Coupling conditions for a class of ”second–order” models for traffic flow

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Abstract

This paper deals with a model for traffic flow based on a system of conservation laws [2]. We construct a solution of the Riemann Problem at an arbitrary junction of a road network. Our construction provides a solution of the full system. In particular, all moments are conserved.

AMS subject classifications: 35Lxx, 35L6

1 Introduction

Macroscopic modelling of vehicular traffic started with the work of Lighthill and Whitham (LWR) [25]. Since then there has been intense discussion and research, see [26, 8, 2, 19, 20, 21, 6, 24] and the references therein. Today, fluid dynamic models for traffic flow are appropriate to describe traffic phenomena as for example congestion and stop-and-go waves [18, 14, 22]. The case of road networks based on the LWR model has been considered in particular in [17, 5, 16]. In a recent preprint [12] Garavello and Piccoli consider a road network based on the Aw–Rascle (AR) model [2] of traffic flow. We thank them for the preprint. Here, in contrast to [12], we

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propose a modelling of the junctions conserving the mass **and** the pseudo-"momentum" $\rho v\, w$. We will discuss below further differences between the two modelings.

We consider a finite directed graph as a model for a road network with unidirectional flow. Each road $i = 1, \ldots, \mathcal{I}$ is modelled by an interval $I_i := [a_i, b_i] \subset \mathbb{R}$ possibly with $a_i = -\infty$ or $b_i = \infty$. Each vertex of the graph corresponds to a junction. For a fixed junction k the set δ_k^- contains all road indices i which are incoming roads, so that $\forall i \in \delta_k^- : b_i = k$. Similarly, δ_k^+ denotes the indices of outgoing roads: $\forall j \in \delta_k^+ : a_j = k$. We skip the index k whenever the situation is clear.

The evolution of $\rho_i(x,t)$ and $v_i(x,t)$ on each road i is given by the AR model [2]

$$\partial_t \rho_i + \partial_x(\rho_i v_i) = 0, \tag{1.1a}$$

$$\partial_t(\rho_i w_i) + \partial_x(\rho_i v_i w_i) = 0, \tag{1.1b}$$

$$w_i = v_i + p_i(\rho_i), \tag{1.1c}$$

where for each i $\rho \mapsto p_i(\rho)$ is a known function ("traffic pressure") with the following properties

$$\forall \rho : \rho p_i''(\rho)) + 2p_i'(\rho) > 0 \text{ and } p_i(\rho) \sim \rho^\gamma \text{ near } \rho = 0 \tag{1.2}$$

and where $\gamma > 0$. The conservative form of (1.1) is

$$\partial_t \begin{pmatrix} \rho_i \\ y_i \end{pmatrix} + \partial_x \begin{pmatrix} y_i - \rho_i p_i(\rho_i) \\ (y_i - \rho_i p_i(\rho_i)) y_i / \rho_i \end{pmatrix} = 0,$$

where $y_i = \rho_i w_i = \rho(v_i + p_i(\rho_i))$. Since w_i and v_i are related by (1.1), we choose to describe solutions in terms of ρ_i and $\rho_i v_i$. For a motivation and a complete discussion of these equations we refer to Section 2 and reference [2], respectively.

We consider weak solutions of the network problem as in [17]: Given a set $i = 1, \ldots, \mathcal{I}$ of smooth functions $\phi_i : [0, +\infty] \times I_i \to \mathbb{R}^2$ having compact support in $I_i = [a_i, b_i]$, which are "smooth" across each junction k, i.e.,

$$\phi_i(b_i) = \phi_j(a_j) \; \forall i \in \delta_k^-, \forall j \in \delta_k^+. \tag{1.3}$$

Then a set of functions

$$U_i = (\rho_i, \rho_i v_i), \quad i = 1, \ldots, \mathcal{I} \tag{1.4}$$

is called a weak solution of (1.1) if and only if equations (1.5) hold for all families of test functions $\{\phi_i\}_{i\in\mathcal{I}}$ with the property (1.3).

$$\sum_{i=1}^{\mathcal{I}} \int_0^{\infty} \int_{a_i}^{b_i} \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} \cdot \partial_t \phi_i + \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} \cdot \partial_x \phi_i dx dt$$

$$- \int_{a_i}^{b_i} \begin{pmatrix} \rho_{i,0} \\ \rho_{i,0} w_{i,0} \end{pmatrix} \cdot \phi_i(x,0) dx = 0, \tag{1.5a}$$

$$w_i(x,t) = v_i(x,t) + p_i^{\dagger}(\rho_i(x,t)). \tag{1.5b}$$

Here, $U_{i,0}(x) = \big(\rho_{i,0}(x), (\rho_{i,0} v_{i,0})(x)\big)$ are the initial data. **The functions $p_i^{\dagger}(\cdot)$ are initially unknown**. The explicit form of each $p_i^{\dagger}$ depends on the initial data and the type of junction. Near any junction k the function $p_i^{\dagger}$ is equal to p_i for all incoming roads. The same is true on all outgoing roads of the junction if there is only *one* incoming road. This is discussed in Sections 3 and 4. In Section 6 we discuss the case where $p_i^{\dagger} \neq p_i$ and give arguments for the necessity of introducing $p_i^{\dagger}$. At this point let us just note that in the general case $p_i^{\dagger}$ depends on a mixture of the incoming flows.

In the case of a single junction we derive from (1.5a), (1.5b) the Rankine-Hugoniot conditions for piecewise smooth solutions

$$\sum_{i\in\delta^-} (\rho_i v_i)(b_i^-, t) = \sum_{i\in\delta^+} (\rho_i v_i)(a_i^+, t), \tag{1.6a}$$

$$\sum_{i\in\delta^-} (\rho_i v_i w_i)(b_i^-, t) = \sum_{i\in\delta^+} (\rho_i v_i w_i)(a_i^+, t). \tag{1.6b}$$

Properties (1.6a) and (1.6b) correspond to conservation of mass and of (pseudo)-"momentum". We remark that the solution constructed in [12] does **not** conserve the (pseudo-) "momentum", see Proposition 2.3 in [12] and therefore is **not** a weak solution in the sense of (1.5a), (1.6a) and (1.6b).

In the next sections we discuss the construction of weak solutions in the sense of (1.5) for initial data constant on each road:

$$(\rho_{i,0}, \rho_{i,0} v_{i,0}) = U_{i,0} = \mathrm{const}_i. \tag{1.7}$$

We consider a single junction. We look for solutions to Riemann problems on each road i as if the road were extended to $]-\infty, \infty[$:

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} + \partial_x \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} = 0, \; U_i(x,0) = \begin{pmatrix} U^- & x < x_0 \\ U^+ & x > x_0 \end{pmatrix}. \tag{1.8}$$

Depending on the road, only one of the Riemann data is defined for $t = 0$:

$$\text{If } i \in \delta^- : U^- = U_{i,0}\,,\ x_0 = b_i \ \text{ and if } i \in \delta^+ : U^+ = U_{i,0}\,,\ x_0 = a_i. \quad (1.9)$$

We construct an (entropy) solution to (1.5) such that all generated waves have non-positive ($i \in \delta^-$) or non-negative ($i \in \delta^+$) speed. Moreover, the solutions satisfy conditions (1.6a) and (1.6b).

We have to impose additional conditions [12] to obtain a unique solution. First, the flux ρv is nonnegative. Next, it has to be distributed according to a priori given ratios, see Section 3 to 7 for further details. Finally, we require that the **total flux be maximized** subject to the other conditions.

The paper is organized as follows. In Section 2 we discuss the general properties of the Riemann problem for the equation (1.1). First, we construct the demand and supply functions, which are necessary to determine the flux at the junction. Refer to [23, 9, 10] for the presentation of supply and demand functions for first-order models. Next, we define admissible states on each road at the junction and finally we construct all intermediate states in the solution of (1.1).

In Section 3 we consider the easiest possible situation, namely, two roads connected by a junction. In Section 4 we extend the results to a junction with one incoming and two outgoing roads. For the results on two incoming and one outgoing road we need a description of the mixture of flows on the outgoing road. Therefore we briefly revisit the main results of [1] and [3] in Section 5. In Section 6 we solve the case of two incoming and two outgoing roads and define homogenized flow. In Section 7 we consider the general case of an intersection with an arbitrary number of incoming and outgoing roads.

2 Preliminary discussion

The conservative variables are ρ_i and $y_i := \rho_i w_i$. We assume $\forall i : 0 \le \rho_i \le \rho_{\max} = 1$ and $\forall i : 0 \le v_i \le v_{\max} = 1$. Furthermore, we set

$$U_i := (\rho_i, \rho_i v_i),\ U := (\rho, \rho v) \quad (2.1)$$

and we skip the subindex i at ρ_i and v_i whenever the intention is clear. The system (1.1) is strictly hyperbolic if $\rho_i > 0$ for all i. The eigenvalues are

$$\lambda_{1,i}(U) = v - \rho p_i'(\rho) \text{ and } \lambda_{2,i}(U) = v. \quad (2.2)$$

The right eigenvectors corresponding to $\lambda_{1,i}$ and $\lambda_{2,i}$ are

$$\mathbf{r}_{1,i} = \begin{pmatrix} 1 \\ -p_i'(\rho) \end{pmatrix} \text{ and } \mathbf{r}_{2,i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let ∇ denote the gradient with respect to (ρ, v). We recall that k is called a genuinely nonlinear characteristic family if $\nabla\lambda_{k,i}(\rho, v)\cdot\mathbf{r}_{k,i}(\rho, v) \neq 0, \forall(\rho, v)$. Depending on the initial data, the associated waves are rarefaction or shock waves. If $\nabla\lambda_{k,i}(\rho, v)\cdot\mathbf{r}_{k,i}(\rho, v) = 0, \forall(\rho, v)$, then k is called a linearly degenerated characteristic family and the associated waves are contact discontinuities. We refer to Definition 7.2.1 and 7.5.1 in [7] for more details.

Here, $k = 1$ is a genuinely nonlinear and $k = 2$ is linearly degenerated characteristic family for all roads i. Moreover, the 1–shock and 1–rarefaction curves coincide and we have a 2-contact discontinuity, see [2]. For each road i the Riemann invariants are

$$w_i(U) = v + p_i(\rho) \text{ and } v_i(U) = v. \tag{2.3}$$

Let us be more specific on the physical interpretation of w and $p(\cdot)$. Other descriptions than (2.3) could be envisioned. In particular, the *additive* role of $p_i(\cdot)$ in w_i (like in the Payne-Whitham model, [26]) is not essential. It was introduced in [2] for "historical" reasons, but it has a draw-back: The associated individual fundamental diagram, see Figure 1 below, implies a zero speed at a maximal (jam) traffic density which is *different* for each category of car-driver pair, i.e., each pairing (w_i, p_i). We keep the above expression (2.3) throughout the paper for sake of simplicity. As noted in [3], the only crucial property of w_i is that it is a *Lagrangian* marker. As an example assume that on each road i, the (pseudo)- pressure is $p_i(\rho) := v_{max} - V_i(\rho)$, where e.g. v_{max} is the maximal speed on all roads and $V_i(\rho)$ is an *equilibrium* speed on road i. Therefore, the *function* $U := (\rho, v) \to w_i(U) = v + p_i(\rho)$ describes the *distance to equilibrium.* The "momentum" equation tells us that each value w is a Lagrangian property, like a label or a color. Hence, when passing from road i to another road j, each driver will preserve its "color". In other words, he will keep the *same value* w, which will now satisfy:

$$w_j(U) = w = w_i(U).$$

This simple observation will be essential in the sequel. In particular, it will lead to a very natural homogenization problem in Section 6.

The classical description by *first* order models is just a *particular case* of our second order model. It corresponds to setting all the w equal to the same constant. So our description can be drastically simplified when no sophisticated information is needed.

We return to the mathematical description. Usually, we draw the level curves of the Riemann invariants (in short the Riemann invariants) in the $(\rho, \rho v)$ plane. An example of the curves is depicted in Figure 1. There is a one-to-one correspondence to the (ρ, y) plane, see [2].

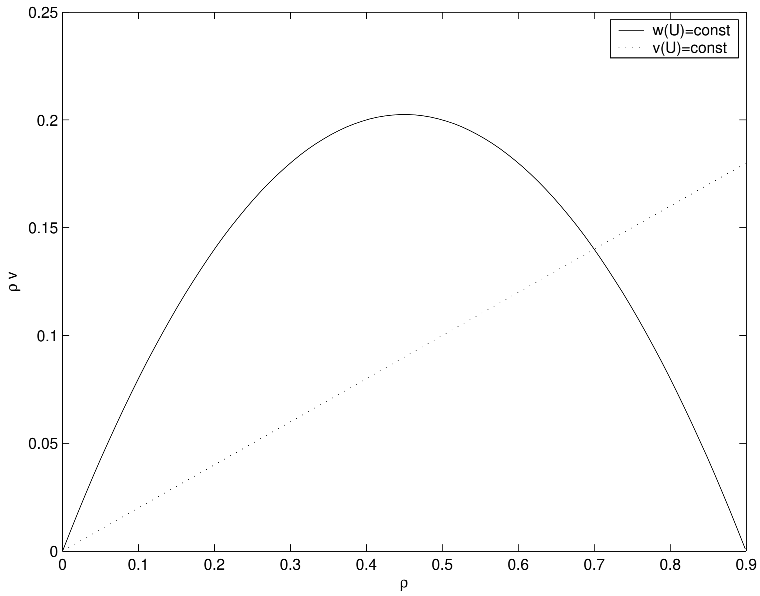

Figure 1: Riemann invariants in the $(\rho, \rho v)$ plane.

For an arbitrary fixed i we discuss the shape of the Riemann invariants in the $(\rho, \rho v)$ plane and characterize important points.

The Riemann invariant $\{v_i(U) = c\}$ is a straight line with slope c passing through the origin. Consider the curve $\{w(U) := w_i(U) = c\}$, where $c \in \mathbb{R}$ denotes a constant. By assumption (1.2) on $p := p_i$ this curve is strictly concave and passes through the origin. Furthermore, if $c > 0$, then the curve $\{w(U) = c\}$ lies in the first quadrant of the $(\rho, \rho v)$ plane for ρ between 0 and a maximal value $\bar{\rho} \in]0, 1]$. The maximal value $\bar{\rho}$ depends on c and $p(\cdot)$. Due to the strict concavity there exists a unique point (i.e. the "sonic point") $\sigma(w, c)$ with $0 < \sigma(w, c) \leq 1$, depending on c and the function $p(\cdot)$. The point $\sigma(w, c)$ maximizes the flux ρv on $\{w(U) = c\}$.

The total flux has to be conserved through an intersection. Therefore, we introduce the functions $r(\rho; w, c)$ and $u(\rho; w, c)$ below. Assume $c > 0$. Then for all $\rho \in [0, \bar{\rho}]$ there exists a unique v such that $w((\rho, \rho v)) = c$. Moreover, there exists a unique pair (r, u) such that

$$
\begin{aligned}
w(r, r\,u) &= w(\rho, \rho\,v), && (2.4a)\\
r\,u &= \rho\,v, && (2.4b)\\
r &\neq \rho \text{ except for } \rho = \sigma(w, c). && (2.4c)
\end{aligned}
$$

In other words (ρ, v) and (r, u) correspond to the same flux and the same level curve of w, see Figure 2 for an example. Hence, for each curve $\{w(U) = c\}$ with $c > 0$ there exists two unique functions $\rho \to r(\rho; w, c)$ and $\rho \to u(\rho; w, c)$ satisfying (2.4) for all $\rho \in [0, \bar{\rho}]$.

Next, we describe the construction of the demand and supply functions for a given curve $\{w(U) = c\}$, $c \geq 0$. As in the case of first-order models, e.g. [23], in the $(\rho, \rho v)$ plane the demand function $d(\rho; w, c)$ is an extension of the **non–decreasing** part of the curve $\{w(U) = c\}$ for $\rho \geq 0$, whereas the supply function $s(\rho; w, c)$ is an extension of the **non-increasing** part of the curve $\{w(U) = c\}$ and $\rho \geq 0$, see Figures 2 and 3 for examples.

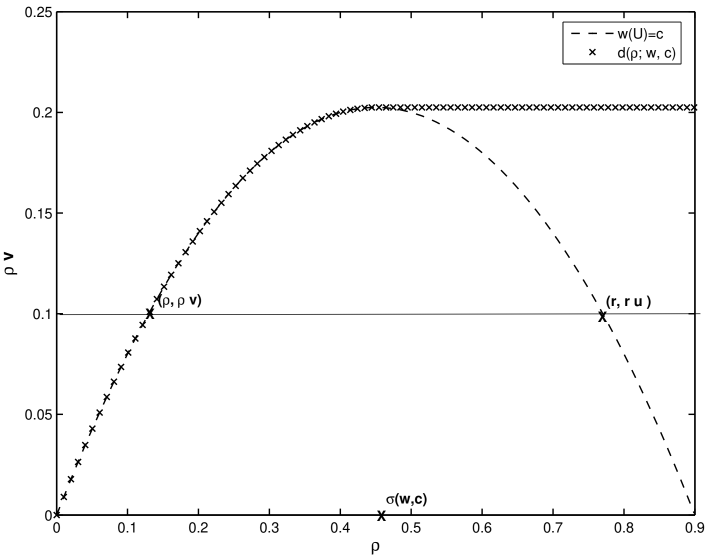

Figure 2: Demand function for given $w(U) = v + p(\rho) = const$. Additionally, $\sigma(w, c)$ and the position of a sample point $(\rho, \rho\,v)$ is and the corresponding $(r, r\,u)$ are shown.

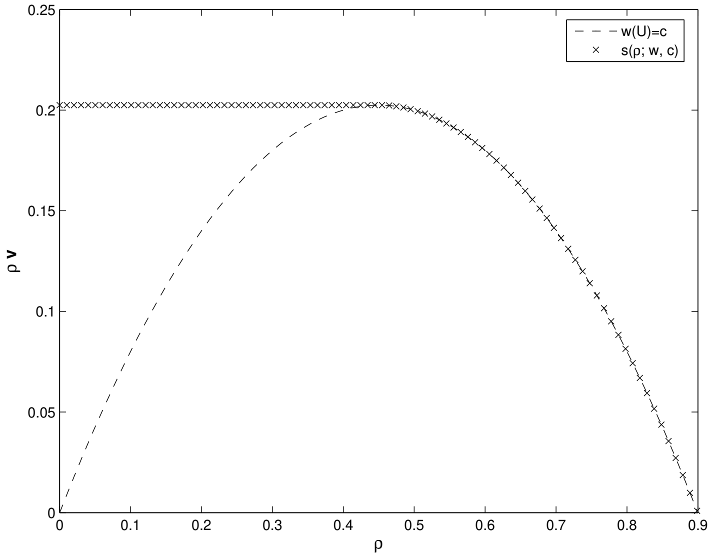

Figure 3: Supply function for a given $w(U) = v + p(\rho) = const$

Now, we consider the Riemann problem (1.8) for a given *incoming* road $i \in \delta^-$. Hence, only the initial datum $U^- = U_{i,0}$ is given. We want to determine all "admissible" states U^+ : A state U^+ is called "admissible" if and only if either the waves of the solution to (1.8) with initial data (U^-, U^+) have negative speed or the solution is constant $U^+ = U^-$. As in [17] we neglect waves of zero speed (stationary waves). Later on U^+ will be an intermediate state in the solution $U_i(\cdot, \cdot)$ on the incoming road i for the full Riemann problem at the junction, i.e., $U_i(x_0-, t) = U^+$.

Proposition 2.1. *Let $U^- = (\rho^-, \rho^- v^-) \neq (0,0)$ be the initial value on an incoming road i. Let the 1-curve through U^- be $w_i(U) = v + p_i(\rho) = w^-$ with $w^- := w_i(U^-)$. Then the "admissible" states $U^+ = (\rho^+, \rho^+ v^+)$ for the Riemann problem must belong to that curve, i.e., $w_i(U^+) = w^-$ and $\rho^+ v^+ \geq 0$. Depending on U^- we distinguish two cases:*

1. *$\rho^- < \sigma(w_i, w^-)$: U^+ is admissible if and only if $\rho^+ > r(\rho^-; w_i, w^-)$ or if $U^+ \equiv U^-$.*

2. *$\rho^- \geq \sigma(w_i, w^-)$: U^+ is admissible if and only if $\sigma(w_i, w^-) \leq \rho^+ \leq 1$.*

If $U^- = (0,0)$ then the admissible state is $U^+ \equiv U^-$.

In all cases the maximal possible flux associated with any admissible state U^+ is $d(\rho^-; w, w^-)$, with $w = w_i$.

Proof. For $U^- \neq (0,0)$ the 2–contact discontinuities are waves with speed $v^- > 0$. Hence, we only have to discuss 1–shock or 1–rarefaction waves. Following [2] a left state U^- can be connected to a right state U^+ by a 1-shock, if and only if $\rho^+ > \rho^-$. The shock speed is then given by the slope of the chord U^-U^+. A left state U^- can be connected to a right state U^+ by a 1–rarefaction wave if and only if $\rho^+ < \rho^-$. Note that in the $(\rho, \rho v)$-plane the slope of the tangent to the curve $\{w_i(U) = c\}$ at a point U is the characteristic speed $\lambda_1(U)$.

By the discussion in the previous section there exists a state U^* with $\rho^* = r(\rho^-; w_i, w^-)$ and $v^* = u(\rho^-; w_i, w^-)$, such that $w_i(U^*) = w^-$. Furthermore, the chord U^-U^* has a zero slope. Hence, we have a 1-rarefaction wave for all states U^+ with $\sigma(w_i, w^-) \leq \rho^+ \leq \rho^-$ and a 1-shock for $\rho^+ \geq \rho^-$.

In both cases the associated flux is not greater than the demand $d(\rho^-; w, w^-)$.

Finally, if $\rho^+ > 0$, then $U^- = (0,0)$ can be connected to U^+ by a 2-contact discontinuity, which has either positive speed or zero speed, c.f. Case 5 in [2]. Hence, only $U^+ \equiv (0,0)$ is admissible. □

Next, we consider the Riemann problem (1.8) for a given *outgoing* road $i \in \delta^+$, a function $w(U) := v + p_i(\rho)$ and a non–negative constant c. Later on, c will of course depend on the initial states on the incoming roads (!), see Section 3 to 7. We look for "admissible" states U^-, i.e., all the states such that the waves of the solution have a positive speed or such that the solution is a constant. Again, we exclude the case of stationary waves. As in the previous case, U^- will be an intermediate state in the solution on the outgoing road i for the full Riemann problem at the junction. Now $U_i(x_0+, t) = U^-$ will hold.

Proposition 2.2. *Consider a state $U^+ \neq (0,0)$ and the level curve of the first Riemann invariant $\{w(U) = c\}$ with an arbitrary non–negative constant c.*

Let $U^\dagger = (\rho^\dagger, \rho^\dagger v^\dagger)$be the point of intersection, if it exists, of the two Riemann invariants $\{v(U) = v^+\}$ and $\{w(U) = c\}$ with $\rho > 0$ and $v > 0$.

Then the "admissible" states U^- for the Riemann problem satisfying $w(U^-) = c$ and $\rho^- v^- \geq 0$ are given by the two cases:

1. $\rho^\dagger \le \sigma(w,c)$: U^- *is admissible if and only if* $0 \le \rho^- \le \sigma(w,c)$.

2. $\rho^\dagger > \sigma(w,c)$: U^- *is admissible if and only if* $0 \le \rho^- < r(\rho^\dagger; w, c)$ *or if* $U^- \equiv U^\dagger$.

Note that the set of admissible states U^- *depends on the existence of the point* $U^\dagger$. *Now assume that either* $U^+ = (0,0)$ *or there is no such point* $U^\dagger$ *with* $\rho^\dagger, v^\dagger > 0$. *Then we set* $U^\dagger = (0,0)$ *and as in Case 1,* U^- *is admissible, if and only if* $0 \le \rho^- \le \sigma(w,c)$.

In all cases the maximal possible flux associated with any "admissible" state U^- *is* $s(\rho^\dagger; w, c)$.

Proof. Due to the range of the eigenvalues we can connect a left state U^- to an intermediate state $U^\dagger$ by a 1-shock or a 1-rarefaction wave of positive speed. Then $U^\dagger$ can be connected to U^+ by a 2-contact discontinuity.

If $U^\dagger$ exists, it is well-defined, since the curves $\{w(U) = c\}$ and $\{v(U) = v^+\}$ have a unique intersection point such that $\rho > 0, \rho v > 0$. If there is no point $U^\dagger$ with $\rho, v > 0$, then the curves have an unique intersection point at $(0,0)$.

Using the same kind of arguments as in Proposition 2.1, we see that either the 1–shock or 1–rarefaction waves connecting U^- and $U^\dagger$ have a positive speed or the solution is constant.

Next, if $\rho^+ = 0$ we set $U^\dagger = U^+$ and can connect to U^- by waves of the first family only, c.f. Case 4 in [2]. ☐

Combining these two results, we obtain

Proposition 2.3. *Consider an incoming (resp. outgoing) road* i*, an initial datum* $U^- := U_{i,0}$ *(resp.* $U^\dagger := U_{i,0}$*) and an arbitrary flux* $q_0 \ge 0$. *Let* $w(U) := v + p_i(\rho)$ *and* $c := w(U_{i,0})$. *Assume*

$$q_0 \le d(\rho_{i,0}; w, c)\ ,\ (\mathrm{resp.} q_0 \le\ \mathrm{s}(\rho_{\mathrm{i},0}; \mathrm{w}, \mathrm{c})).$$

By Propositions 2.1 and 2.2, there exists a unique state U^+ *(resp.* U^-*), such that the corresponding Riemann problem (1.8) admits a solution such that* $w(U^+) = c$ *and* $\rho^+ v^+ = q_0$ *(resp.* $w(U^-) = c$ *and* $\rho^- v^- = q_0$*)* and *either all the waves have negative (resp. positive) speed, or the solution is a constant on the corresponding road.*

The reader is advised to pay attention to the notation. In the *full* solution to the Riemann problem at a junction we will have

$$\text{for } i \in \delta^- :\ U_i^+ = U_i(x_0-, t) \text{ and for } i \in \delta^+ :\ U_i^- = U_i(x_0+, t). \qquad (2.5)$$

Unfortunately, it seems hard to avoid this possibly misleading notation.

To summarize, Proposition 2.3 describes the set of "admissible" states for the Riemann data on incoming and outgoing roads. These states, including $U^\dagger$, defined as in Proposition 2.2, will be intermediate states in the solution of the full problem, satisfying (1.6a) **and** (1.6b). We now turn to the study of the first case.

3 One incoming and one outgoing road

The simplest possible network contains two roads connected by a junction, i.e., one road with two different road conditions.

Proposition 3.1. *Consider two roads $i = 1, 2$ with $a_1 = -\infty, b_1 = a_2$ and $b_2 = \infty$ and initial data $U_{i,0} = (\rho_{i,0}, \rho_{i,0}v_{i,0}), i = 1, 2$ constant.*

Then there exists a unique solution $U_i(x, t)$ of the Riemann problem at the junction (1.8) and (1.9) with the properties (1) and (2). We refer to equation (3.2) and to the end of the proof for a description of the structure of this solution.

1. *$U_i(x, t)$ is a weak solution of the network problem (1.5a-1.5b), where $p_i^\dagger \equiv p_i, i = 1, 2$ given in (1.1). Furthermore (1.6a-1.6b) are satisfied, and $\rho_i(x, t)v_i(x, t) \geq 0$, $i =$ 1, 2.*
2. *The flux $(\rho_1 v_1)(b_1^-, t)$ is maximal at the interface, subject to the above conditions*

Proof. Let $U_1^- := U_{1,0}, U_2^+ := U_{2,0}$ and $w_i(U) = v + p_i(\rho)$ for $i = 1, 2$. As described in Section 2 we construct the *demand* function for the incoming road

$$d(\rho) := d(\rho; w_1, w_1(U_1^-)),$$

and the *supply* function for the outgoing road

$$s(\rho) := s(\rho; w_2, w_1(U_1^-)). \tag{3.1}$$

Note that the supply function is an extension of the non-increasing part of the curve $\{w_2(U) = w_1(U_1^-)\}$. The expression (3.1) of the supply function $s(\cdot)$ involves the **function** w_2 and the **value** $w_1(U_1^-)$, since the cars which are initially on road 1 and which have moved on road 2, have kept their Lagrangian "color" $w_1(U_1^-)$.

By Proposition 2.2 we obtain $U_2^\dagger$ either as the intersection of the curves $\{v_2(U) = v_2^+\}$ and $\{w_2(U) = w_1(U_1^-)\}$ or by $U_2^\dagger = (0,0)$. Then we solve the maximization problem

$$\begin{aligned} &\max q_1 \text{ subject to} \\ &0 \leq q_1 \leq d(\rho_1^-), \\ &0 \leq q_1 \leq s(\rho_2^\dagger). \end{aligned}$$

Denote by $\tilde{q}$ the point where the maximum is attained. Of course the above is equivalent to $\tilde{q} = \min\{d(\rho_1^-), s(\rho_2^\dagger)\}$, but we will need the general form later.

Now, as in Proposition 2.3 there exists U_1^+ and U_2^-, such that $\rho_1^+ v_1^+ = \rho_2^- v_2^- = \tilde{q}$.

Knowing the states U_1^+ and U_2^-, we solve the two Riemann problems:

$$i = 1,2: \qquad \partial_t \begin{bmatrix} \rho_i \\ v_i \end{bmatrix} + \partial_x \begin{bmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{bmatrix} = 0, \tag{3.2a}$$

$$i = 1: \quad U_1(x,0) = \begin{bmatrix} U_1^- \equiv U_{1,0} & x < b_1 \\ U_1^+ & x \geq b_1 \end{bmatrix}, \tag{3.2b}$$

$$i = 2: \quad U_2(x,0) = \begin{bmatrix} U_2^- & x \leq a_2 \\ U_2^+ \equiv U_{2,0} & x > a_2 \end{bmatrix}, \tag{3.2c}$$

to obtain weak entropy solutions $U_1(x,t)$ and $U_2(x,t)$. Each solution consists of at most two waves: a 1–rarefaction or a 1-shock wave associated with the first eigenvalue, followed by a 2–contact discontinuity associated with the second eigenvalue.

The conditions (1.6a-1.6b) are satisfied since

$$\tilde{q} = \rho_1^+ v_1^+ = \rho_2^- v_2^-,$$

and

$$w_1(U_1^+) = w_1(U_1^-) = w_2(U_2^-) = w_2(U_2^\dagger).$$

□

An example of a solution in the (x,t) plane is depicted in Figure 4.

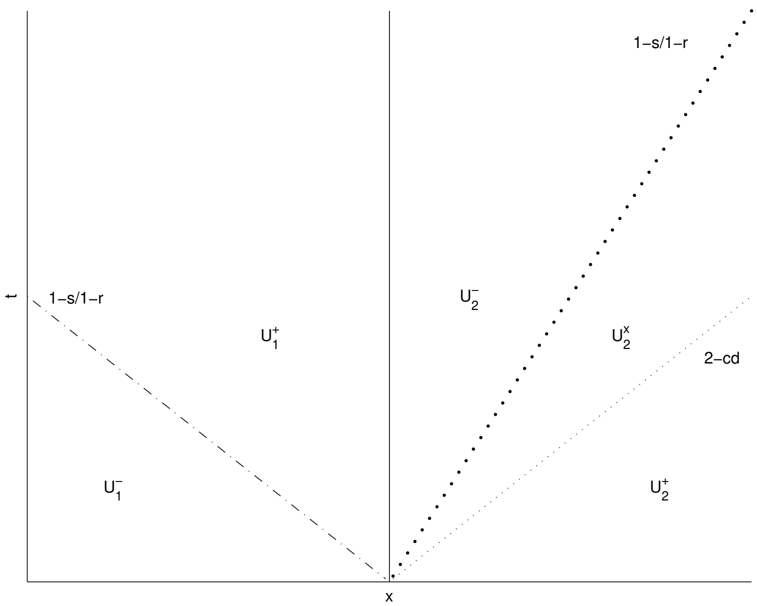

Figure 4: Possible solution to the Riemann problems of road 1 (left part) and road 2 (right part). $1-s/1-r$ stands for 1-shock or 1-rarefaction wave connecting the left and right states. Similarly, 2-cd denotes the 2-contact discontinuity.

4 One incoming and two outgoing roads

We now consider the case of one incoming and two outgoing roads. We cannot expect to obtain a unique solution without imposing additional assumptions on the distribution of the flux among the outgoing roads. One could impose an optimization criterion, such as maximizing the total flux at the interface [17, 5].

Here, we impose the proportions (α and $(1-\alpha)$) of cars which go from road 1 to roads 2 and 3. This condition was introduced first in [5] for the first order LWR model and in [12] for the AR model. In the case of first order models, the car distribution at junctions has also been studied in [23, 9] and many others.

Proposition 4.1. *Consider three roads $i = 1, 2, 3$ with $a_1 = -\infty, b_1 = a_2 = a_3$ and $b_2 = b_3 = \infty$ and constant initial data $U_{i,0} = (\rho_{i,0}, \rho_{i,0} v_{i,0}), i = 1, 2, 3$. Let $0 \leq \alpha \leq 1$ be given.*

Then there exists a unique solution $U_i(x,t), i = 1,2,3$ of the Riemann problem at the junction (1.8) and (1.9) with the following properties (1) and (2). A description of its structure can be found at the end of the proof.

1. *$U_i(x,t)$ is a weak solution of the network problem (1.5a-1.5b) wherein $p_i^\dagger \equiv p_i$ for all $i = 1,2,3$.*

   Furthermore (1.6a-1.6b) are satisfied, and $\rho_i(x,t)v_i(x,t) \geq 0,\ ,\ i = 1,2,3$.

2. *For all $t > 0$ the flux is distributed in proportions α and $1-\alpha$ between roads 2 and 3:*

$$\alpha(\rho_1 v_1)(b_1^-,t) = (\rho_2 v_2)(a_2^+,t), \tag{4.1a}$$
$$(1-\alpha)(\rho_1 v_1)(b_1^-,t) = (\rho_3 v_3)(a_3^+,t), \tag{4.1b}$$

3. *The flux $(\rho_1 v_1)(b_1^-,t)$ is maximal at the interface, subject to the above conditions.*

Proof. Let $U_1^- = U_{1,0}$, $U_i^+ = U_{i,0}, i = 2,3$ and for $i = 1,2,3$ let $w_i(U) := v + p_i(\rho)$. As in Section 2 we construct the demand function

$$d(\rho) := d(\rho; w_1, w_1(U_1^-))$$

and the two supply functions

$$s_2(\rho) := s(\rho; w_2, w_1(U_1^-)),\ s_3(\rho) := s(\rho; w_3, w_1(U_1^-))$$

For $i = 2,3$ we obtain the points $U_i^\dagger$ as intersection of $\{v(U) = v_i^+\}$ with $\{w_i(U) = w_1(U_1^-)\}$ or as $U_i^\dagger = (0,0)$, c.f. Proposition 2.2. We solve the maximization problem

$$\max q_1 \text{ subject to} \tag{4.2a}$$
$$0 \leq q_1 \leq d(\rho_1^-), \tag{4.2b}$$
$$0 \leq \alpha q_1 \leq s_2(\rho_2^\dagger), \tag{4.2c}$$
$$0 \leq (1-\alpha)q_1 \leq s_3(\rho_3^\dagger).. \tag{4.2d}$$

Denote by $\tilde{q}$ the point where the maximum is attained. Of course the above is equivalent to $\tilde{q} = \min\{d(\rho_1^-), s_2(\rho_2^\dagger)/\alpha, s_3(\rho_3^\dagger)/(1-\alpha)\}$. By Proposition 2.3

we conclude

$$\exists U_1^+ \text{ such that } \rho_1^+ v_1^+ = \tilde{q},\ w_1(U_1^+) = w_1(U_1^-), \tag{4.3a}$$
$$\exists U_2^- \text{ such that } \rho_2^- v_2^- = \alpha\tilde{q},\ w_2(U_2^-) = w_1(U_1^-), \tag{4.3b}$$
$$\text{and } \exists U_3^- \text{such that } \rho_3^- v_3^- = (1-\alpha)\tilde{q},\ w_3(U_3^-) = w_1(U_1^-). \tag{4.3c}$$

Clearly, the conditions (1.6a-1.6b) and (4.1a-4.1b) are satisfied by (4.3). Again, each solution $U_i(x,t)$ consist of a juxtaposition of rarefaction or shock waves associated with the first eigenvalue and a contact discontinuity associated with the second eigenvalue of (1.1). The construction is similar to equation (3.2) in Proposition (3.1). In the limit cases $\alpha = 0$ or $\alpha = 1$ we are exactly in the setting of Proposition 3.1. □

Before studying the more surprising case of two incoming and one outgoing roads in Section 6, we must recall a few basic facts on the Lagrangian version of the model and the corresponding homogenized system.

The reader is advised to take a look at the first part of Section 5 and then to move to Section 6. The second part of Section 5 deals with details on the homogenization and can be read after Section 6.

5 The Lagrangian model and its homogenized version

The Lagrangian formulation is introduced in [1]. A formal derivation is given in [28] and a mathematical study in [13]. The homogenization of this system is studied in [3]. Proofs of statements below can be found in the above references.

Consider a single road with $p_i := p$. Then it turns out that the weak entropy solutions of

$$\rho_t + (\rho v)_x = 0, (\rho w)_t + (\rho v w)_x = 0, w = v + p(\rho),$$

correspond to the weak entropy solutions of the equivalent system in (mass) Lagrangian coordinates (X,t):

$$\tau_t - v_X = 0,\ w_t = 0,\ w = v + P(\tau), \tag{5.1}$$

with $\tau := 1/\rho, P(\tau) := p(\rho)$. Here, X is the Lagrangian (mass) coordinate, defined by $\partial_x X = \rho$ and $\partial_t X = -\rho v$. The existence of $X(\cdot,\cdot)$ follows from the mass conservation equation. For some unspecified t_0, $X(x, t_0) := \int_0^x \rho(y, t_0)\, dy$, where we implicitly defined ρ as the dimensionless density, i.e., the *fraction of space* occupied by the cars, see [1]. Therefore X is the position of each car if all cars were parked "nose to tail".

As in [1] consider two different approximations of the system (5.1):

(i) The fully discrete solution of (5.1) constructed with the Godunov scheme, with space and time steps ΔX and Δt.
(ii) The semi-discrete approximation, namely the (infinite) system of ODEs

$$\partial_t \begin{bmatrix} \tau_j \\ w_j \end{bmatrix} - \begin{bmatrix} \frac{v_{j+1}-v_j}{\Delta X} \\ 0 \end{bmatrix} = 0,$$

where ΔX is the length of a car (fixed for simplicity). It is easy to see that this system can be rewritten in the form

$$\dot{x_j} = v_j,\ \dot{w_j} = 0 \text{ with } \tau_j = (x_{j+1} - x_j)/\Delta X, w_j = v_j + P(\tau_j). \tag{5.2}$$

In other words, the *semi*-discretisation of (5.1) is *exactly* the "Follow-the-Leader model" [15].

The *rigorous* results of convergence in [1, 13] are as follows :

(a) When ΔX and Δt tend to zero with a fixed ratio, and satisfy the CFL stability condition, a subsequence of the fully discrete (Godunov) solution converges to an entropy weak solution of (5.1). This limit is viewed as a coarse graining limit, i.e., a "zooming" with the same ratio in X and t ("hyperbolic scaling").

(b) Next, when Δt tends to zero, with ΔX fixed, the Godunov solution converges to the unique solution of the microscopic Follow-the-Leader (FLM) system.

(c) Finally, when ΔX tends to zero, this microscopic (FLM) solution converges to an entropy weak solution to (5.1).

These results were essentially based on uniform *a priori* BV-estimates (estimates on the total variation) for the Godunov solution. Indeed, this Lagrangian scheme preserves the total variation of the two Riemann invariants if the initial data are BV-functions.

The case of initial data with large oscillations in w, i.e., oscillations in the characteristics of car-driver pairs, is studied in [3]. Oscillations in w generate also oscillations in τ. Note that oscillations in v would be unrealistic (and dangerous!), and would be immediately cancelled by the genuinely non-linear eigenvalue λ_1.

In the above-mentioned (hyperbolic) "zooming", the oscillations in w are wilder and wilder as the zoom parameter goes to 0. Therefore, the corresponding sequence of functions converges only *weakly* to some limit. The above results can be extended and uniqueness can be proved in this more general setting. The modification involves a homogenized relation between v, w and τ, which uses the language of Young measures, see [27, 4, 11].

Let us briefly recall a few basic facts on Young measures, adapted to our context. The reader is advised to take a look at the practical example given in Section 6.

We introduce a (Lagrangian) grid (X_j) and define $U_j = (\tau_j, w_j)$ and $U_{\Delta X}(X,t) := \sum_j U_j(t)\chi_j(X)$, where χ_j is the characteristic function on $I_j := (X_{j-1/2}, X_{j+1/2})$. For any $\Delta X > 0$, let $U_j^0 = (\tau_j^0, w_j^0)$ be uniformly bounded for all j, and $U_{\Delta X}^0(X) := \sum_j U_j^0\chi_j(X)$ be be the corresponding sequence of piecewise-constant initial data. Of course, this sequence is uniformly bounded in L^∞ when $\Delta X \to 0$.
Therefore [27, 4], there exists a subsequence, still denoted by $U_{\Delta X}^0(\cdot)$, and a family of probability measures $\nu_{X,t}$ in the (v,w) plane, depending on X, such that the weak-$*$ limit of *any* continuous function $F(v_{\Delta X}^0,\, w_{\Delta X}^0)$ is equal a.e. to

$$< \nu_{X,t},\, F(v,\, w) >:= \int F(v,w)\, d\nu_{X,t}(v,\, w). \tag{5.3}$$

Since the sequence $(v_{\Delta X})$ does *not* oscillate, the same subsequence converges pointwise to some strong limit $v^*(X,t)$. Hence, equation (5.3) can be rewritten

$$< \nu_{X,t},\, F(v,\, w) >=< \mu_X,\, F(v^*(X,t),\, w) >:= \int F(v^*(X,t), w)\, d\mu_X(w),$$

where the probability measures μ_X describe the weak limit of all functions in the single variable w. Therefore μ depends on X, but not on t.

The main result in [3] can be stated as follows.

Proposition 5.1. *(i) Under the above assumptions, the (sub)sequence of entropy weak solutions corresponding to the above (sub)sequence converges in L^∞ weak-$*$ to the unique "entropy" weak solution $U^* = (\tau^*, w^*)$, of the* **homogenized** *problem*

$$\partial_t \tau^* - \partial_X v^* = 0,\ \partial_t w^* = 0\,. \tag{5.4}$$

(ii) Furthermore, $(v_{\Delta X})$ converges almost everywhere and the limit state can be characterised as

$$\tau^*(X,t) = \int P^{-1}(w - v^*(X,t))d\mu_X(w), \tag{5.5a}$$

$$w^*(X,t) = w^*(X,0) = \int w d\mu_X(w)\,, \tag{5.5b}$$

where μ_X is the Young measure associated with the sequence $(w_{\Delta X})$.

Moreover, there is a similar result of homogenisation for a *multi-class* Follow-the-Leader Model, similar to (5.2), with *oscillating* data w_j. We again refer to the above reference for more details. Proposition 6.1 in the next section deals with a practical example of the above result.

6 Two incoming and one outgoing road

As in Section 4, we need an additional assumption to obtain a unique solution at the junction. We introduce a "mixture–rule", which describes, how cars of the incoming road mix when they enter the outgoing road. One of the most natural assumptions is an equal priority rule:

The cars of both incoming roads enter the outgoing road alternately.

Note that other assumptions on the mixture of cars are also possible. The discussion below remains valid with obvious changes according to a different mixture rules.

Proposition 6.1. *Consider three roads $i = 1, 2, 3$ with $a_1 = a_2 = -\infty, b_1 = b_2 = a_3$ and $b_3 = \infty$ and constant initial data $U_{i,0} = (\rho_{i,0}\rho_{i,0}v_{i,0}), i = 1, 2, 3.$*

Then there exists a unique solution $U_i(x,t), i = 1, 2, 3$ of the Riemann problem at the junction (1.8) and (1.9) with the following properties.

1. *$U_i(x,t)$ is a weak solution of the network problem (1.5a-1.5b), where $p_i^\dagger \equiv p_i$ for the incoming roads $i = 1, 2$.*

   For the outgoing road $i = 3$, we obtain two different expressions for $p_3^\dagger$, depending on the position (x,t):

   In the triangle $\{(x,t) : a_3 \le x \le a_3 + v_{3,0}t\}$ of the $x - t$ plane, we consider the homogenised solution *described below. Therefore, $p_i^\dagger(\cdot) :=$*

$p_i^*(\cdot)$ is given by equations (6.3) to (6.6). This solution depends on the applied mixture principle, the initial data on $U_{1,0}, U_{2,0}$ and the road conditions p_3. The triangle is bounded at any fixed time $t > 0$ by $x = a_3$ and $x = a_3 + tv_{3,0}$.

In the remaining part of the outgoing road we have $p_3^\dagger \equiv p_3$.

2. *The equations (1.6a-1.6b) are satisfied, with $\rho_i(x,t)v_i(x,t) \geq 0$, $1 \leq i \leq 3$. In particular $U_3(a_3^+, t)$ satisfies*

$$w_3^\dagger(U_3(a_3^+,t)) := w_3^*(U_3(a_3^+,t)) := v_3(a_3^+,t) + p_3^*(\rho_3(a_3^+,t)) = \bar{w},$$

where $\bar{w}$ is the homogenized value*:*

$$\bar{w} := \frac{1}{2}\left(w_1(U_{1,0}) + w_2(U_{2,0}),\right). \tag{6.1}$$

3. *The two incoming fluxes are equal (equal priority rule), and the total flux $2(\rho_1 v_1)(b_1^-, t) = 2(\rho_2 v_2)(b_2^-, t) = (\rho_3 v_3)(a_3^+, t)$ is maximal subject to the other conditions.*

Before giving the proof of this result, let us motivate the definition of (6.1) and the necessity of dealing with a function p_3^*. Consider the discrete Follow-the-Leader Model (5.2), with oscillating $w_j = v_j + P(\tau_j)$:

$$\partial_t \begin{bmatrix} \tau_j \\ w_j \end{bmatrix} - \begin{bmatrix} \frac{v_{j+1} - v_j}{\Delta X} \\ 0 \end{bmatrix} = 0.$$

More precisely, consider a microscopic situation on the outgoing road 3. As in the introduction of this section, assume that the cars coming from each incoming road pass the junction in an alternating way.

Although w was *constant* on each of the roads 1 and 2, the outgoing flow is obviously oscillating. In fact, in *Lagrangian* coordinates,

$$w_j^0 = \begin{bmatrix} w_1 & j \text{ even} \\ w_2 & j \text{ odd} \end{bmatrix},$$

where the constants w_1 and w_2 are given by the two *incoming* flows. The corresponding function P on the outgoing road, is the function $P_3(\tau) := p_3(1/\tau)$. Then the piecewise-constant approximation $w_{\Delta X}$ alternately takes the two values w_1 and w_2. Consequently, for any continuous function F,

$$F(w_{\Delta X}) \rightharpoonup^* (F(w))^* := \frac{1}{2}(F(w_1) + F(w_2)) = \int F(w) d\mu_X(w),$$

$$\text{where } \mu_X := \frac{1}{2}\left(\delta_{w_1} + \delta_{w_2}\right). \tag{6.2}$$

The value of w has to be given by (6.1), since one car out of two comes from each road 1 or 2 (think of black and white cars producing a grey homogenized flow), *and* since any Lagrangian interval of length ΔX contains one car. Recall that we assumed that all cars have the same length. This assumption could be relaxed, and the formulas would be modified in an obvious way.

Therefore, in the limit $\Delta X \to 0$, the cars passing through the junction have the average property associated with the Young measure μ_X in (6.2). By Section 5, the corresponding homogenized solution is the unique weak entropy solution of (5.4), where τ^* is given by (5.5a), i.e. here by

$$\tau^*(X,t) = \frac{1}{2}\,(P_3^{-1}(w_1 - v^*(X,t)) \,+\, P_3^{-1}(w_2 - v^*(X,t))), \tag{6.3}$$

which (by monotonicity of P_3) defines a one-to-one relation between $v := v^*(X,t)$ and $\tau := \tau^*(X,t)$:

We **choose** to rewrite (6.3) in the form

$$v = w - P_3^*(\tau),\; w := \bar{w}, \tag{6.4}$$

where $\bar{w}$ is given by (6.1). In other words, we *define* P_3^* so that, for each $\tau = \tau^*$, the value $v = v^*$ defined by (6.4) is the *unique* solution of equation (6.3) to the unknown v. This (convenient) notation could be misleading for an *arbitrary* value w. Indeed, the homogenized relation between v and τ depends on μ_X, see (5.5a). Therefore, it depends on the local proportions of cars coming from each incoming road. In other words, (6.4) would be *wrong* for any value $w \neq \bar{w}$. However, see below, on the relevant portion of road 3, the homogenized w *only* takes the value $\bar{w}$.

Now, three questions arise:

(i) How do we express this in Eulerian coordinates?
(ii) In Eulerian coordinates, what is the portion of road 3 concerned with this homogenized flow?
(iii) Does this solution respect the Rankine-Hugoniot relations (1.6a), (1.6b) at the interface $x = b_1 = b_2 = a_3$, and how is it connected with the downstream flow on road 3?

(i) First, see [1], we can rewrite (5.4), (6.3) in Eulerian coordinates, to get the equivalent system (even for weak entropy solutions):

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{6.5a}$$

$$\partial_t(\rho w) + \partial_x(\rho v w) = 0, \tag{6.5b}$$

$$w(U) \equiv w^*(U) = v + p_3^*(\rho), \tag{6.5c}$$

with $w(U) \equiv \bar{w}$ and

$$p_3^*(\rho) := P_3^*(\rho^{-1}) \tag{6.6}$$

defined by (6.4). Again, for *arbitrary* values of w, we would not recover the correct homogenized solution.

(ii) In the (x, t) plane, at time $t > 0$, the portion of road 3 concerned with this self-similar, homogenised flow is a triangle bounded by $x = b_1 = b_2 = a_3$ and by $x = a_3 + t\, v_{3,0}$. Here, $v_{3,0}$ is the initial datum on road 3.

(iii) On the above portion of road 3, our solution satisfies (6.5), (6.6) and the value of w is *a constant* and is equal to the corresponding average value given by (6.1).

The boundary data specified below preserve the conservation of mass at the intersection and satisfy the equal priority rule on the mixture of the cars:

$$\rho_3\, v_3 = \rho_1\, v_1 + \rho_2\, v_2 = 2\, \rho_1\, v_1$$

Therefore, combining with (6.1), we see that $\rho_3\, v_3\, w_3 = \rho_1\, v_1\, w_1 + \rho_2\, v_2\, w_2$, i.e., we recover (1.6b): our solution *also* satisfies the conservation of $y = \rho\, w$ at the junction. Roughly speaking, e.g. the total number of white cars is also preserved at the intersections!

Now we can give the proof of Proposition 6.1.

Proof. Let $U_i^- = U_{i,0}$ for $i = 1, 2$ and $U_3^+ = U_{3,0}$. Denote by $w_i(U) = v + p_i(\rho)$. Let the demand functions d_1 and d_2 be defined by

$$d_1(\rho) := d(\rho; w_1, w_1(U_1^-)),\ d_2(\rho) := d(\rho; w_2, w_2(U_2^-)).$$

With all the previous remarks in mind and again $w_3^\dagger(U) = v + p_3^\dagger(U)$ and $p_3^\dagger(\cdot) := p_3^*(\cdot)$, we consider the following supply function

$$s_3(\rho) := s_3^\dagger(\rho) := s(\rho; w_3^\dagger, \bar{w}),\ \bar{w} = \frac{1}{2}\left(w_1(U_1^-) + w_2(U_2^-)\right)$$

As in Proposition 2.2 we obtain the intermediate state $U_3^\dagger = (\rho_3^\dagger, \rho_3^\dagger v_3^\dagger)$ as the intersection of $\{v_3(U) = v_3^+\}$ and $\{w_3^\dagger(U) = \bar{w}\}$, or as $U_3^\dagger = (0, 0)$. Then we solve for q_1, q_2

$$\begin{array}{cccc} & \max q_1 + q_2 & \text{subject to} & \\ 0 \leq & q_i & \leq d_i(\rho_i^-), & i = 1, 2, \\ 0 \leq & q_1 + q_2 & \leq s_3(\rho_3^\dagger), & \\ q_1 & = & q_2. & \end{array}$$

Clearly, $\tilde{q} = q_1 = q_2 = \min\{s_3(\rho_3^\dagger)/2, d_1(\rho_1^-), d_2(\rho_2^-)\}$ is the unique solution. As in the Proposition (2.3) we conclude

$$\exists U_i^+ \text{ such that } \rho_i^+ v_i^+ = \tilde{q},\ w_i(U_i^+) = w_i(U_i^-)\ i = 1, 2,$$
$$\exists U_3^- \text{ such that } \rho_3^- v_3^- = 2\tilde{q},\ w_3^\dagger(U_3^-) = \bar{w}.$$

We recall that $U_i(b_i-, t) = U_i^+$ for $i = 1, 2$ and $U_3(a_3+, t) = U_3^-$.

Then the conditions (1.6a-1.6b) are satisfied. Using the considerations above, the function $p_3^\dagger$ is defined in the triangle $\{(x, t) : a_3 \leq x \leq a_3 + tv_{3,0}\}$ of the $x - t$ plane.

Each solution $U_i(x, t)$ is a juxtaposition of either a rarefaction or a shock wave and a contact discontinuity.

In particular, on the outgoing road $i = 3$, the states U_3^- and $U_3^\dagger$ are connected by a rarefaction or a shock wave associated with the first eigenvalue of system (1.1), with $p_i = p_3^\dagger = p_3^*$. Then $U_3^\dagger$ is connected to $U_3^+ = U_{3,0}$ by a contact discontinuity associated with the second eigenvalue $\lambda_2 = v_{3,0}$, which is **independent** of p_i. Hence, out of the above mentioned triangle, $U_3(x, t) \equiv U_{3,0}$. $\square$

An example of a solution is depicted in Figure 5 and 6.

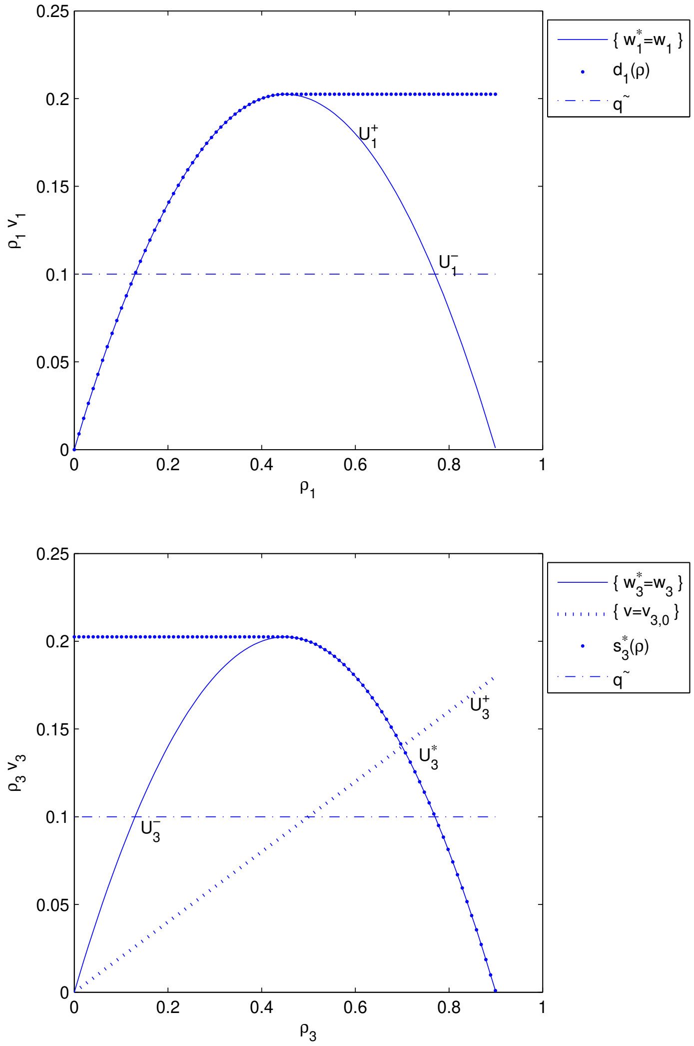

Figure 5: An example of intermediate states on road 1 (top part) and road 3 (bottom part) in case where $\tilde{q} = d_2(\rho_{2,0}; w_2, w_2(U_{2,0}))$, i.e., $\tilde{q} < d_1(\cdot)$ and $\tilde{q} < s_3^\dagger(\cdot)$ respectively. In this case the solution U_2 on road 2 is a constant $U_2(x,t) = U_{2,0} \equiv U_2^-$ and therefore omitted from the plots. In the drawings U_3^*, s_3^*, w_3^* and w_1^* stand for $U_3^\dagger, s_3^\dagger, w_3^\dagger$ and $w_1^\dagger$, respectively.

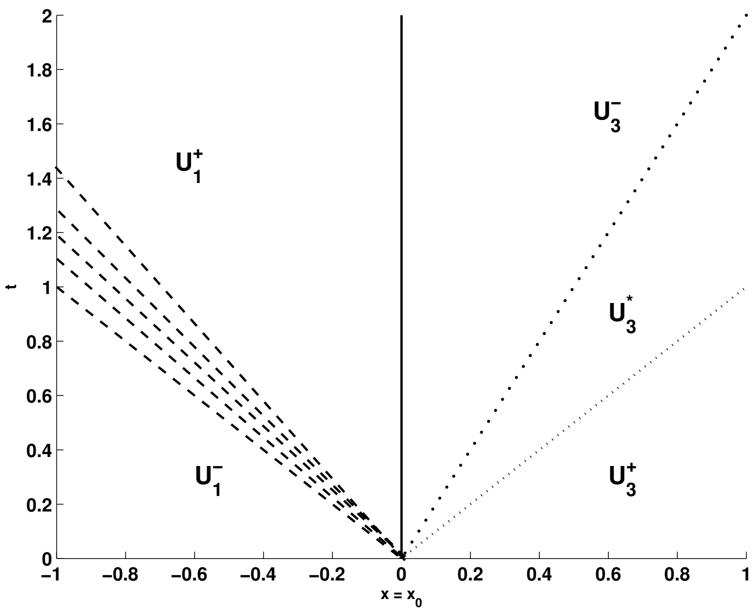

Figure 6: Plot of the solution U_1 and U_3 in the $x-t-$plane with data as in Figure 5. Left to the interface, $U_1^- \equiv U_{1,0}$ is connected by a 1-rarefaction to U_1^+. Right to the interface U_3^- is connected by 2–shock to $U_3^* \equiv U_3^\dagger$ and this state in turn is connected to $U_3^+ \equiv U_{3,0}$ by a 2–contact discontinuity. We omit the solution U_2 since it is constant.

7 Arbitrary number of incoming and outgoing roads

We combine the results of Section 4 to 6 to treat the general case. We consider a fixed junction with m incoming roads $\delta^- = \{1, \dots, m\}$ and n outgoing roads $\delta^+ = \{m+1, \dots, m+n\}$. We assume constant initial data $U_{i,0}$ for all i and we look for solutions to the Riemann problem (1.8) and (1.9).

In Section 4 to 6 we imposed additional conditions to obtain a unique solution. Here, as in Section 6 we introduce a mixture principle for the outgoing traffic which is an extension of the equal priority rule, c.f. assumption (H4) below. However, the stated results can be adapted to other mixture rules.

For a set of functions $U_i(x,t) = (\rho_i(x,t), \rho_i(x,t)v_i(x,t))\,,\ i \in \delta^- \cup \delta^+$ we introduce the following abbreviations:

$$\mathbf{q}_i := \rho_i(b_i^-, t)v_i(b_i^-, t),\ \forall i \in \delta^-, \tag{7.1a}$$
$$\mathbf{q}_j := \rho_j(a_j^+, t)v_j(a_j^+, t),\ \forall j \in \delta^+. \tag{7.1b}$$

Next, we introduce real numbers $q_{ji} \in \mathbb{R}$ for $j \in \delta^+$ and $i \in \delta^-$ corresponding to the (a priori unknown) *actual fluxes* of cars coming from road i and going to road j. Since the number of cars entering and leaving the junction is the same,

$$\mathbf{q}_i = \sum_{j \in \delta^+} q_{ji},\ \mathbf{q}_j = \sum_{i \in \delta^-} q_{ji}. \tag{7.2}$$

We look for a solution $U_k(x,t)$ which satisfies the following assumptions and constraints.

(H1) Preferred choice of the drivers:

As in [12] we are given a matrix A,

$$A = (\alpha_{ji})_{j \in \delta^+, i \in \delta^-} \in \mathbb{R}^{n \times m}, \tag{7.3}$$

such that $0 \le \alpha_{ji} \le 1$ and $\sum_{j \in \delta^+} \alpha_{ji} = 1,\ \forall i \in \delta^-$.

We introduce $\mathbf{a}_j := \sum\limits_{i \in \delta^-} \alpha_{ji}$ for notational convenience. We impose the constraint:

$$q_{ji} = \alpha_{ji}\mathbf{q}_i \qquad \forall j \in \delta^+, i \in \delta^-. \tag{7.4}$$

(H2) Relation for $w_j^\dagger$ (on the outgoing roads)

$$\forall j \in \delta^+ : w_j^\dagger(U_j(a_j^+, t)) = \sum_{i\in\delta^-} \frac{q_{ji}}{\mathbf{q}_j} w_i(U_i(b_i^-, t)). \tag{7.5}$$

As in the previous Sections $w_i(U_i(b_i-, t)) = w_i(U_{i,0}), \forall i \in \delta^-$ and for all $j \in \delta^+$:

$$w_j^\dagger(U_j(a_j^+, t)) := v_j(a_j^+, t) + p_j^\dagger(\rho_j(a_j^+, t)) = \bar{w}_j.$$

The functions $p_j^\dagger$ and the homogenized values $\bar{w}_j$ have to specified later.

(H3) Bounds on the actual fluxes:

$$0 \le \mathbf{q}_i \le d_i(\rho_{i,0}) \ \forall i \in \delta^- \tag{7.6a}$$
$$0 \le \mathbf{q}_j \le s_j(\rho_j^\dagger) \ \forall j \in \delta^+ \tag{7.6b}$$

Here d_i denotes the demand function on road i, i.e., $d_i := d_i(\rho; w_i, w_i(U_{i,0}))$ where $w_i(U) = v + p_i(\rho)$, and $s_j := s_j(\rho; w_j^\dagger, \bar{w}_j)$ is the supply function on road j. The functions $w_j^\dagger$ and the homogenized values $\bar{w}_j$ are specified later and depend on the applied mixture rule. Finally, $(\rho_j^\dagger, \rho_j^\dagger v_j^\dagger)$ is the intermediate state on road j, i.e., the unique intersection of the curves $\{v_j(U) = v_{j,0}\}$ and $\{w_j^\dagger(U) = \bar{w}_j\}$.

In order to define a unique solution, we have to impose a further constraint, e.g., a maximization criterion as in [17, 12]. Here, as in Section 6, we choose to impose the following rule.

(H4) The mixture rule:

The **actual** incoming fluxes $(\mathbf{q}_i)_{i\in\delta^-}$ are proportional to a **given** non-negative vector $(\tilde{q}_i)_{i\in\delta^-}$. The equal priority rule introduced in Section 6 is a particular subcase, with $(\tilde{q}_i)_{i\in\delta^-} = (1, \ldots, 1)$. So we impose in general

$$\mathbf{q}_i = \tilde{q}\,\tilde{q}_i \ge 0 \tag{7.7}$$

where $\tilde{q} > 0$ is *a priori unknown*, but $(\tilde{q}_i)_{i\in\delta^-}$ is given.

Theorem 7.1. *Consider a junction with m incoming and n outgoing roads, with constant initial data $U_{i,0} = (\rho_{i,0}, \rho_{i,0} v_{i,0})$ for all $i \in \delta^- \cup \delta^+$ under the assumptions (H1) to (H4).*

Then there exists a unique solution $\{U_i(x,t)\}_{i \in \delta^- \cup \delta^+}$ to the Riemann problems (1.8),(1.9), which is described below, and which satisfies the following properties.

1. *$\{U_i(x,t)\}_{i \in \delta^- \cup \delta^+}$ is a weak entropy solution of the network problem (1.5a-1.5b) and for $i \in \delta^- : p_i^\dagger \equiv p_i$.*

   For the outgoing roads $j \in \delta^+$ we obtain two different expressions for $p_j^\dagger$, depending on the region. In the $x-t$-plane in a triangle near the junction, we consider the homogenized solution and hence $p_j^\dagger(\cdot) = p_j^*(\cdot)$ defined below in (7.14). This triangle is defined by $\{(x,t) : a_j \leq x \leq t v_{j,0}\}$ for any fixed time $t > 0$. Beyond this triangle we have $p_j^\dagger(\cdot) \equiv p_j(\cdot)$.

2. *The constraints (7.4) to (7.6) are satisfied and the homogenized values $\bar{w}_j$ are given by:*

$$\bar{w}_j := \sum_{i \in \delta^-} \frac{q_{ji}}{\mathbf{q}_j} w_i(U_{i,0}) \; \forall j \in \delta^+. \tag{7.8}$$

   The ratios $q_{ji}/\mathbf{q}_j$ are defined below in equation (7.15).

3. *Moreover, the incoming fluxes satisfy (7.7) and they are maximal subject to the other conditions.*

For simplicity we restrict ourselves the case of the equal priority rule. Obviously the proof can be extended to the general case (7.7). Note that the matrix A plays the same role as in [12], but we do **not** need the same restrictions on A.

Proof. With the discussion in Section 6 in mind we consider the following supply functions for $j \in \delta^+$:

$$s_j(\rho) := s(\rho; w_j^\dagger; \bar{w}_j), \tag{7.9}$$

$$\bar{w}_j := \sum_{i \in \delta^-} \frac{q_{ji}}{\mathbf{q}_j} w_i(U_{i,0}), \tag{7.10}$$

where $w_i(U) = v + p_i(U)$ $\forall i \in \delta^-$ and where $w_j^\dagger(U) = v + p_j^\dagger(U)$ and $p_j^\dagger(\cdot) := p_j^*(\cdot)$ for all $j \in \delta^+$. For each $j \in \delta^+$ $p_j^*(\cdot)$ is defined as in Section 6. Namely, we first define the function

$$P_j(\tau) := p_j(1/\tau). \tag{7.11}$$

Then we set

$$v \to \tau := \sum_{i\in\delta^-} \frac{q_{ji}}{\mathbf{q}_j} P_j^{-1}(w_i(U_{i,0}) - v). \tag{7.12}$$

Next, we *choose* to define a new invertible function P_j^* by rewriting the relation (7.12) under the form

$$\tau := \left(P_j^*\right)^{-1}(\bar{w}_j - v), \tag{7.13}$$

which we **only** use with the particular value $\bar{w}_j$ defined in (7.8). Finally, we set

$$p_j^\dagger(\rho) := p_j^*(\rho) := P_j^*(1/\rho). \tag{7.14}$$

Of course this construction assumes that the proportions $q_{ji}/\mathbf{q}_j$ are known. Here, thanks to the crucial assumption (H4), we can determine them:

$$\frac{q_{ji}}{\mathbf{q}_j} = \frac{\alpha_{ji}\mathbf{q}_i}{\sum_{i\in\delta^-} q_{ji}} = \frac{\alpha_{ji}\tilde{q}_i}{\sum_{i\in\delta^-} \alpha_{ji}\tilde{q}_i} \; \forall i \in \delta^-, \forall j \in \delta^+. \tag{7.15}$$

In particular in the case of the equal priority rule $\tilde{q}_i = 1, \forall i \in \delta^-$ holds true. Therefore, $\mathbf{q}_i = \tilde{q}$, $\mathbf{q}_j = \mathbf{a}_j\tilde{q}$ and $q_{ji}/\mathbf{q}_j = \alpha_{ji}/\mathbf{a}_j$ for $i \in \delta^-, j \in \delta^+$ and for some unknown $\tilde{q} \in \mathbb{R}$.

Before we turn to the determination of $\tilde{q}$ we define $U_j^\dagger$. As in Proposition 2.2 we obtain for each j the intermediate state $U_j^\dagger$ as intersection of $\{v_j(U) = v_{j,0}\}$ and $\{w_j^\dagger(U) = \bar{w}_j\}$.

Now, we obtain $\tilde{q}$ as unique solution to the following maximization problem

$$\max_{q\in\mathbb{R}} q \;\text{ subject to} \tag{7.16a}$$

$$0 \le \mathbf{q}_i = q \le d_i(\rho_{i,0}; w_i; w_i(U_{i,0})),\; \forall i \in \delta^-, \tag{7.16b}$$

$$0 \le \mathbf{q}_j = \mathbf{a}_j q \le s_j(\rho_j^\dagger; w_j^\dagger; \bar{w}_j),\; \forall j \in \delta^+, \tag{7.16c}$$

where the functions $s_j(\cdot), w_j^\dagger(\cdot)$ and the values $\bar{w}_j$ are well–defined since the proportions $q_{ji}/\mathbf{q}_j$ are known.

We conclude as before

$$\exists U_i^+ \text{ such that } \rho_i^+ v_i^+ = \tilde{q},\ w_i(U_i^+) = w_i(U_{i,0})\ \forall i \in \delta^+,$$
$$\exists U_j^- \text{ such that } \rho_j^- v_j^- = \mathbf{a}_j \tilde{q},\ w_j^\dagger(U_j^-) = \bar{w}_j\ \forall j \in \delta^+.$$

The conditions (1.6a-1.6b) are satisfied. Also (7.5) and (7.6) are fulfilled.

Again, each $U_i(x,t)$ consists of a juxtaposition of rarefaction or shock waves associated with the first eigenvalue and, for $i \in \delta^+$, an additional contact discontinuity associated with the second eigenvalue. Furthermore, the solution satisfies on the incoming roads $i \in \delta^- : U_i^+ = U_i(b_i-, t)$ and on the outgoing roads $j \in \delta^+ : U_j^- = U_i(a_j+, t)$.

For general $\tilde{q}_i$, equations (7.16b) and (7.16c),respectively, become

$$0 \le \mathbf{q}_i = \tilde{q}_i\ q \le d_i(\rho_{i,0}; w_i, w_i(U_{i,0}))\ \ \forall i \in \delta^-,$$
$$\text{and } 0 \le \mathbf{q}_j = \left(\sum_{i \in \delta^-} \alpha_{ji} \tilde{q}_i \right) q \le s_j^\dagger(\rho_{j,0}; w_j^\dagger, \bar{w}_j)\ \ \forall j \in \delta^+.$$

□

8 Conclusion

In this paper, we have introduced coupling conditions for the AR traffic flow model. Contrary to [12] the total "momentum" (e.g. the total number of white cars!) is conserved at each junction. We have presented the full solution to Riemann problems for different cases and have given a microscopic motivation and validation of the approach. Last, we have discussed the general case of arbitrary numbers of incoming and outgoing roads. The most striking fact is the role of the homogenized flow on part of the outgoing roads. It is worth to note that, even with Riemann data, and with the **same** function $p_j \equiv p$ on **all** the roads, after some time, due to the mixture of cars at each junction, the flow is associated with a **new homogenized** pseudo-pressure $p_j^\dagger$, which depends on the proportions of the mixture.

As we already said in Section 2, the model presented is too sophisticated for real life applications. But it contains as particular case the classical first-order models.

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