Correction of Second Order Hyperbolic Model.
Links with the "Follow the Leader" Models

Michel Rascle

Laboratoire J.A. Dieudonné UMR-CNRS n.6621
Université de Nice-Sophia Antipolis

Email: rascle@math.unice.fr
http://www-math.unice.fr/ rascle

Introduction.

3 classes of models:

Microscopic "Follow-the-leader" models :

(Large) ODE systems, see Gazis-Herman-Rothery (1961) and many others....
Example :

$$\dot{x}_i = v_i, \qquad (1)$$

$$\dot{v}_i = C(v_i)^m \frac{v_{i+1} - v_i}{(x_{i+1} - x_i)^{\gamma+1}} + \frac{A}{T_r}\left[V_{eq}\left(\frac{\Delta X}{x_{i+1} - x_i}\right) - v_i\right]$$

In principle, $\exists$ a delay in 2^{nd} equation, vanishing at the limit. We neglect it.
If $A > 0$, last term : (macroscopic) relaxation term, with $T_r >> T_{reaction}$, Klar-Kühne-Wegener.
Here, we take $m = 0$ and in general, $A = 0$.

Kinetic Models:

see Prigogine-Herman, Paveri-Fontana,
Klar-Wegener, Nelson, Illner, Sopasakis, Dolbeault ...

$$\partial_t f + v \cdot \partial_x f = C(f, f)$$

Nice mathematical problems ...

Big particles : small "Avogadro number": not much room for kinetic scale.

Collision terms sometimes nonlocal in x.

Molecular chaos assumption questionable...

However,

Successful prediction of multi-modal fundamental diagrams.

Formal derivation of our fluid model !...
Klar-Wegener, 2000.

Macroscopic models :

"First order" :

Lighthill-Whitham-Richards (LWR)

$$\partial_t \rho + \partial_x(\rho v) = 0\,;\, v = V(\rho).$$

Equilibrium model, V given, e.g. $V(\rho] = 1 - \rho$.
Scalar conservation law, often with a concave flux :
Shocks when braking, Rarefactions when accelerating.

"Second order" :

I. Payne-Whitham (PW) :

Gas dynamics system, P : "pressure" (but no momentum conservation !!):

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t v + v\partial_x v + \rho^{-1}\tilde{P}'(\rho)\partial_x \rho = 0 \end{cases}$$

Can add a diffusion and/or a relaxation term :

$$(\tau^{-1})\,(V_{eq}(\rho) - v) + \;\nu\partial_x^2 v$$

in RHS of second equation ($=$ anticipation equation).

Université Nice SOPHIA ANTIPOLIS

II. Critics: C. Daganzo ('95): Requiem for 2^{nd} order ...

PW model is a crazy model ! Indeed,

In some cases, cars are going backward !!

Part of the information travels faster than cars : !!

(since $\lambda_2 := v + c > v$, c : sound speed).

III. Aw-Rascle (SIAP) : Resurrection ... ?

Correction of PW : New 2^{nd} order ("AR") model and

Study of Riemann Problem (see below).

Question: what's wrong in PW ?

Rewrite 2^{nd} equation :

$$\partial_t v + \ v\, \partial_x v = -(\rho)^{-1} p'(\rho) \partial_x \rho + \text{Diffusion} + \text{Relax.}$$

Question : **does a driver respond to a** space (PW) or time **derivative of the density ???**

Our correction: TIME derivative :

Replace $\partial_x \rho$ by $(\partial_t \rho + v \partial_x \rho)$!!

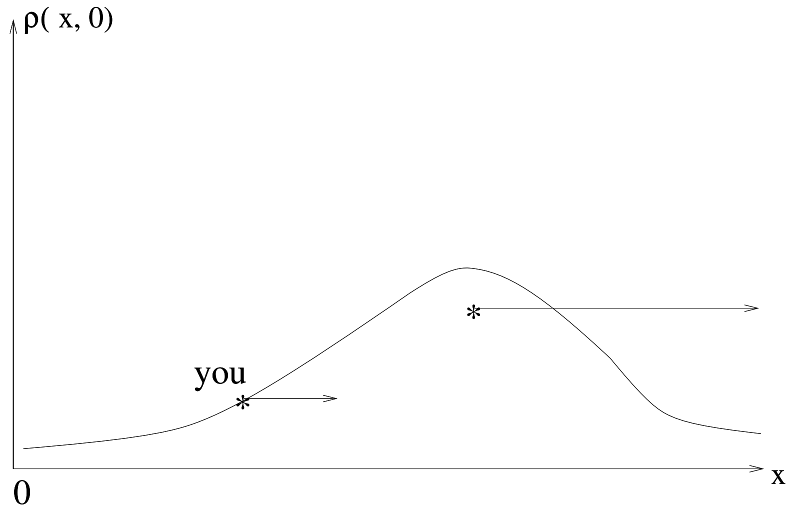


Figure 1: **In such a case, would you brake, or accelerate?**

With this simple fixing, no more paradox :

$$v \geq 0 \qquad\qquad \lambda_2 = v.$$

Remark: No diffusion !!, but relaxation is O.K. [and $\Rightarrow$ **formal** diffusion (Chapman-Enskog)].

New 2$^{\text{nd}}$ order ("AR") model and Study of Riemann Problem (see below). Related works include :

J. Greenberg, SIAP 01 :

Initial Value Problem (IVP) for same model, adding a special relaxation term (in "characteristic" case).

Aw-Klar-Materne-Rascle (SIAP 02) : Derivation ... :

Initial Value Problem (IVP) for the model with relaxation (in the general "(sub)characteristic" case), and rigorous derivation of this new model from Follow-the-Leader models.

Independently, M. Zhang (Trans. Res., B, 02 ?):

"A model devoid of ..." rediscovered same model and gave a formal derivation from Follow-the-Leader models : different communities, same ideas "in the air"...

P. Bagnerini, M. Rascle, To appear SIMA

Homogenization of same model for Multiclass Traffic Flow. Same links with multiclass Follow -the-Leader models.

Incomplete list !!.

see e.g. M. Zhang, Tong Li, R. Colombo, Benzoni-Colombo ...

Summary.

- (1) : Microscopic Follow the Leader ODE system (1).
- (2) : our "AR" 2^{nd} order model (Eulerian) $\leftrightarrow$ (2') (Lagrangian).
- (3) : its (Lagrangian) Godunov discretization.

Then :

- Solution of (3): uniform estimates in L^∞ and BV.
- (3)$\equiv$ also (1^{rst} order, explicit) time discretization of (1).
- When ΔX and $\Delta t \to 0$, fixed ratio, CFL, (3) $\to$ (2).

- When $\Delta t \to 0$, with fixed ΔX, (3) converges to (1)$\to$: same estimates for (1).
- Now, (1) = 1/2-discretization of (2'). When $\Delta X \to 0$, (1)$\to$ (2').

In this sense, (2) or (2') = hydrodynamic limit of the Follow-the-Leader system.

Outline.

- 1. Introduction.
- 2. Our model.
- 3. Riemann Problem. Comparison with PW.
- 4. The model with relaxation.
- 5. Homogenized model (with P. Bagnerini).

Our model.

In Payne-Whitham (PW), in "anticipation equation" :

$$\partial_t v + \ v\, \partial_x v + (\rho)^{-1} \partial_x p(\rho) = \ 0,$$

replace $\partial_x p$ by convective derivative $(\partial_t + v\partial_x)$ of (new) p.

New model :

$$\partial_t \rho + \ \partial_x(\rho v) = \ 0,$$

$$\partial_t (v + p(\rho)) + \ v\, \partial_x (v + p(\rho)) = \ 0.$$

(Pseudo)-pressure $= p(\rho) = -V(\rho) + \text{constant}$, V : equilibrium velocity : $V'(\rho) \leq 0$.

In conservative form , 2^{nd} equation $\rightarrow$

$$\partial_t(\rho w) + \ \partial_x(v \rho w) = \ 0\ , \quad w := \ v + p(\rho).$$

Riemann Problem : Examples.

Strictly hyperbolic system: (e.g.) in variables $U = (\rho, v)$

$$\partial_t U + A(U)\,\partial_x U = 0,$$

$$A(U) := \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix}.$$

Real eigenvalues :

$$\lambda_1 = v - \rho\, p'(\rho) \leq \lambda_2 = v,$$

distinct except at vacuum : $\rho = 0$ if e.g. $p(\rho) = \rho^\gamma$, $0 < \gamma \leq 1$.

Diagonalize : Riemann invariants :

$$w = v + p(\rho) = v - V(\rho) + \mathrm{Const}\ , \ z = v.$$

w : distance to equilibrium. v : velocity. For smooth solutions,

$$\partial_t w + \lambda_2\, \partial_x w = 0, \quad \lambda_2 = v,$$

$$\partial_t z + \lambda_1\, \partial_x z = \partial_t v + (v - \rho p'(\rho))\, \partial_x v = 0.$$

Eigenvalues :

λ_1 : genuinely nonlinear (GNL) : strictly monotone in a 1-wave.

λ_2 : linearly degenerate (LD) : constant in a 2-wave.

Classically, λ_2 is LD, $\rightarrow$ $z = v =$constant across a 2-wave,

and $w =$ constant across a 1-rarefaction wave.

Here w also $=$ constant across a 1- shock : much more exceptional : Systems " à la Temple " : shock curves and rarefaction curves coincide.

Solution of Riemann Problem : Given $U_\pm = (\rho_\pm, v_\pm) \in I\!R^2$, draw the curves

$$w = w_- := w(U_-) \text{ and } z = v = v_+ := z(U_+),$$

Unique intersection : $U_0 \leftrightarrow (w_-, v_+)$.

Connect $U_- \leftrightarrow U_0$ by 1-wave : $w = w_-$ and
shock wave (braking) if $v_0 = v_+ < v_-$
or rarefaction wave (acceleration) if $v_- < v_0 = v_+$.

and $U_0 \leftrightarrow U_+$ by $2-$ wave : contact discontinuity, rear cars follow leading car, same speed $v_0 = v_+$.

Application : comparison with PW.

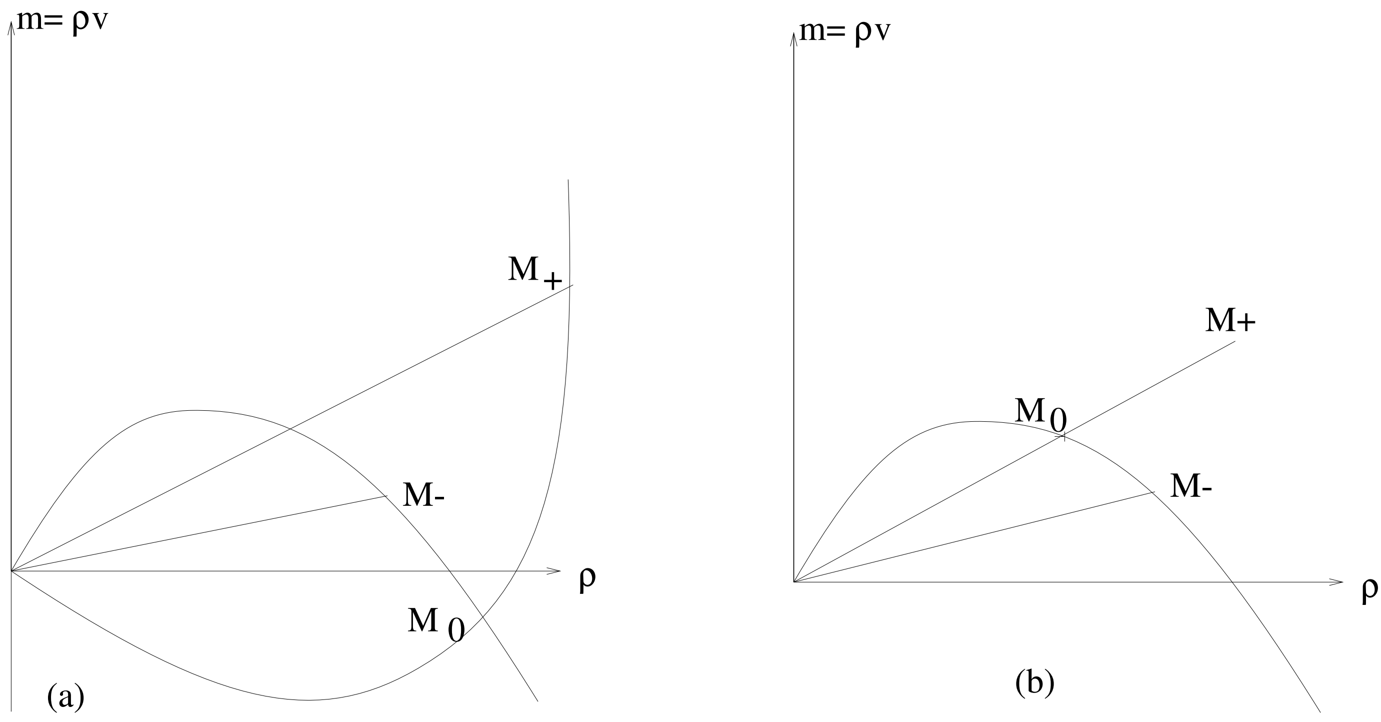


Figure 2: **Example of Riemann Problems for the PW model (a), and for our model (b). In (a) cars behind instantaneously reach negative speeds (!), although cars ahead travel *faster* !**

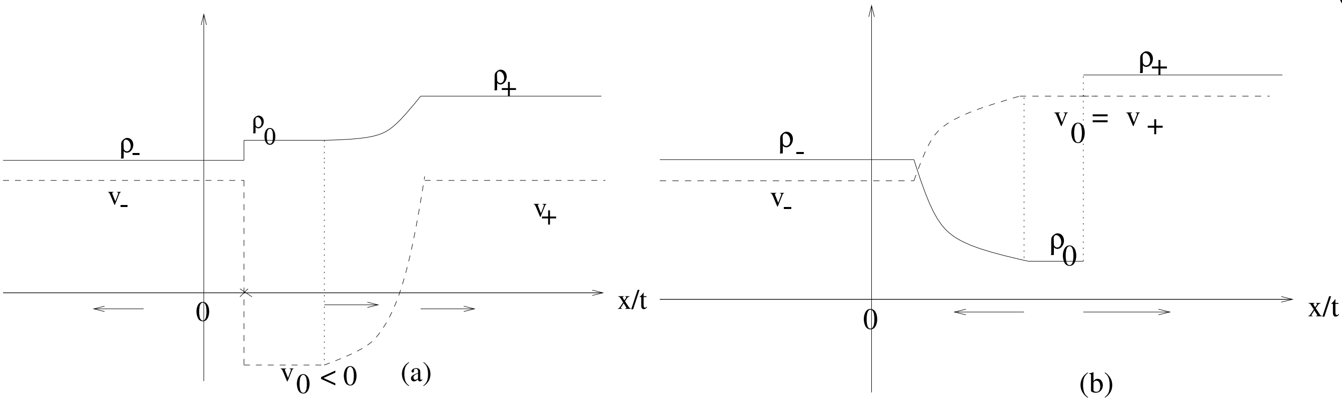


Figure 3: **Negative velocities appear with the PW model (a), not with our model (b).**

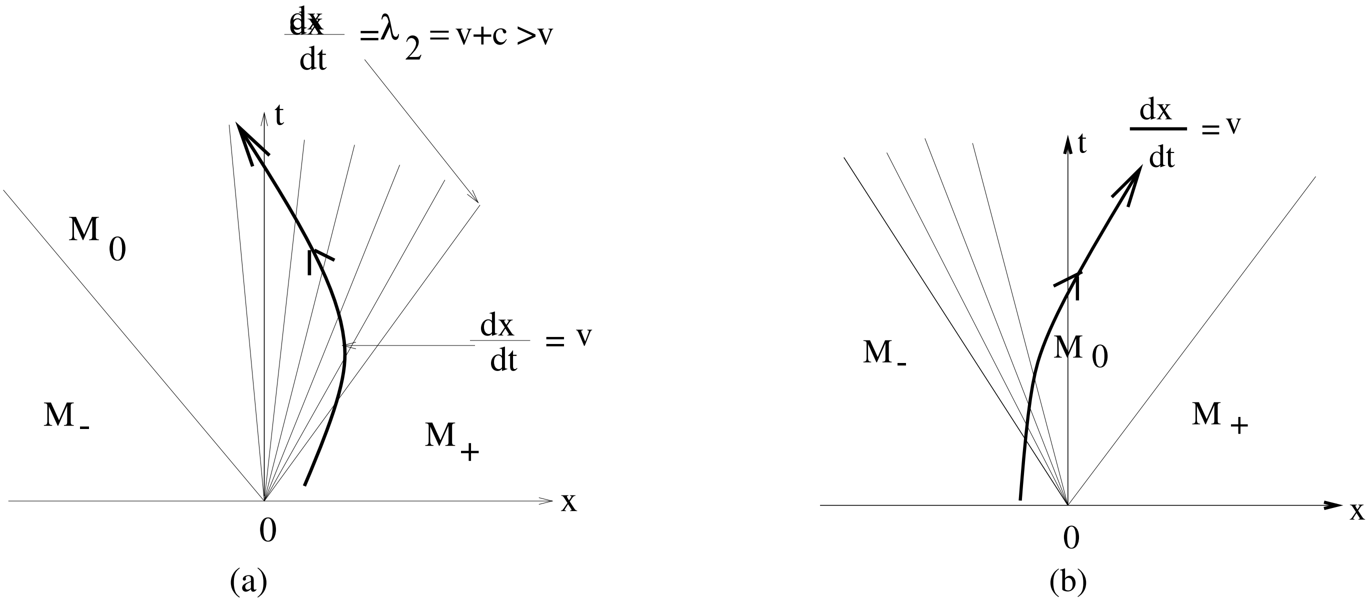


Figure 4: **The trajectories** $\frac{dx}{dt} = v$ **cross the 2 (resp 1)-rarefaction wave from the right to the left with PW model (a) from the left to the right with our model(b) : part of the information travels *faster* than cars ahead, and modifies their trajectories (!) in PW model (a), not in our model (b).**

Université Nice SOPHIA ANTIPOLIS

Remarks.

All qualitative requirements now satisfied :

velocities and densities - remain ≥ 0 and bounded (invariant region), except near $\rho = 0$ for $p(\rho) = ln\rho$.

No wave travels faster than the speed of the vehicles.

Model always predicts natural waves : shocks when braking, contacts when following, rarefactions, possibly with vacuum, when accelerating.

Only (milder, curable) drawbacks :

1. If U_+ at vacuum, maximal speed in rarefaction wave depends on U_- : unrealistic ??. Fixing : either maximal speed (reachable by all cars) or add relaxation term.

2. Initial data must be in invariant region. If not, cars could hit each other. Fixing: relaxation (?) and/or add a singularity for p at $\rho = \rho_m = 1$, cf Colombo ...

The model with relaxation.

$$\partial_t \rho \ + \ \partial_x(\rho v) \ = \ 0,$$

$$\partial_t(\rho(v \ + \ p(\rho))) + \ v\partial_x(\rho(v \ + \ p(\rho))) = \rho(T_{eq}^{-1} \ (V_{eq}(\rho) - v),$$

T_{eq} : relaxation time, $V_{eq}(\rho)$ equilibrium velocity.

Weak entropy solutions :

Additional conservation laws of this system : entropy - entropy flux pairs of functions with scalar values :

$$(\eta(U), \ q(U)) := (\eta(\rho, v), \ q(\rho, v)).$$

For any convex entropy $\eta(U)$, impose

$$\partial_t \eta \ + \ \partial_x q \ \leq T_{eq}^{-1} \partial_v \eta \ (V_{eq}(\rho) - v).$$

Q :(strictly ??) convex , convex / w ??? ... Extension of Krushkov entropies : nonconvex, but entropy ineq. ... and minimal at equilibrium.... cf Aw's PhD

Remark : modeling needs a second heuristic choice function V.

Crucial assumtion: Subcharacteristic condition, Whitham, Chen-Liu-Levermore ... Here :

$$-p'(\rho) \leq V'(\rho) \leq 0.$$

Example of equilibrium velocity

$$V(\rho) = \alpha\ (p(\rho_m) - \ p(\rho))\ ,\ 0 \leq \alpha \leq 1,$$

$\rho_m = 1$, maximal (normalized) density, and

$0 < \alpha < 1$: (strictly) subcharacteristic case.
$\alpha = 0$ or 1 : characteristic case, cf J. Greenberg ($\alpha = 1$).

Role of subcharacteristic condition : $T_{eq} \to 0$, Chapman-Enskog expansion... At leading order :

$$\partial_t \rho + \partial_x(\rho\ V(\rho)) = \ -T_{eq}\partial_x(V_1(\rho)) \sim \ dx(D(\rho)\ \partial_x \rho).$$

Nonlinear diffusion coefficient

$$D(\rho) := -T_{eq}^{-1}\ (V'(\rho) + p'(\rho))V'(\rho)$$

must be ≥ 0. Also $\to$ the sum of TV of v and w controlled, cf Natalini ...

Université Nice SOPHIA ANTIPOLIS

Formal regularization without infinite speed of propagation and cars going backward...

Also essential condition for stability and convergence of numerical schemes and of zero relaxation limit, i.e. convergence of " AR" to Lighthill- Whitham-Richards (LWR) when $T_{eq} \to 0$.

Some related papers:
J. Greenberg (SIAP) , Aw (PhD), Schochet, Lattanzio-Marcati ...

Chen-Liu-Levermore, Natalini, Jin-Xin, Katsoulakis-Tzavaras ...

Same + Vasseur, Chen-Rascle : initial layer problem for non BV case.

Case where (sub)characteristic condition not satisfied: Greenberg'-Klar-Rascle: SIAP '02: model with hysteresis (cf Van der Waals fluids) : bi-modal equilibrium function with V_{eq} not monotone.

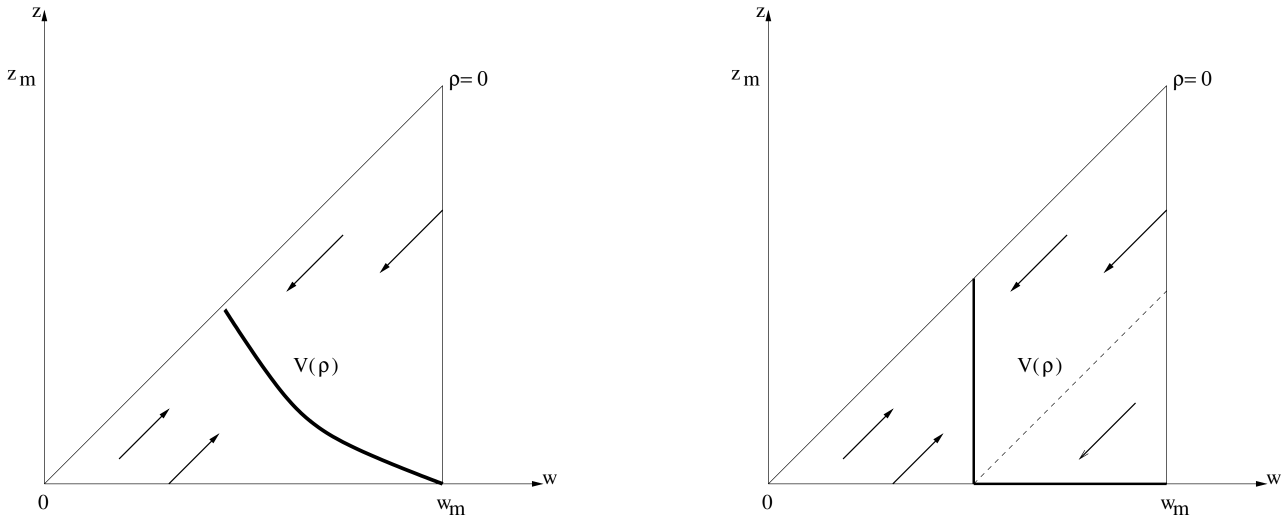


Figure 5: **Invariant region R and equilibrium curve** $v = V_{eq}(\rho)$ **for** $\gamma > 0$**, in the** $(w, z) = (w, v)$ **plane, in the subcharacteristic case (left), and in the characteristic case (right).**

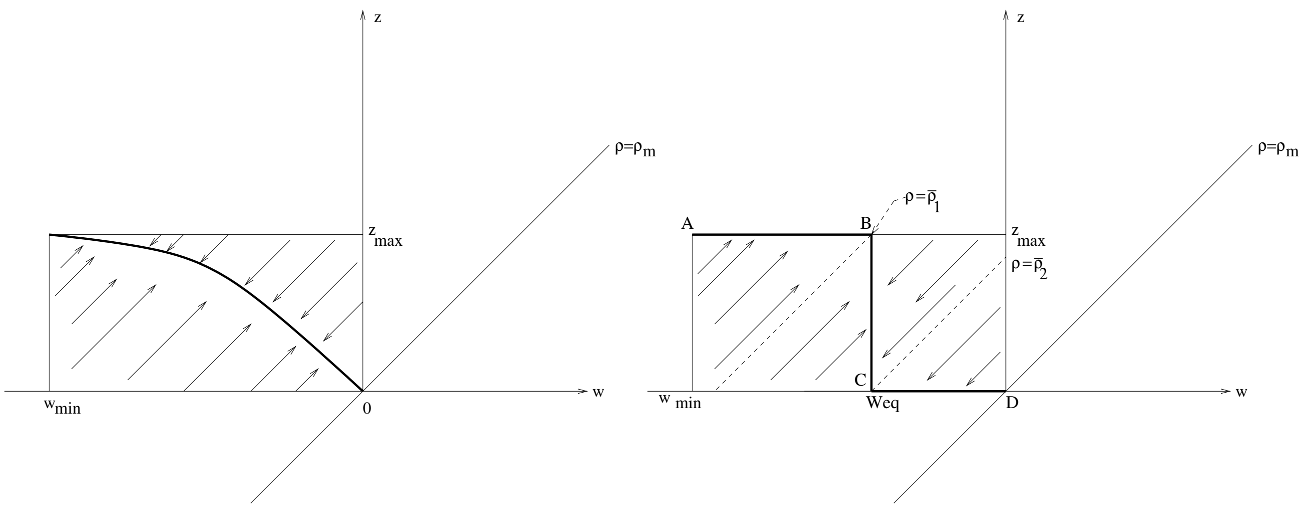


Figure 6: **Invariant region R and equilibrium curve** $v = V_{eq}(\rho)$ **for** $\gamma = 0$**, in the** (w, v) **plane, in the subcharacteristic case (left), and in the characteristic case (right).**

A Multiclass Homogenized Model for Traffic Flow.

P. Bagnerini and Michel Rascle

Laboratoire J.A. Dieudonné UMR-CNRS n.6621
Université de Nice-Sophia Antipolis

Email: pbagneri or rascle@math.unice.fr
pbagneri@dm.unife.it

Introduction

Goal: Macroscopic multiclass Model(cars, trucks etc.):

Introduce two coefficients : w, a, which characterize each class.

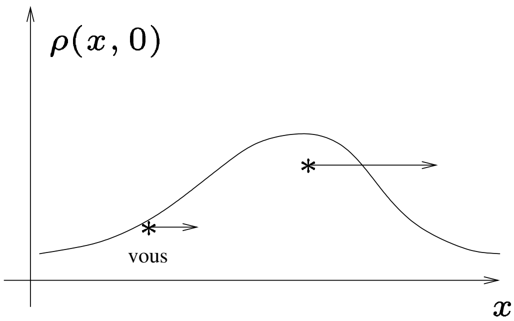
Long stretch of road: oscillating distribution of vehicle classes.
Average value : $< \mu_x(a), a >$

Prototype: $\mu_x = \sum_i \alpha_i(x)\delta_{(a_i,w_i)}$. Once v is known, we know average density ρ, and density ρ_i of each class.

Payne-Whitham model.(PW):

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t v + v\partial_x v + \rho^{-1}\tilde{P}'(\rho)\partial_x \rho = 0 \end{cases}$$

Daganzo, “Requiem for second-order fluid approximation to traffic flow” (’95)



Here, speed increasing x, decreasing in t along your trajectory.
Q : brake ou accelerate? Aw-Rascle 2000:

$$\partial_x \tilde{P}(\rho) \longleftarrow (\partial_t + v\,\partial_x)\tilde{P}(\rho)$$
$$\partial_t(v + \tilde{P}(\rho)) + \partial_x(v + \tilde{P}(\rho)) = 0$$

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t \rho w + \partial_x v\rho w = 0 \end{cases} \qquad w = v + \tilde{P}(\rho) = v - \tilde{V}(\rho) + \text{Const}$$

w : distance to equilibrium.

$\tilde{V}(\rho)$:= equilibrium speed : anticipation term.

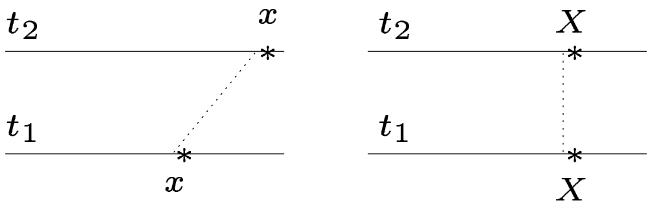
Multiclass Model.

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t \rho w + \partial_x v\rho w = 0 \\ \partial_t \rho a + \partial_x v\rho a = 0 \end{cases}$$

$$\begin{cases} \text{AR} \quad w = v + \tilde{P}(\rho) := \text{distance to equilibrium.} \\ \text{Here,} \quad w = v - \tilde{V}(\rho, a). \text{ e.g. } w = v + a\tilde{P}(\rho) \end{cases}$$

Coefficients $a \in [0,1]$ and w ... characterize each class, e.g.

$$w = v - \tilde{V}(\rho, a) := v + a\tilde{P}(\rho) \ , \ a \in [1/2, 1].$$



Lagrangian mass coordinates (X, t) :

$$T = t \qquad X = \int^x \rho(y,t)dy \qquad \tau = 1/\rho$$

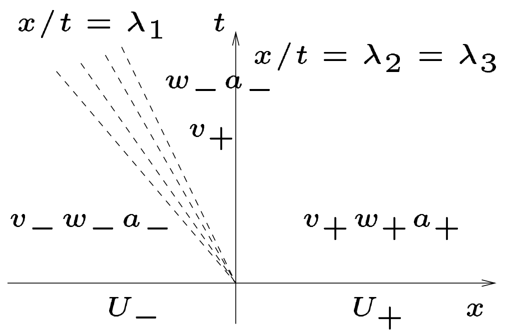
$$\begin{cases} \partial_t \tau - \partial_X v = 0 \qquad v = w - aP(\tau) \\ \partial_t w = 0 \\ \partial_t a = 0. \end{cases}$$

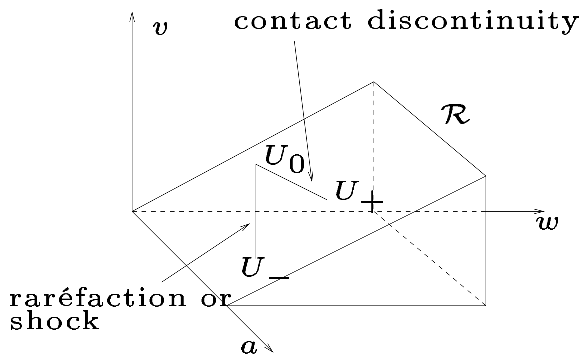
e.g. $V(\tau, a) = -aP(\tau) + \text{Const} \; ; \; P'(\tau) < 0, \qquad P''(\tau) > 0$

$$P(\tau) := \tilde{P}(1/\rho) = \begin{cases} \frac{v_{ref}}{\gamma} \frac{1}{\tau^\gamma}, & \gamma > 0, \\ -v_{ref} \ln(\tau), & \gamma = 0, \end{cases}$$

$$\text{VNL} \;\; \lambda_1 = aP'(\tau) < 0, \qquad \text{LD} \;\; \lambda_2 = \lambda_3 = 0$$

(Strict) Riemann Invariants: $v \leftrightarrow \lambda_1$, $(w, a) \leftrightarrow \lambda_2 = \lambda_3 = 0$.





Entropy-flux pairs:

$$\eta(v, w, a) = \int_0^v \frac{q'(s)}{aP'(\tau(s, w, a))} ds + \eta_0(w, a), \; q = q(v)$$

$\forall \eta$ convex / to τ ($\Leftrightarrow \forall q = q(v)$ concave / to v),

$$\partial_t \eta + \partial_x q \leq 0 \;\text{in}\, \mathcal{M}(\mathbb{R} \times]0, \infty[)$$

$$\partial_t \tau - \partial_x (w - aP(\tau)) = 0$$

Godunov Scheme.

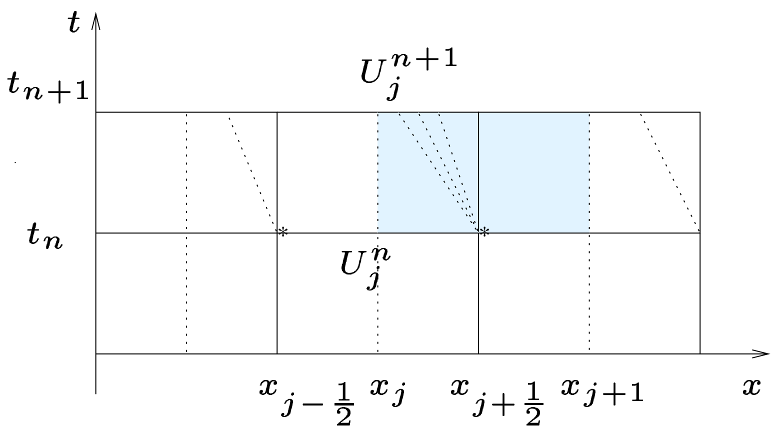
In Lagrangian coordinates (x,t) : $\partial_t U + \partial_x F(U) = 0$

$$U_j^{n+1} := U_j^n - \frac{\Delta t}{\Delta x}[F_{God}(U_j^n, U_{j+1}^n)) - F_{God}(U_{j-1}^n, U_j^n)] =$$

$$= U_j^n + \frac{\Delta t}{\Delta x}(v_{j+1}^n - v_j^n, 0, 0)$$

$$\begin{cases} \partial_t \tau_h - \partial_x v_h = 0 & \text{in } \mathbb{R}\times]t_n, t_{n+1}[\\ \partial_t w_h = 0 \\ \partial_t a_h = 0 \end{cases}$$

$$\begin{cases} U_h(x,t) & h \leftrightarrow (\Delta x, \Delta t) : \text{ scaling parameter : } h \to 0. \\ \Delta x : \text{ car length.} \end{cases}$$



Université Nice SOPHIA ANTIPOLIS

Limit $h \to 0$.

$$\begin{cases} \partial_t \tau_h - \partial_x v_h \sim 0 & (\equiv 0 \quad \text{in } \mathbb{R}\times]t_n, t_{n+1}[\,) \\ \partial_t w_h = 0 \\ \partial_t a_h = 0 \end{cases}$$

$$(v_h w_h, a_h) \stackrel{*}{\rightharpoonup} (v^*, w^*, a^*) L^\infty(\mathbb{R} \times \mathbb{R}_+)$$

$$f(v_h, w_h, a_h)(x,t) \stackrel{*}{\rightharpoonup} f^* \neq f(v^*, w^*, a^*) \text{ dans } L^\infty$$

Young Measure: $\exists (\nu_{x,t})_{x,t}$ probab. measures : $\forall f$ continuous,

$$f(v_h, w_h, a_h)(x,t) \stackrel{*}{\rightharpoonup} f^* := < \nu_{x,t}(v,w,a), f(v,w,a) >$$

Initial data : $(v_h^0, w_h^0, a_h^0, \tau_h^0)$: piecewise constant, BV for v, oscillating for (w, a, τ), with $w_h^0 = v_h^0 + a_h^0 P(\tau_h^0)$. Moreover, assume that $(\nu_{x,0})_x$ is associated to the whole sequence (v_h^0, w_h^0, a_h^0).

Convergence strong for (v_h). Therefore $\nu_{x,t}$ = tensor product

$$\nu_{x,t} = \delta(v - v^*(x,t)) \otimes \mu_x(w,a),$$

μ_x well defined by the whole sequence (w_h^0, a_h^0).

Université Nice Sophia Antipolis

Limits :

$$w = v + aP(\tau) \qquad \tau_h = \mathcal{T}(v_h, w_h, a_h) := P^{-1}((w_h - v_h)/a_h)$$

$$\tau_h \overset{*}{\rightharpoonup} \tau^*(x,t) =< \nu_{x,t}(v,w,a), P^{-1}((w-v)/a) >=$$

$$=< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) >$$

Theorem 1.

$U_h := (v_h, w_h, a_h)$; $\nu_{x,0} := \delta(v - v_*^0(x)) \otimes \mu_x(w,a)$, associated to the whole sequence U_h^0. Then, when Δt and $\Delta x \to 0$, with $\Delta t/\Delta x = C$ and CFL, the whole sequence CV in L^∞ weak-* to the unique weak "entropy" solution of homogenized system :

$$\begin{cases} \partial_t \tau^* - \partial_x v^* = 0 \\ \partial_t w^* = \partial_t a^* = 0 \end{cases}$$

with:

$$\tau^*(x,t) =< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) > .$$

Initial data :

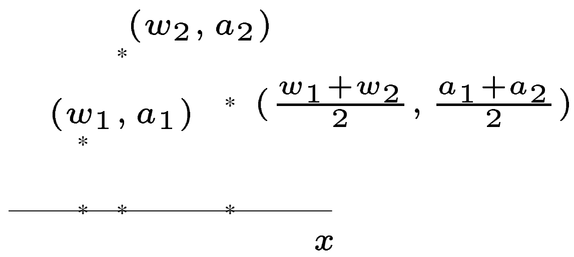
$$\begin{cases} \tau_*^0 =< \mu_x(w,a), P^{-1}((w - v_0^*)/a) >, \\ (w_*^0,\ a_*^0) = < \mu_x(w,a), (w,\ a) > . \end{cases}$$

"Entropy" : $\forall k$,

$$\partial_t < \mu_x(w,a), |\mathcal{T}(v^*(x,t), w, a,) - \mathcal{T}(k, w, a,)| > -\partial_x |v^*(x,t) - k| \le 0$$

In fact, $| < \mu_x(w,a), \mathcal{T}(...) > - < \mu_x(w,a), \mathcal{T}(...) > |$, since $\mathcal{T}(., w, a)$ **increasing.**

Example : $\mu_x(w,a) = \sum_i \alpha_i(x)\delta_{w_i,a_i}$:
$\tau^*(x,t) =< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) :=$
$\sum_i \alpha_i(x)P^{-1}((w_i - v^*(x,t))/a_i)$.



Once v^* is known (by Thm 1), we know average τ^*, and
each τ_i^* corresponding to same macroscopic speed v^* (= BV function) : τ^* depends on v^*.

Sketch of proof...)

In the limit (subsequence...) :

$$\partial_t \tau^* - \partial_x v^* = 0$$

$$\tau^*(x,t) == < \mu_x(w,a), \mathcal{T}(w,a,v^*(x,t)) > := \mathcal{T}^*(x, v^*(x,t))$$

Scalar Equation , coupled with $w_t = a_t = 0$.

L^∞ estimates, $\tau \geq 1$, $v \geq 0$... : Invariant regions.

BV estimates in x and t for v , in t for τ.
$\rightarrow$ $L^1(dx)$ equicontinuity in t. Trace at $t = 0$...

Entropy inequalities hold for all pair $q = q_k(v)$ concave /to v $\leftrightarrow \eta_k$ (convex / to τ), with

$$\eta_k = |\mathcal{T}(v,w,a) - \mathcal{T}(k,w,a)| = |\tau - \mathcal{T}(k,w,a)|.$$

Université Nice SOPHIA ANTIPOLIS

Weak-* limit : measure-valued solution (DiPerna, Szepessy...)

$$\partial_t < \mu_x(w,a), |\mathcal{T}(v^*(x,t),w,a,)-\mathcal{T}(k,w,a,)| > -\partial_x|v^*(x,t)-k| \le 0$$

In fact, $| < \mu_x(w,a), \mathcal{T}(...) > \ - \ < \mu_x(w,a), \mathcal{T}(...) > |$,
since $\mathcal{T}(.,w,a)$ **increasing.**

Knowing $\mu_x(w,a)$, v^* is the entropy solution of:

$$\partial_t \mathcal{T}^*(x,v^*) - \partial_x v^* = 0 \ , \ \tau^* := \mathcal{T}^*(x,v^*) \ , \ \text{i.e.}$$
$$\partial_t |\mathcal{T}^*(x,v^*) - \mathcal{T}^*(x,k)| - \partial_x |v^* - k| \le 0, \quad \forall k \, .$$

Here, inversion of the roles of x and t (of v and τ)
$\to$ entropy condition in conservative form.

Uniqueness à la Krushkov (weak regulariy / to x, but ∂_t) !!

Doubling of variables: first $y \to x$, then $s \to t$.

... L^1 contraction $\to$ uniqueness of the solution....

Cf ...

- Scalar non autonomous equations : cf e.g. Baiti-Jensen, Klingenberg- Risebro, Towers, Seguin-Vovelle ...
- Systems: cf Bressan *et al*, Bianchini ....
- Uniqueness of $mv-$solutions: Arguments of DiPerna, Szepessy don't work directly.
- Homogenization of Hamilton-Jacobi eq: cf Lions-Papanicolaou-Varadhan, Namah-Roquejoffre, Evans *et al*, Fathi .... Periodic case, smooth in x...
- Corrector terms (periodic case) : E-Serre ...

Link with Multiclass microscopic model.

Continuous model and semi-discrétisation :

$$\begin{cases} \partial_t \tau - \partial_x v = 0 \\ \partial_t w = 0 \\ \partial_t a = 0 \end{cases} \qquad (1) \qquad\qquad \begin{cases} \dot{\tau}_j(t) = \frac{v_{j+1} - v_j}{\Delta x} \\ \dot{w}_j(t) = 0 \\ \dot{a}_j(t) = 0 \end{cases} \qquad (2)$$

$$\tau_j = (x_{j+1} - x_j)/\Delta x \qquad \dot{\tau}_j = (v_{j+1} - v_j)/\Delta x$$

$$w \overset{e.g.}{=} v + aP(\tau) \ , \ v_j = v_{j-1/2} \ , \ \Delta x : \text{vehicle length.}$$

Multiclass "Follow-the-Leader" model : $\equiv$ (2)

$$\begin{cases} \dot{x}_j & = v_j, \\ \dot{v}_j & = -a_j \, P'(\tau_j) \, \frac{v_{j+1} - v_j}{\Delta x} \ = \ - \, a_j P'(\tau_j) \, \dot{\tau}_j \quad (2) \\ \dot{a}_j & = 0. \end{cases}$$

(Explicit) Discretization of (2) $\equiv$ Godunov scheme for (1):

$$\begin{cases} \tau_j^{n+1} : & = \tau_j^n + \frac{\Delta t}{\Delta x}(v_{j+1}^n - v_j^n), \\ w_j^{n+1} & = w_j^n, \qquad\qquad (3) \\ a_j^{n+1} & = a_j^n. \end{cases}$$

Université Nice SOPHIA ANTIPOLIS

Homogenized Multiclass Model (1_{hom}) :

Theorem 1.

$U_h := (v_h, w_h, a_h)$; $\nu_{x,0} := \delta(v - v_*^0(x)) \otimes \mu_x(w,a)$, associated to the whole sequence U_h^0. Then, when Δt and $\Delta x \to 0$, with $\Delta t / \Delta x = C$ and CFL, the whole sequence CV in L^∞ weak-* to the unique weak "entropy" solution of homogenized system :

$$\begin{cases} \partial_t \tau^* - \partial_x v^* = 0 \\ \partial_t w^* = 0 \\ \partial_t a^* = 0 \end{cases}$$

with:

$$\begin{cases} \tau^*(x,t) =< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) > \\ \overset{\text{e.g.}}{=} \sum_i \alpha_i(x) P^{-1}((w_i - v^*(x,t))/a_i). \end{cases}$$

Initial data :

$$\begin{cases} \tau_*^0 =< \mu_x(w,a), P^{-1}((w - v_0^*)/a) >, \\ (w_*^0,\ a_*^0) = < \mu_x(w,a), (w,\ a) > . \end{cases}$$

Theorem 2

(i) When $\Delta t \to 0$ with fixed Δx, (3) $\to$ (2) : the whole sequence

$(v_h, \tau_h, w_h, a_h) \overset{\text{weak-*}}{\rightharpoonup} (v_\Delta, \tau_\Delta, w_\Delta, a_\Delta)$

$\equiv (v_j, \tau_j, w_j, a_j)(t)$ for $x \in (x_j, x_{j+1})$ $\quad \equiv$ solution of (2).

(ii) Same estimates : BV in x and t for v_Δ, in t for τ_Δ.

(iii) Entropy inequalities $\to$: $\forall$ concave $q(v)$,

$\frac{d}{dt}\eta(v_j(t), w_j, a_j) + (q(v_{j+1})(t) - q(v_j)(t))/\Delta x \le 0.$

Theorem 3

(i) When $\Delta x \to 0$ (2) $\rightharpoonup$ (1_{hom}) : the whole sequence

$(v_\Delta, \tau_\Delta, w_\Delta, a_\Delta) \overset{\text{weak-*}}{\rightharpoonup}$ to the unique entropy solution of homogenized model (1_{hom}).

Sketch of Proof.

Strong CV for $v_\Delta, q(v_\Delta)$, weak-* for $(\tau_\Delta, w_\Delta, a_\Delta)$ $\to$ (another (?)) mv-solution.

Same μ_x, entropy inequalities : uniqueness of v^* $\to$ same limit.

Result = Commutation of limits.

Homogenization of τ as a function of v, with oscillating w, a.