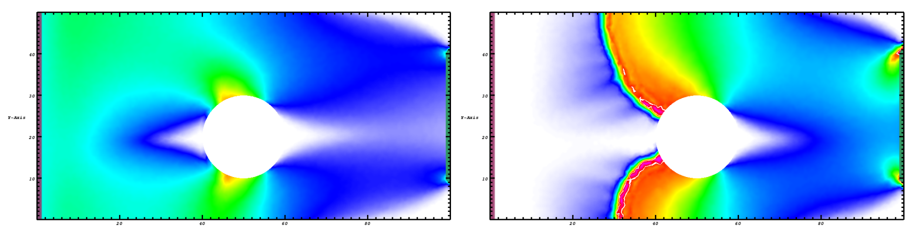

Fig. 3. Plots of the density at $t < 120$ (left) and $t > 120$(right) , using the full scheme of section 3 with $\beta(\rho) = 1.0 - \frac{\rho}{\rho_\infty}$ and $\rho_\infty = 10$

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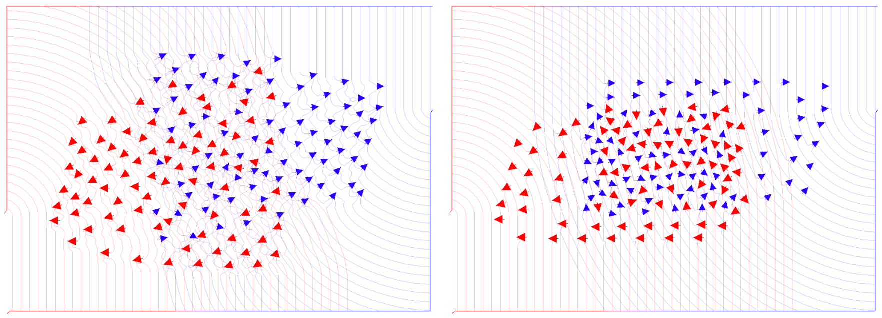

Fig. 4. Choice 1 (left) vs choice 2 (right). Snapshots at computational midtime. For comparison, the speeds are normalized in both algorithms.

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