VALIDITY OF NONLINEAR GEOMETRIC OPTICS FOR ENTROPY SOLUTIONS OF MULTIDIMENSIONAL SCALAR CONSERVATION LAWS

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Abstract. Nonlinear geometric optics with various frequencies for entropy solutions only in L^∞ of multidimensional scalar conservation laws is analyzed. A new approach to validate nonlinear geometric optics is developed via entropy dissipation through scaling, compactness, homogenization, and L^1–stability. New multidimensional features are recognized, especially including nonlinear propagations of oscillations with high frequencies. The validity of nonlinear geometric optics for entropy solutions in L^∞ of multidimensional scalar conservation laws is justified.

1. Introduction

We are concerned with nonlinear geometric optics for entropy solutions of multidimensional scalar conservation laws:

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathbf{F}(u) = 0, \qquad u \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n, \tag{1.1}$$

where $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n$ is a smooth flux function. Consider the Cauchy problem (1.1) with Cauchy data:

$$u|_{t=0} = u_0^\varepsilon(\mathbf{x}) := \underline{u} + \varepsilon u_1(\phi_1/\varepsilon^{\alpha_1}, \cdots, \phi_n/\varepsilon^{\alpha_n}), \tag{1.2}$$

where u_1 is a periodic function of each of its n arguments whose period is denoted by $P = [0,1]^n$ (without loss of generality), $\underline{u}$ is a constant ground state, the linear phases $\Phi := (\phi_1, \cdots, \phi_n)$:

$$\phi_i := \sum_{j=1}^n J_{ij} x_j \tag{1.3}$$

are linearly independent with constant matrix $J = (J_{ij})_{1 \le i,j \le n}$, and

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, \infty)^n$$

is the magnitude indices of frequency of the initial oscillations. We seek a geometric optics asymptotic expansion:

$$u^\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon v^\varepsilon(t, \mathbf{x}). \tag{1.4}$$

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In this paper, we develop a new approach including several basic frameworks and new tools in Sections 2-5 to validate weakly nonlinear geometric optics via entropy dissipation through compactness, scaling, homogenization, and L^1–stability, and we apply this approach first to the one-dimensional case in Section 4 and then to the multidimensional case in Section 5 to recognize new multidimensional features and validate nonlinear geometric optics for multidimensional scalar conservation laws by extending the ideas and techniques in Section 4.

To illustrate multidimensional features clearly in nonlinear geometric optics, we focus now on the two dimensional case. Let $u := u^\varepsilon$ be the Krushkov solution of the Cauchy problem:

$$\partial_t u + \partial_{x_1} f_1(u) + \partial_{x_1} f_2(u) = 0, \tag{1.5}$$

$$u(0, x_1, x_2) = u_0(x_1, x_2) \equiv \underline{u} + \varepsilon\, u_1\, (\phi_1/\varepsilon^\alpha,\ \phi_2/\varepsilon^\beta), \tag{1.6}$$

where, as for the general setting (1.1)–(1.4), the linear phases (ϕ_1, ϕ_2):

$$\phi_1 := a_1 x_1 + a_2 x_2, \quad \phi_2 := b_1 x_1 + b_2 x_2 \tag{1.7}$$

are linearly independent, and

$$\alpha_1,\ \alpha_2\ \geq\ 0.$$

The Krushkov solution is an L^∞ function $u = u(t, x_1, x_2)$ satisfying

$$\partial_t|u-k| + \partial_{x_1}(sign(u-k)(f_1(u)-f_1(k))) + \partial_{x_2}(sign(u-k)(f_2(u)-f_2(k))) \leq 0 \tag{1.8}$$

in the sense of distributions for any $k \in \mathbb{R}$. For the Krushkov solution u, we look for an asymptotic expansion:

$$u = u^\varepsilon(t, x_1, x_2) := \underline{u} + \varepsilon\, v^\varepsilon\, (t, x_1, x_2). \tag{1.9}$$

After, if necessary, a linear change of coordinates: $x_1 \to x_1 - a_0 t,\ \ x_2 \to x_2 - b_0 t$, we may assume

$$f_1'(\underline{u}) = f_2'(\underline{u}) = 0.$$

Since ϕ_1 and ϕ_2 are linearly independent, we can rewrite equation (1.5) in these coordinates, even though they are not necessarily orthonormal, and perform a formal asymptotic expansion to obtain

$$\partial_t v + \varepsilon\,(a\partial_{\phi_1} + b\partial_{\phi_2})v^2 + \varepsilon^2\,(c\partial_{\phi_1} + d\partial_{\phi_2})v^3 = \varepsilon^3(\partial_{\phi_1}R_1 + \partial_{\phi_2}R_2), \tag{1.10}$$

where

$$\begin{aligned} a &:= (a_1 f_1''(\underline{u}) + a_2 f_2''(\underline{u}))/2, \quad & b &:= (b_1 f_1''(\underline{u}) + b_2 f_2''(\underline{u}))/2,\\ c &:= (a_1 f_1'''(\underline{u}) + a_2 f_2'''(\underline{u}))/6, \quad & d &:= (b_1 f_1'''(\underline{u}) + b_2 f_2'''(\underline{u}))/6, \end{aligned} \tag{1.11}$$

and $R_j := R_j(v, \underline{u}, \varepsilon), j = 1, 2$, are Lipschitz functions in v, $\underline{u}$, and ε with the form:

$$R_1(v, \underline{u}, \varepsilon) = -\tfrac{1}{6}\int_0^1 (1-\theta)^3 (a_1 f_1^{(4)} + a_2 f_2^{(4)})(\underline{u} + \varepsilon\theta v)d\theta\, v^4, \tag{1.12}$$

$$R_2(v, \underline{u}, \varepsilon) = -\tfrac{1}{6}\int_0^1 (1-\theta)^3 (b_1 f_1^{(4)} + b_2 f_2^{(4)})(\underline{u} + \varepsilon\theta v)d\theta\, v^4. \tag{1.13}$$

Define

$$M := \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} f_1''(\underline{u})/2 & f_1'''(\underline{u})/6 \\ f_2''(\underline{u})/2 & f_2'''(\underline{u})/6 \end{pmatrix}. \tag{1.14}$$

We assume that the matrix $M = M(\underline{u})$ is invertible. It is equivalent to require that both

$$J := \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \frac{D(\phi_1,\ \phi_2)}{D(x_1,\ x_2)} \tag{1.15}$$

and

$$N := \begin{pmatrix} f_1''(\underline{u})/2 & f_1'''(\underline{u})/6 \\ f_2''(\underline{u})/2 & f_2'''(\underline{u})/6 \end{pmatrix} \tag{1.16}$$

be invertible. Note that the invertibility of (1.15) is a corollary of the linear independence of the phases (ϕ_1, ϕ_2), and the invertibility of (1.16) is an assumption of genuine nonlinearity and "genuine multidimensionality", which particularly implies that the second derivatives of the fluxes f_1 and f_2 are not proportional on any interval in a neighborhood of $\underline{u}$. This type of nonlinear assumptions is enough to obtain compactness; indeed, this nonlinear assumption is the strongest nonlinear assumption near the constant ground state $\underline{u}$. The compactness of solution operators can be achieved by applying the compensated compactness method and the averaging lemma. In this regard, we refer to Chen-Frid [2, 3], Engquist-E [12], and Lions-Perthame-Tadmor [22].

The main problems concerned in this paper include

(i). identification of the formal limit V of $v = v^\varepsilon$ as a function of t and the fast variables, which turn out to be, for the simplest cases,

$$X_1 := \phi/\varepsilon^{min(\alpha_1,1)}, \quad X_2 := \psi/\varepsilon^{min(\alpha_2,1)}; \tag{1.17}$$

(ii). justification of this asymptotics, that is, the strong convergence of v^ε to V in L^1_{loc}:

$$v^\varepsilon - V \to 0 \qquad \text{strongly in } L^1_{loc} \tag{1.18}$$

in the two systems of fast coordinates (t, X_1, X_2) and of slow coordinates (t, x_1, x_2), where V is the profile.

We emphasize that our purpose here is not necessarily to obtain the sharpest possible results on the convergence rates, say in (1.18). In particular, contrarily to the one-dimensional case, there is even no available result on decay *rates* of the total variation of the solution to (1.5) with periodic initial data since genuine nonlinearity of the flux function fails, although there are some results of strong convergence to a constant state, see [2, 3, 12]. Precisely, our goal here is to develop a new approach and apply it to prove rigorously the results of strong convergence like (1.18) for entropy solutions, by *only* using entropy dissipation through compactness, scaling, homogenization, and L^1–stability, without relying on the BV structure upon solutions.

There are too many cases to give here a precise statement of the results. Let us just mention for the moment that, in the one-dimensional case with only one phase ϕ and with X defined as in (1.17), there are three subcases for entropy solutions in L^∞:

(i). If $\alpha > 1$, then the initial oscillation is so fast that it is "canceled" by the nonlinearity, that is, v^ε converges strongly to σ with

$$\sigma = \overline{u}_1 := \text{ mean value of } u_1 \text{ over the period}; \tag{1.19}$$

(ii). If $\alpha = 1$ that is the natural situation of weakly nonlinear geometrical optics (WNLGO), then v^ε converges strongly to the profile σ in the fast variable X, which is uniquely determined by the Cauchy problem:

$$\partial_t \sigma + a\partial_X \sigma^2 = 0, \qquad \sigma(0, X) = u_1(X); \tag{1.20}$$

(iii). If $\alpha < 1$, then the initial oscillation is so slow that, in the variable $X = \phi/\varepsilon^\alpha$, v^ε converges strongly to σ, which is determined by

$$\partial_t \sigma = 0, \qquad \sigma(0, X) = u_1(X), \tag{1.21}$$

that is, $\sigma(t, X) = u_1(X)$.

These results are proved in Section 4. In the multidimensional case, the situation is much more complicated: real new multidimensional features are involved and multidimensional phenomena occur (see Section 5, especially the examples in the two-dimensional case), although some cases are a combination of these three different possibilities. In particular, the links between the linear phases $(\phi_1, \cdots, \phi_n)$ and the fluxes $\mathbf{F}$ through formulas, such as (1.10) to (1.14) for the two-dimensional case, lead to a number of interesting cases which deserve to be described more precisely. Furthermore, we develop a new approach in Sections 2-5 to validate weakly nonlinear geometric optics first for the one-dimensional case in Section 4 and then to extend the ideas and techniques from Section 4 to deal with the multidimensional case in Section 5. An important tool to preserve the L^1_{loc}–convergence after the triangular change of variables depending on ε is introduced in Lemma 3.1 and a "quasi" LU factorization of matrix M to include all new cases is formulated in Lemma 5.1. We also recognize new phenomena including the blowup of the gradients of the geometric optics asymptotic expansions; in contrast to the classical geometric optics expansions with the gradients of order 1 since the amplitude is of order ε and the frequency of order ε^{-1}. There are essentially two ways to obtain such very high oscillations for the two-dimensional case:

(i). A phase gradient is orthogonal to the second flux derivative on the ground state;

(ii). There are some precise arithmetic relations between the coefficients of the matrix M.

For dimension $n \geq 3$, we can have higher oscillations with combinations of these two ways, or with orthogonality to the second and third flux derivatives, among others.

For related results on nonlinear geometric optics, see DiPerna-Majda [11] for one-dimensional 2×2 hyperbolic systems of conservation laws; also see Cheverry [6, 7], Guès [15], Hunter-Majda-Rosales [16], Junca [18, 19], Joly-Métivier-Rauch [17], Majda-Rosales [23], and the references cited therein.

2. Geometric Optics, Compactness, and L^1–Stability

In this section, we introduce some basic frameworks to validate nonlinear geometric optics for entropy solutions only in L^∞ of multidimensional conservation laws.

2.1. Basic Properties of Nonlinear Geometric Optics Expansions.

Basic Properties of Nonlinear Geometric Optics Expansions. Consider the nonlinear geometric optics expansion (1.4) of solutions of the Cauchy problem (1.1)–(1.2). Set $v^\varepsilon(t, \mathbf{x}) = \frac{u^\varepsilon(t,\mathbf{x}) - u}{\varepsilon}$. We first derive some basic properties of the sequence $v^\varepsilon(t, \mathbf{x})$.

Lemma 2.1. Let $u_1 \in L^\infty$. Assume that, for each fixed $\varepsilon > 0$, $u^\varepsilon(t, \mathbf{x})$ is the entropy solution of the Cauchy problem (1.1)–(1.2). Then

$$\|v^\varepsilon\|_{L^\infty} \le \|u_1\|_{L^\infty} \qquad \text{for any } \varepsilon > 0. \tag{2.1}$$

Proof. Using Krushkov's uniqueness theorem, $u^\varepsilon(t, \mathbf{x})$ is the periodic entropy solution of the Cauchy problem (1.1)–(1.2) for any fixed $\varepsilon > 0$, since u_1 is periodic.

First, taking the convex entropy $(u - \underline{u})^p$ of (1.1) for even $p \ge 2$, we obtain from the entropy inequality that

$$\partial_t (u^\varepsilon - \underline{u})^p + \mathrm{div}_{\mathbf{x}} \left(p \int^{u^\varepsilon} (\xi - \underline{u})^{p-1} \mathbf{F}'(\xi)\, d\xi \right) \le 0$$

in the sense of distributions. Equivalently,

$$\partial_t (u^\varepsilon - \underline{u})^p + \mathrm{div}_{\Phi} \left(p\, J \int^{u^\varepsilon} (\xi - \underline{u})^{p-1} \mathbf{F}'(\xi)\, d\xi \right) \le 0$$

in the sense of distributions, where $\Phi = J\mathbf{x}$. Integrating with respect to Φ and using the periodicity, we obtain

$$\int_{P^\varepsilon} |u^\varepsilon(t, J^{-1}\Phi) - \underline{u}|^p d\Phi \le \int_{P^\varepsilon} |u_0(\phi_1/\varepsilon^{\alpha_1}, \cdots, \phi_n/\varepsilon^{\alpha_n}) - \underline{u}|^p d\phi_1 \cdots d\phi_n,$$

where $P^\varepsilon = \{(\phi_1, \cdots, \phi_n) : (\phi_1/\varepsilon^{\alpha_1}, \cdots, \phi_n/\varepsilon^{\alpha_n}) \in P\}$. This is equivalent to

$$\int_{P^\varepsilon} |v^\varepsilon(t, J^{-1}\Phi)|^p d\Phi \le \int_{P^\varepsilon} |u_1(\phi_1/\varepsilon^{\alpha_1}, \cdots, \phi_n/\varepsilon^{\alpha_n})|^p d\phi_1 \cdots d\phi_n$$

for any even $p \ge 2$. Taking power $1/p$ in both sides and letting $p \to \infty$, we conclude (2.1).

Lemma 2.2. If $u^\varepsilon(t, x_1, x_2)$ is the entropy solution of (1.5)–(1.6), then $v^\varepsilon(t, x_1, x_2)$ is the entropy solution of (1.10) with initial data

$$v^\varepsilon|_{t=0} = u_1(\phi_1/\varepsilon^{\alpha_1}, \phi_2/\varepsilon^{\alpha_2}). \tag{2.2}$$

Proof: Notice that u^ε satisfy (1.8). Choosing $l = \frac{k-u}{\varepsilon}$ with the linear transformation into the (ϕ_1, ϕ_2)-coordinates, we have

$$\begin{aligned} \partial_t |v^\varepsilon - l| \quad &+ \quad \varepsilon(a\partial_{\phi_1} + b\partial_{\phi_2})(sign(v^\varepsilon - l)((v^\varepsilon)^2 - l^2)) \\ &+ \quad \varepsilon^2(c\partial_{\phi_1} + d\partial_{\phi_2})(sign(v^\varepsilon - l)((v^\varepsilon)^3 - l^3)) \\ &+ \quad \varepsilon^3(\partial_{\phi_1}(sign(v^\varepsilon - l)(R_1(v^\varepsilon, \underline{u}, \varepsilon) - R_1(l, \underline{u}, \varepsilon))) \\ &- \quad \partial_{\phi_2}(sign(v^\varepsilon - l)(R_2(v^\varepsilon, \underline{u}, \varepsilon) - R_2(l, \underline{u}, \varepsilon)))) \le 0, \end{aligned} \tag{2.3}$$

which implies that $v^\varepsilon(t, x_1, x_2)$ is the entropy solution of (1.10) and (2.2).

Remark 2.1. The same proof implies that Lemma 2.2 also holds for (1.1)–(1.2) with $n \ge 3$, which will be used in Section 5.

2.2. **Compactness of Approximate Solutions.** We now present several compactness lemmas, which can be achieved by compactness arguments and Young measures with the aid of entropy dissipation of the solutions.

Lemma 2.3. Assume that $u(t, \mathbf{x})$ is the unique entropy solution in L^∞ of (1.1) with initial data

$$u|_{t=0} = u_0 \in L^\infty, \tag{2.4}$$

where the solution is understood in the sense of distributions with initial data included in the integral entropy inequality. Let the Young measure $\nu_{t,\mathbf{x}}(\lambda)$ be a measure-valued solution to (1.1) and (2.4) with $\nu_{0,\mathbf{x}} = \delta_{u_0(\mathbf{x})}$, i.e.,

$$\partial_t\langle\nu_{t,\mathbf{x}}, \eta(\lambda)\rangle + \mathrm{div}_{\mathbf{x}}\langle\nu_{t,\mathbf{x}}, \mathbf{q}(\lambda)\rangle \le 0, \qquad \nu_{0,\mathbf{x}} = \delta_{u_0(\mathbf{x})}, \tag{2.5}$$

for any convex entropy pair $(\eta, \mathbf{q})$. Then

$$\nu_{t,\mathbf{x}}(\lambda) = \delta_{u(t,\mathbf{x})}(\lambda), \qquad a.e.\ (t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

That is, if $u^\varepsilon(t,\mathbf{x})$ is a sequence of uniformly bounded approximate solutions to (1.1) and (2.4) so that the corresponding Young measure $\nu_{t,\mathbf{x}}$ satisfies (2.5), then $u^\varepsilon(t,\mathbf{x})$ converges strongly to the unique entropy solution $u(t,\mathbf{x})$ of (1.1) and (2.4).

This result can be obtained by combining DiPerna's argument in [10] with the monotonicity argument as in Chen-Rascle [4]; also see Szepessy [27].

In particular, Lemma 2.3 implies that, if $u^\varepsilon(t,\mathbf{x})$ are the entropy solutions of

$$\partial_t u^\varepsilon + \mathrm{div}_{\mathbf{x}}(\mathbf{F}(u^\varepsilon) + \mathbf{G}_\varepsilon(u^\varepsilon)) = 0, \qquad u|_{t=0} = u_0 \in L^\infty,$$

where $\mathbf{G}_\varepsilon' \to 0$ strongly in $L^\infty_{loc}(\mathbb{R})$ when $\varepsilon \to \infty$, then $u^\varepsilon(t,\mathbf{x})$ converges strongly to the unique entropy solution $u(t,\mathbf{x})$ of (1.1) and (2.4).

On the other hand, the nonlinearity of the flux function can yield the compactness of solution operators. We start with the one-dimensional case.

Lemma 2.4. Consider the Cauchy problem for one-dimensional conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad u|_{t=0} = u_0(x). \tag{2.6}$$

Assume that there is no interval (α, β) in which f is affine. Then the entropy solution operator $u(t,\cdot) = S_t u_0(\cdot) : L^\infty \to L^1_{loc}$, determined by (2.6), is compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$. Furthermore, if a uniformly bounded sequence $u^\varepsilon(t,x)$ satisfies that

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \quad \text{is compact in} \quad H^{-1}_{loc}, \tag{2.7}$$

then $u^\varepsilon(t,x)$ strongly converges to an L^∞ function $u(t,x)$ a.e.

The first proof of this lemma was given in Tartar [28] by using infinite entropy-entropy flux pairs. A simpler proof can be found in Chen-Lu [5] by using only two natural entropy-entropy flux pairs.

The corresponding multidimensional version of Lemma 2.4 is the following.

Lemma 2.5. Consider the Cauchy problem (1.1)–(1.2). Assume that, for any $(\tau, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^n$ with $\tau^2 + |\mathbf{k}|^2 = 1$,

$$meas\{v \in \mathbb{R} \ : \ \tau + \mathbf{F}'(v) \cdot \mathbf{k} = 0\} = 0. \tag{2.8}$$

Then the entropy solution operator $u(t,\cdot) = S_t u_0(\cdot) : L^\infty \to L^1_{loc}$, determined by (1.5), is compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^n)$. In particular, if

$$\det(\mathbf{F}''(v), \cdots, \mathbf{F}^{(n+1)}(v)) \neq 0 \qquad \text{for any } v \in \mathbb{R}, \tag{2.9}$$

then the entropy solution operator of (1.1) is compact from $L^\infty(\mathbb{R}^n)$ to $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^n)$.

Proof. The first part of this lemma is essentially due to Lions-Perthame-Tadmor [22]; its complete proof can be found in Chen-Frid [3].

For the second part, it suffices to prove that, for any $(\tau, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^n$ such that $\tau^2 + |\mathbf{k}|^2 = 1$, the set $E := \{v \in \mathbb{R} \,:\, \tau + \mathbf{F}'(v) \cdot \mathbf{k} = 0\}$ is countable, so $meas(E) = 0$. Let $h(v) \equiv \tau + \mathbf{F}'(v) \cdot \mathbf{k}$.

If $\mathbf{k} = (0, \cdots, 0)$, then $h(v) = \pm 1$ and $v \notin E$.

If $\mathbf{k} \neq (0, \cdots, 0)$, then, for any $v \in \mathbb{R}$, there exists $j \in \{1, \cdots, n\}$ such that

$$\frac{d^j h(v)}{dv^j} \neq 0.$$

Otherwise,

$$\mathbf{k} \perp span(\mathbf{F}''(v), \cdots, \mathbf{F}^{(n+1)}(v)) = \mathbb{R}^n,$$

i.e., $\mathbf{k} = (0, \cdots, 0)$. Therefore, the zeros of $h(v)$ are isolated.

Remark 2.2. Lemma 2.5 is also true if the genuine nonlinearity assumption (2.9) is imposed only on the constant state $\underline{u}$ for which the compactness, locally near $\underline{u}$, can be achieved.

Remark 2.3. Lemma 2.5 also holds if there exist $2 \leq i_1 < i_2 < \cdots < i_n$ such that

$$\det(\mathbf{F}^{(i_1)}(v), \cdots, \mathbf{F}^{(i_n)}(v)) \neq 0,$$

followed by a similar proof.

Lemma 2.6. Let $\mathbf{F}, \mathbf{G}_\varepsilon \in C^1(\mathbb{R}; \mathbb{R}^n)$ such that $\mathbf{F}$ satisfies the nondegeneracy condition (2.8) and $\mathbf{G}'_\varepsilon(u) \to 0$ strongly in $L^\infty_{loc}(\mathbb{R})$ as $\varepsilon \to 0$. Assume that $u_0^\varepsilon(\mathbf{x})$ is a uniformly bounded sequence converging weak-star to $u_0(\mathbf{x})$. Let $u^\varepsilon(t, \mathbf{x})$ and $u(t, \mathbf{x})$ be the entropy solutions of the Cauchy problems:

$$\begin{aligned} &\partial_t u^\varepsilon + \mathrm{div}_{\mathbf{x}}(\mathbf{F}(u^\varepsilon) + \mathbf{G}_\varepsilon(u^\varepsilon)) = 0, \qquad u^\varepsilon(0, \mathbf{x}) = u_0^\varepsilon(\mathbf{x}), \\ &\partial_t u + \mathrm{div}_{\mathbf{x}} \mathbf{F}(u) = 0, \qquad u(0, \mathbf{x}) = u_0(\mathbf{x}), \end{aligned}$$

respectively. Then the sequence $u^\varepsilon(t, \mathbf{x})$ strongly converges to $u(t, \mathbf{x})$ in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^n)$.

Proof: Let $\mathbf{F}_\varepsilon := \mathbf{F} + \mathbf{G}_\varepsilon$. Then the kinetic formulation of $u^\varepsilon(t, \mathbf{x})$ for (1.1) is

$$\partial_t \chi_\varepsilon + \mathbf{F}'_\varepsilon(\xi) \cdot \nabla_{\mathbf{x}} \chi_\varepsilon = \partial_\xi m_\varepsilon$$

with m_ε uniformly bounded in $\mathcal{M}_{loc}$ since u^ε are uniformly bounded in L^∞. We can rewrite the kinetic formulation as

$$\partial_t \chi_\varepsilon + \mathbf{F}'(\xi) \cdot \nabla_{\mathbf{x}} \chi_\varepsilon = \partial_\xi m_\varepsilon - \mathbf{G}'_\varepsilon(\xi) \cdot \nabla_{\mathbf{x}} \chi_\varepsilon.$$

Then, using Theorem 3.1 of [25] (pp. 124) (also [3, 22]), we get the compactness of the sequence $u^\varepsilon(t, \mathbf{x})$ in L^1_{loc}. Therefore, up to a subsequence, $u^\varepsilon \to w$ strongly in L^1_{loc}. Passing to the limit in the weak formulation of (1.1) and the entropy inequality, we find that, for any $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^n)$,

$$\int_0^\infty \int_{\mathbb{R}^n} (w \partial_t \varphi + \mathbf{F}(w) \cdot \nabla_{\mathbf{x}} \varphi)(t, \mathbf{x})\, dt d\mathbf{x} + \int_{\mathbb{R}^n} u_0(\mathbf{x}) \varphi(0, \mathbf{x})\, d\mathbf{x} = 0, \tag{2.10}$$

and

$$\partial_t |w - k| + div(sign(w - k)(\mathbf{F}(w) - \mathbf{F}(k))) \leq 0 \qquad \text{for any } k \in \mathbb{R} \tag{2.11}$$

in the sense of distributions. A priori, we could have lost the initial data in passing to the limit in the entropy inequalities. Now Vasseur's result in [29] indicates that any solution of (2.10)–(2.11) satisfying (2.8) has a strong trace on $t = 0$. Since the weak formulation of (1.1) implies that u_0 is a weak trace of w, we conclude that u_0 is the strong trace of w on $t = 0$, which implies $w \equiv u$ by the Krushkov's uniqueness theorem. Since the weak limit is unique, then the whole sequence $u^\varepsilon(t, \mathbf{x})$ converges to $u(t, \mathbf{x})$.

2.3. **L^1-Stability with respect to the Flux Functions.** We have

Lemma 2.7. Let $\mathbf{F} \in C^1(\mathbb{R}; \mathbb{R}^n)$. Let $u \in BV_{loc} \cap L^\infty(\mathbb{R}^n; \mathbb{R})$ and $v \in L^\infty(\mathbb{R}^n; \mathbb{R})$ be periodic entropy solutions with period P of the Cauchy problems:

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathbf{F}(u) = 0, \qquad u(0, \mathbf{x}) = u_0(\mathbf{x});$$
$$\partial_t v + \mathrm{div}_{\mathbf{x}} \mathbf{G}(v) = 0, \qquad v(0, \mathbf{x}) = v_0(\mathbf{x}),$$

respectively. Then, for any $0 < t \leq T$,

$$\int_P |u - v|(t, \mathbf{x}) d\mathbf{x} \leq \int_P |u_0 - v_0|(\mathbf{x}) d\mathbf{x} + LT |\nabla_{\mathbf{x}} u_0(\cdot)|_{\mathcal{M}(P)}.$$

where $L := \max_{1 \leq j \leq n}(\max\{|(\mathbf{F}'_j - \mathbf{G}'_j)(u)| \,:\, |u| \leq \max(\|u_0\|_\infty, \|v_0\|_\infty)\})$.

See [1, 26] for the non-periodic case and [19] for the periodic initial data with respect to one space variable. We can extend the proof of [19] to the case of periodic initial data with respect to each space variable.

We will use this lemma with initial data $u_0^\varepsilon := u_0(x_1/\varepsilon^{\alpha_1}, x_2/\varepsilon^{\alpha_2})$. If $u_0 \in BV(P)$ is periodic with period P, then $\partial_{x_1} u_0^\varepsilon$ is of order $1/\varepsilon^{\alpha_1}$ and $\partial_{x_2} u_0^\varepsilon$ is of order $1/\varepsilon^{\alpha_2}$ in the space $\mathcal{M}_{loc}(\mathbb{R}^2)$.

3. Scaling, L^1–Convergence, and Homogenization

In this section, we introduce new tools to validate nonlinear geometric optics for entropy solutions only in L^∞ of multidimensional conservation laws. These tools are about the changes of variables that preserve the L^1_{loc}–convergence, weak oscillating limits, and uniqueness of the profiles.

3.1. **Scaling and L^1-Convergence.** We first formulate the following useful lemma.

Lemma 3.1 (L^1-convergence of periodic functions and triangular scaling). Let $u^\varepsilon \in L^1(P; \mathbb{R})$ be a sequence of periodic functions with period $P = [0, 1]^n$. Let $A^\varepsilon = (a^\varepsilon_{ij})_{1 \leq i,j \leq n}$ be a sequence of lower triangular $n \times n$ matrices such that

$$\min_{1 \leq i \leq n} \liminf_{\varepsilon \to 0} |a^\varepsilon_{ii}| > 0. \tag{3.1}$$

Set $v^\varepsilon(\mathbf{x}) := u^\varepsilon(A^\varepsilon \mathbf{x})$. Then, when $\varepsilon \to 0$, $u^\varepsilon(\mathbf{x})$ converges strongly to 0 in $L^1_{loc}(\mathbb{R}^n)$ *if and only if* $v^\varepsilon(\mathbf{x})$ converges strongly to 0 in $L^1_{loc}(\mathbb{R}^n)$.

Proof: We first note the following facts:

1. For any $a \in \mathbb{R}$, the translation operator $\lambda \mapsto \varphi(\lambda) := \lambda + a$ is obviously one to one from $[0, 1) \equiv \mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$.

2. If $A = (a_{ij})_{1 \leq i,j \leq n}$ is an invertible and lower triangular $n \times n$ matrix, then

$$A \Pi_{1 \leq i \leq n} [0, 1/a_{ii}) = [0, 1)^n \qquad \text{in } (\mathbb{R}/\mathbb{Z})^n\,, \tag{3.2}$$

where $[0, a) := \{\lambda a \,:\, \lambda \in [0, 1)\}$ is independent of the sign of a, and A in (3.2) is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ associated with the matrix A.

This can be seen by induction on the dimension. If $n = 1$, the result is trivial. Now, assume that $n \geq 2$ and the result is true for $n-1$. Let B be the submatrix $(a_{ij})_{1\leq i,j\leq n-1}$. For any $X \in \Pi_{1\leq i\leq n}[0, 1/a_{ii})$ with $X = (\lambda_1/a_{11}, \cdots, \lambda_n/a_{nn})$ and $Y := (\lambda_1/a_{11}, \cdots, \lambda_{n-1}/a_{(n-1)(n-1)})$ for $\lambda_i \in [0, 1)$, then

$$AX = \begin{pmatrix} BY \\ \left(\sum_{i\leq n-1} \lambda_i a_{ni}/a_{ii}\right) + \lambda_n \end{pmatrix}.$$

Therefore, by induction, for $Z = (Z_1, \cdots, Z_n), Z_j \in (0, 1), j = 1, \cdots, n$, there exists a unique $Y \in \Pi_{1\leq i\leq n-1}[0, 1/a_{ii})$ such that $BY = (Z_1, \cdots, Z_{n-1})$. Now, fix Y and let $a := \sum_{i\leq n-1} \lambda_i a_{ni}/a_{ii}$. Using Fact 1, there exists a unique $\lambda_n \in [0, 1)$ such that $a + \lambda_n = Z_n$. Therefore, for any $Z \in [0, 1)^n$, there exists a unique $X \in \Pi_{1\leq i\leq n}[0, a_{ii}^{-1})$ such that $AX = Z$ in $(\mathbb{R}/\mathbb{Z})^n$.

We now use Facts 1 and 2 to complete the proof of Lemma 3.1. First, there exists $\eta > 0$ and $\delta > 0$ such that, when $\varepsilon \in (0, \eta)$,

$$\min_{1\leq i\leq n} |a^\varepsilon_{ii}| > \delta.$$

Choose $R > 1 + 1/\delta$ and $\Omega := (-R, R)^n$. Then, when $\varepsilon \in (0, \eta)$, we have

$$1/|a^\varepsilon_{ii}| < 1/\delta, \qquad [|a^\varepsilon_{ii}|R] > |a^\varepsilon_{ii}|,$$

since $[|a^\varepsilon_{ii}|R] > |a^\varepsilon_{ii}|R - 1$ and $|a^\varepsilon_{ii}|R - 1 > |a^\varepsilon_{ii}|$ from $R - 1 > \delta^{-1} > |a^\varepsilon_{ii}|^{-1}$, where $[a]$ is the integer part of a real number a such that $[a] \leq a < [a] + 1$. Furthermore, since $[|a^\varepsilon_{ii}|R] \leq R|a^\varepsilon_{ii}|$ and $R|a^\varepsilon_{ii}| > 1$, we have

$$[|a^\varepsilon_{ii}|R] + 1 < 2R|a^\varepsilon_{ii}|.$$

Set $B^\varepsilon := \Pi_{1\leq i\leq n}[0, 1/a^\varepsilon_{ii})$ and $k^\varepsilon := (k_i/a^\varepsilon_{ii})_{1\leq i\leq n}$ for any $k \in \mathbb{Z}^n$. Define

$$\begin{aligned} M^\varepsilon &:= \#\{k \in \mathbb{Z}^n : (k^\varepsilon + B^\varepsilon) \cap \overline{\Omega} \neq \emptyset\} < \Pi_{1\leq i\leq n}(2([|a^\varepsilon_{ii}|R] + 1)) < 4^n R^n |detA^\varepsilon|, \\ N^\varepsilon &:= \#\{k \in \mathbb{Z}^n : k^\varepsilon + B^\varepsilon \subset \overline{\Omega}\} \geq \Pi_{1\leq i\leq n}(2[|a^\varepsilon_{ii}|R]) > 2^n |detA^\varepsilon|. \end{aligned}$$

Fact 2 implies that

$$A^\varepsilon(k^\varepsilon + B^\varepsilon) = A^\varepsilon k^\varepsilon + A^\varepsilon B^\varepsilon = [0, 1)^n \quad \text{in } (\mathbb{R}/\mathbb{Z})^n.$$

Then we have

$$\begin{aligned} \int_\Omega |v^\varepsilon(\mathbf{x})| d\mathbf{x} \qquad &\leq M^\varepsilon \int_{B^\varepsilon} |u^\varepsilon(A^\varepsilon \mathbf{x})| d\mathbf{x} \leq \frac{M^\varepsilon}{|detA^\varepsilon|} \int_{(0,1)^n} |u^\varepsilon(\mathbf{y})| d\mathbf{y} \\ &\leq (4R)^n \int_{(0,1)^n} |u^\varepsilon(\mathbf{y})| d\mathbf{y}, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |v^\varepsilon(\mathbf{x})| d\mathbf{x} \qquad &\geq N^\varepsilon \int_{B^\varepsilon} |u^\varepsilon(A^\varepsilon \mathbf{x})| d\mathbf{x} \geq \frac{N^\varepsilon}{|detA^\varepsilon|} \int_{(0,1)^n} |u^\varepsilon(\mathbf{y})| d\mathbf{y} \\ &\geq 2^n \int_{(0,1)^n} |u^\varepsilon(\mathbf{y})| d\mathbf{y}, \end{aligned}$$

which concludes the proof.$\square$

Remark 3.1. In the proof of this lemma, we have also established the following useful inequalities: If A is a lower invertible triangular $n \times n$ matrix, $\delta := \min_k |A_{kk}| > 0$, and $u \in L^1(\mathbb{R}^n; \mathbb{R})$ is a 1-periodic function in each variable, then, for any $R > 1 + 1/\delta$,

$$2^n \int_{(0,1)^n} |u(y)|dy \le \int_{(-R,R)^n} |u(Ax)|dx \le (4R)^n \int_{(0,1)^n} |u(y)|dy. \tag{3.3}$$

Remark 3.2. In Lemma 3.1, condition (3.1) is *necessary* for preserving the strong L^1_{loc}-convergence in the rescaled triangular change of variables $y := A^\varepsilon x$. For instance, we choose

$$w(x_1, x_1) = \sin(x_1 - x_2),$$

and then

$$u^\varepsilon(x_1, x_2) := w(x_1/\varepsilon, x_1/\varepsilon + \varepsilon x_2) \to 0 \qquad \text{in } L^1_{loc},$$

but

$$w(x_1/\varepsilon, x_2/\varepsilon)$$

does not converge to 0 in L^1_{loc} when $\varepsilon \to 0$.

3.2. **Weak Oscillating Limits in $L^2_{loc}(\mathbb{R}^n; \mathbb{C})$.** In this Section as well as in Section 3.3, we are essentially going to use Fourier analysis. Therefore, it is more convenient to deal with complex-valued functions. We want to study weak limits of sequences of oscillating data, with mixed scalings of space variables through a lower triangular matrix T^ε. This result will be used in Section 5, especially in Theorem 5.1, where the matrix L^ε with entries $(L^\varepsilon)_{ij} = \varepsilon^{\mu_j - \beta_i} L_{ij}$ satisfy all the conditions below (see Section 5).

Let T^ε be a lower *triangular* $n \times n$ *matrix* for $0 < \varepsilon \le 1$ such that

$$\min_{1 \le k \le n} \left(\liminf_{\varepsilon \to 0} |T^\varepsilon_{kk}| \right) > 0 \tag{3.4}$$

which, moreover, can be decomposed into two lower triangular matrices: the constant part T_0 and the oscillating part T^ε_1:

$$T^\varepsilon = T_0 + T^\varepsilon_1 \tag{3.5}$$

satisfying that, for all $1 \le p, q \le n$,

$$(T_0)_{p,q} \ne 0 \Longrightarrow (T^\varepsilon_1)_{p,q} = 0, \tag{3.6}$$

$$(T^\varepsilon_1)_{p,q} = t_{p,q}/\varepsilon^{\alpha_{p,q}} \qquad \text{for } \alpha_{p,q} > 0, t_{p,q} \in \mathbb{R}. \tag{3.7}$$

We exclude the case $T^\varepsilon_1 \equiv 0$, which is a trivial case: no oscillation.

Therefore, we can rewrite T^ε_1:

$$T^\varepsilon_1 = \sum_{k=1}^{m} A_k/\varepsilon^{\gamma_k}, \tag{3.8}$$

where m is a positive integer, $\gamma_1 > \gamma_2 > \cdots > \gamma_m$, and all A_k are nonzero lower triangular matrices. Define

$$K := \bigcap_{k=1}^{m} Ker(\,^tA_k) = \bigcap_{\varepsilon > 0} Ker(\,^tT^\varepsilon_1),$$

where we denote ^tA as the transposed matrix of A. Then, for any $\alpha \in \mathbb{R}^n$, there are only two cases:

(i). $\alpha \in K \Longrightarrow {}^tT_1^\varepsilon \alpha = 0$ for all $\varepsilon > 0$;
(ii). $\alpha \notin K \Longrightarrow \lim_{\varepsilon \to 0} \| {}^tT_1^\varepsilon \alpha\| = \infty$ for the norm $\|\cdot\|$ in $\mathbb{R}^n$.
Furthermore, studying (3.8) with respect to ε^{-1}, we find that, for sufficiently small ε,

$$K = Ker({}^tT_1^\varepsilon).$$

We will use the following Hilbert space of 1-periodic functions in each space variable: $L_p^2 \simeq L^2((0,1)^n;\mathbb{C})$. On L_p^2, we define the orthogonal projection:

$$\begin{array}{llll} \widetilde{P}: & L_p^2 & \mapsto & L_p^2 \\ & \exp(2\pi i\alpha\cdot x) & \mapsto & \begin{cases} \exp(2\pi i\alpha\cdot x) & \text{if } \alpha \in K, \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{3.9}$$

We now state the following result, which will be used in Section 5. This result is classical when the functions are smooth.

Lemma 3.2 (Oscillating sequences in L_p^2). Let $u \in L_p^2$, i.e., u be periodic with period 1 in each space variable. Let T^ε be a family of lower triangular $n \times n$ matrices satisfying (3.4)–(3.7). Then, with $\widetilde{u} := \widetilde{P}(u)$, we have

$$u\,(T^\varepsilon x) \rightharpoonup \widetilde{u}(T_0 x) \qquad \text{when } \varepsilon \to 0,$$

where $\widetilde{u}(T_0x)$ is well defined even if T_0 is degenerate. Furthermore, if $u^\varepsilon \in L_p^2$ and converges strongly to u in L^1_{loc} when $\varepsilon \to 0$, we also have

$$u^\varepsilon\,(T^\varepsilon x) \rightharpoonup \widetilde{u}(T_0 x).$$

Proof: The proof is divided into four steps.

Step 1. $\widetilde{u}(T_0x)$ *is well defined almost everywhere.* Let Π_K be the orthogonal projection from $\mathbb{R}^n$ to K. Since the spectrum of $\widetilde{u}$ is included in K, that is,

$$\widetilde{u}(x) = \widetilde{u}(\Pi_K(x)), \tag{3.10}$$

then, $\widetilde{u}(T_0x)$ is well defined a.e. if and only if $K = \Pi_K(R(T_0))$, where $R(T_0)$ is the range of T_0. From assumptions (3.4)–(3.7), we easily obtain

$$Ker({}^tT_0) \cap Ker({}^tT_1^\varepsilon) = \{0\}, \quad \dim Ker({}^tT_0) + \dim Ker({}^tT_1^\varepsilon) = n; \tag{3.11}$$

$$Ker({}^tT_0) \oplus Ker({}^tT_1^\varepsilon) = \mathbb{R}^n; \tag{3.12}$$

$$\dim R(T_0) = \dim Ker({}^tT_1^\varepsilon). \tag{3.13}$$

Furthermore, for small ε, $K = Ker({}^tT_1^\varepsilon)$. Then, from equality (3.13), we get $dim K = \dim(R(T_0))$. Thus, we have

$$K^\perp \cap R(T_0) = \{0\} \Longleftrightarrow K = \Pi_K(R(T_0)).$$

Now, from (3.12), $\{0\} = (Ker({}^tT_0) \cup K)^\perp = (Ker({}^tT_0))^\perp \cap K^\perp = R(T_0) \cap K^\perp$, and the result follows.

Step 2. We now first prove the weak convergence result for polynomial trigonometric functions. Let $\mathcal{TP}_1$ be the linear space of trigonometric polynomials in $I\!R^n$ with period 1 in each variable, that is, $\mathcal{TP}_1 := span\{\exp(2\pi i\alpha\cdot x) \;:\; \alpha \in Z\!\!Z^n\}$. Take $u(x) := \exp(2\pi i\alpha\cdot x)$. Then

$$u(T^\varepsilon x) := \exp(2\pi i\alpha\cdot(T_0x)) \times \exp(2\pi i\alpha\cdot(T_1^\varepsilon x)).$$

We have two cases:

(i). If $\alpha \in K$, then we get ${}^tT_1^\varepsilon\alpha \equiv 0$, which implies

$$u(T^\varepsilon x) := \exp(2\pi i\alpha\cdot(T_0x)) = \widetilde{u}(T_0x).$$

(ii). If $\alpha \notin K$, then $\| \,{}^tT_1^\varepsilon \alpha\| \to \infty$ when $\varepsilon \to 0$, which implies

$$\exp(2\pi i\alpha \cdot (T_1^\varepsilon x)) = \exp(2\pi i(\,{}^tT_1^\varepsilon \alpha) \cdot x) \rightharpoonup 0,$$

and thus

$$u(T^\varepsilon x) \rightharpoonup 0 = \widetilde{u}(T_0 x) \qquad \text{when } \varepsilon \to 0.$$

By linearity, Lemma 3.2 is also true for any function in $\mathcal{TP}_1$.

Step 3. We now conclude by the density of $\mathcal{TP}_1$ in L_p^2 thanks to Step 1. Let $u \in L_p^2$. Then, for any small $\delta > 0$, there exists $v \in \mathcal{TP}_1$ such that

$$\int_{(0,1)^n} |u - v|^2(x)dx < \delta^2.$$

For any $\phi \in C_0^\infty(\mathbb{R}^n)$, we now prove that the following quantity is small:

$$\begin{aligned} \left|\int_{\mathbb{R}^n} (u(T^\varepsilon x) - \widetilde{u}(T_0 x))\phi(x)dx\right| \quad &\le \quad \left|\int_{\mathbb{R}^n} (u - v)(T^\varepsilon x)\phi(x)dx\right| \\ &\quad + \left|\int_{\mathbb{R}^n} (v(T^\varepsilon x) - \widetilde{v}(T_0 x))\phi(x)dx\right| \\ &\quad + \left|\int_{\mathbb{R}^n} (\widetilde{v}(T_0 x) - \widetilde{u}(T_0 x))\phi(x)dx\right| \\ &:= \quad A^\varepsilon + B^\varepsilon + C^\delta. \end{aligned}$$

Indeed, let $B := (-R, R)^n$ be a bounded set containing the compact support of ϕ and $\delta > 0$ be any small constant.

First, for ε small enough, $A^\varepsilon < C_1\delta$ where the constant C_1 depends only on $\min_k(\liminf_{\varepsilon\to 0} |T_{kk}^\varepsilon|)$ and ϕ, since the L^2-norm on $(u - v)$ over a period $(0,1)^n$ can be controlled by using Lemma 3.1 (see Remark 3.1).

Second, $B^\varepsilon < \delta$ for ε small enough, since $v \in \mathcal{TP}_1$.

Third, since $\widetilde{P}$ is an orthogonal projection, we have

$$\int_{(0,1)^n} |\widetilde{u} - \widetilde{v}|^2(x)dx \le \int_{(0,1)^n} |u - v|^2(x)dx < \delta^2.$$

Setting $w := \widetilde{u} - \widetilde{v}$, then the L^1-norm of w is less than any δ on the unit square $P := (0,1)^n$. We need to compute the L^1-norm of $w(T_0 x)$ on the bounded subset B of $\mathbb{R}^n$. Since w admits a trace on K, we see that

$$C_\delta < C_2\delta,$$

where C_2 comes from the Fubini Theorem and the change of variables on $K \cap B$. Therefore, the sum of these three terms is less than $(C_1 + 1 + C_2)\delta$ for ε small enough, with constants C_1 and C_2 independent of ε. This concludes the proof of this step.

Step 4. Now, if u^ε converges to u strongly in L^1_{loc} when $\varepsilon \to 0$, we find from Lemma 3.1 that

$$u^\varepsilon(T^\varepsilon x) - u(T^\varepsilon x) \to 0$$

strongly in L^1_{loc}, which concludes the proof. □

3.3. **Uniqueness of the Profiles.** In Section 5, we will use an algorithm which defines some profiles. In this subsection, we provide some tools to prove that the profile is *unique*, and therefore is *independent* of the particular chosen triangulation. We first recall some basic facts on the class of almost periodic functions.

3.3.1. *Almost Periodic Functions.* We first introduce

$$\mathcal{TP} := span\{\exp(i\alpha \cdot x) : \alpha \in I\!\!R^n\}$$

that is the linear space of trigonometric polynomials in $I\!\!R^n$. For any measurable set $Q \subset I\!\!R^n$ with $meas(Q) = |Q| > 0$ and any $f, g \in \mathcal{TP}$, we define the natural scalar product on $\mathcal{TP}$:

$$\langle f, g \rangle := \lim_{T \to \infty} \frac{1}{T^n |Q|} \int_{T.Q} f(x) \overline{g(x)} dx, \qquad \langle f \rangle := \langle f, 1 \rangle ,$$

where $T.Q := \{Tq : q \in Q\}$.

It is well known that $\langle f \rangle$ is independent of the choice of Q. In particular, this property implies the scale invariance of the mean:

$$A \in LG_n(I\!\!R) \Longrightarrow \langle f(Ax) \rangle = \langle f(x) \rangle , \tag{3.14}$$

where $LG_n(I\!\!R)$ is the linear group of invertible $n \times n$ matrices. We use the usual norm associated with this scalar product: $\|f\|_{ap}^2 := \langle f, f \rangle$ and the natural Hilbert space L^2_{ap}. We use C^0_{ap} and L^1_{ap} to denote the closure of $\mathcal{TP}$ associated with the L^∞-norm and the L^1-almost periodic norm: $\|f\|_{1,ap} := \langle |f| \rangle$, respectively. For any $U \in L^2_{ap}$, we define the spectrum of U:

$$Sp[U] := \{\alpha \in I\!\!R^n : c_\alpha[U] := \langle U(x), \exp(i\alpha \cdot x) \rangle \neq 0\}.$$

Then the spectrum of each U is countable, and U satisfies Parseval's equality:

$$\|U\|_{ap}^2 = \sum_{\alpha \in Sp[U]} |c_\alpha[U]|^2 .$$

Denote by $L^2_p((0,1)^n)$ the classical set of 1-periodic functions in each space variable. We recall that a prototype of quasi-periodic functions is a function v such that there exists a matrix M and a periodic function u such that $v(x) = u(Mx)$. Note that all periodic functions are quasi-periodic but the converse is false and, similarly, all quasi-periodic functions are almost-periodic but the converse is also false. Also, for $u \in L^1_p((0,1)^n)$, we have

$$\langle |u| \rangle = \int_{(0,1)^n} |u(x)| dx. \tag{3.15}$$

For more details, see [8].

3.3.2. *Uniqueness of the Profiles.* We now show the uniqueness of the profiles.

Lemma 3.3. Let $U, V \in L^\infty_p((0,1)^n; \mathbb{C})$. Set

$$d_U := \dim span\{Sp[U]\}, \qquad d_V := \dim span\{Sp[V]\}.$$

Assume that

(i). $U(A^\varepsilon x) = V(B^\varepsilon x) + R^\varepsilon(x)$,
(ii). $A^\varepsilon, B^\varepsilon \in C^0((0,1); LG_n(\mathbb{R}))$,
(iii). $\lim_{\varepsilon \to 0} \langle |R^\varepsilon(x)| \rangle = 0$.

Then there exists a matrix C such that

$$U(y) = V(Cy) \qquad a.e. \text{ in } y. \tag{3.16}$$

More precisely, $rank\, C = d_U$ and, if $n > d_U$, U admits a trace on $span\{Sp[U]\}$ and V a trace on $span\{Sp[V]\}$ so that equality (3.16) is satisfied by these traces.

Proof. First, we rescale:

$$y = A^\varepsilon x, \qquad C^\varepsilon = B^\varepsilon (A^\varepsilon)^{-1}, \qquad \tilde{R}^\varepsilon(y) = R^\varepsilon(x),$$

and use the invariance of the mean to have

$$U(y) = V(C^\varepsilon y) + \tilde{R}^\varepsilon(y) \quad \text{ and } \quad \lim_{\varepsilon\to 0} \left\langle |\tilde{R}^\varepsilon(x)| \right\rangle = 0, \tag{3.17}$$

where

$$V(C^\varepsilon y) = \sum_{\beta\in Sp[V]} c_\beta[V] \exp\left(i\beta\cdot(C^\varepsilon y)\right) \quad \text{ in } L^2_{ap}.$$

Now, for any $\alpha \in Sp[U]$, define

$$\delta := |c_\alpha[U]|/2 > 0, \qquad I := \{\beta : \quad |c_\beta[V]| > \delta\}.$$

Then I is a *finite* set since $\displaystyle\sum_{\alpha\in Sp[U]} |c_\alpha[U]|^2 < \infty$. Using equality (3.17), we can calculate $\langle U(y), \exp(i\alpha\cdot y)\rangle$ to obtain

$$c_\alpha[U] = c^\varepsilon_\beta[V] + r^\varepsilon, \tag{3.18}$$

where $\lim_{\varepsilon\to 0} r^\varepsilon(x) = 0$ and $\alpha = {}^tC^\varepsilon \beta^\varepsilon$, that is, $\beta^\varepsilon = ({}^tC^\varepsilon)^{-1}\alpha$ since C^ε is one to one. Take now $\eta > 0$ such that, when $\varepsilon < \eta$, $|r^\varepsilon| < \delta$. Thus, if $\varepsilon < \eta$, β^ε must be in I. Since β^ε is continuous with respect to ε and I is finite, then β^ε must be a constant when $\varepsilon < \eta$.

Furthermore, if $d_U = n$, taking $\{\alpha^1, \cdots, \alpha^n\} \in Sp[U]$ which is a basis of $\mathbb{R}^n$, for ε small enough, we obtain $\beta^1, \cdots, \beta^n$ such that $\alpha^k = {}^tC^\varepsilon\beta^k$ for all k. Therefore, for sufficiently small ε, C^ε becomes a constant C. Then we rewrite equality (3.17) as follows:

$$U(y) = V(Cy) + \tilde{R}^\varepsilon(y).$$

Passing to the limit in ε yields the conclusion for the generic case $n = d_U$, i.e., U depends on each variable in y.

In the case $d_U < n$, we use the previous case and choose $\{\alpha^1, \cdots, \alpha^{d_U}\} \subset Sp[U]$ as a base of $span\{Sp[U]\}$ to obtain

$${}^t((C^\varepsilon)^{-1}) Sp[U] \subset Sp[V].$$

By symmetry in (3.17), the converse inclusion is also true. Therefore,

$$Sp[U] = {}^tC^\varepsilon Sp[V] \qquad \text{for small } \varepsilon > 0,$$

and again ${}^tC^\varepsilon$ is constant on $span\{Sp[V]\}$. We also have relation (3.18) and then the equality among the Fourier coefficients of each profile. Therefore, $d_U = d_V$ and V depends only on the variables in $span\{Sp[V]\}$. Notice that, at the limit, ${}^tC^\varepsilon$ **only** needs be constant on $span\{Sp[V]\}$, not necessarily on the whole space, see Remark 5.3. This completes the proof. ☐

4. Validity of Nonlinear Geometric Optics in L^∞: 1-D Case

For the one-dimensional case with $f \in C^3$, without loss of generality, we can take $\phi = x$ and consider the following Cauchy problem:

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = 0, \qquad u|_{t=0} = \underline{u} + \varepsilon u_1(x/\varepsilon^\alpha), \tag{4.1}$$

where $u_1 \in L^\infty$ is periodic with period $P = [0,1]$.

First, from Lemmas 2.1 and 2.2, we have

$$\|v^\varepsilon\|_{L^\infty} \le \|u_1\|_{L^\infty} < \infty,$$

and $v^\varepsilon(t,x)$ is the entropy solution of the Cauchy problem:

$$\partial_t v + \varepsilon \partial_x (a v^2) = \varepsilon^2 \partial_x R(v, \underline{u}, \varepsilon), \qquad v|_{t=0} = u_1(x/\varepsilon^\alpha), \tag{4.2}$$

with $a = f''(\underline{u})/2$ and

$$R(v, \underline{u}, \varepsilon) = \frac{1}{2}\int_0^1 (1-\theta)^2 f'''(\underline{u} + \varepsilon\theta v)\, d\theta\, v^3, \tag{4.3}$$

which is a Lipschitz function in $v, \underline{u}$, and ε. Then we have the following theorem.

Theorem 4.1. Let $f \in C^3$.

(i). If $\alpha = 1$, then

$$\int_0^1 |u^\epsilon(t,x) - \underline{u} - \epsilon\sigma(t, x/\epsilon)| dx = o(\varepsilon),$$

where the profile $\sigma(t,x)$ is uniquely determined by the Cauchy problem for the inviscid Burgers equation:

$$\partial_t \sigma + a\partial_X \sigma^2 = 0, \qquad \sigma|_{t=0} = u_1(X),$$

which is the validity of classical weakly nonlinear geometric optics.

(ii). If $\alpha < 1$, then

$$\int_0^1 |u^\epsilon(t,x) - \underline{u} - \epsilon u_1(x/\epsilon^\alpha)| dx = o(\varepsilon),$$

which means that the slow initial oscillation propagates linearly.

(iii). If $\alpha > 1$, then

$$\int_0^1 |u^\epsilon(t,x) - \underline{u}| dx = o(\varepsilon),$$

which means that the fast initial oscillation is canceled by the nonlinearity of the flux function.

Proof. We now prove this theorem in the three cases, separately.

1. Case $\alpha = 1$. Consider the following perturbation problem:

$$\partial_t V + a\partial_X V^2 = \varepsilon \partial_X R(V, \underline{u}, \varepsilon), \qquad V|_{t=0} = u_1(X). \tag{4.4}$$

We want to prove that, as $\varepsilon \to 0$, the solution sequence V^ε of (4.4) is determined by the profile σ governed by the Cauchy problem of the inviscid Burgers equation:

$$\partial_t \sigma + a\partial_X \sigma^2 = 0, \qquad \sigma|_{t=0} = u_1(X) \in L^\infty \text{ periodic with period } P. \tag{4.5}$$

Notice that the unique solution $\sigma(t,X)$ of (4.5) is in $BV(\mathbb{R}^2_+)$ although $u_1(X)$ is only L^∞. We now divide three steps to prove this fact.

Step 1. Since V^ε is an entropy solution of (4.4) for any fixed $\varepsilon > 0$, then, for any $\eta \in C^2, \eta'' \geq 0$, we conclude that

$$\partial_t \eta(V^\varepsilon) + \partial_X q(V^\varepsilon) - \varepsilon \partial_X \Big(\int^{V^\varepsilon} \eta'(\xi) R_\xi(\xi, \underline{u}, \varepsilon) d\xi \Big)$$

is a nonpositive, uniformly bounded Radon measure sequence. This implies that

$$\partial_t \eta(V^\epsilon) + \partial_X q(V^\epsilon)$$

is compact in $H^{-1}_{loc}(\mathbb{R}^2_+)$ from Lemma 2.1 and Murat's Lemma [24] with the aid of the argument in Chen-Frid [2]. Then, using the compactness lemma (Lemma 2.4) for scalar conservation laws, we conclude that there exists $\sigma(t, X)$ such that

$$V^\varepsilon(t, X) \to \sigma(t, X) \qquad a.e.$$

and $\sigma(t, X)$ is uniquely determined by (4.5). Then we have

$$\int_0^t \int_0^1 |V^\varepsilon(\tau, X) - \sigma(\tau, X)| dX d\tau \to 0 \qquad \text{as } \varepsilon \to 0. \tag{4.6}$$

Step 2. Notice that $|R'_v(V, \underline{u}, \varepsilon)| \leq C$. Then, using Lemma 2.7, we have from (4.4) that, for $0 < t < T < \infty$,

$$\int_0^1 |V^\varepsilon(t, X) - \sigma(t, X)| dX \leq \varepsilon C T |\partial_x u_1|_{\mathcal{M}(P)}. \tag{4.7}$$

Using a standard mollifier to smooth u_1 such that $\varepsilon \partial_x u_1^\varepsilon \to 0$ in L^1_{loc}, we conclude that, for all $t \in (0, \infty)$,

$$\Delta(\varepsilon, t) := \int_0^1 |V^\varepsilon(t, X) - \sigma(t, X)| dX \to 0 \qquad \text{as } \varepsilon \to 0. \tag{4.8}$$

Step 3. Now we return to our problem for v^ε of (4.2) with the aid of the result (4.8) for (4.4)–(4.5). For fixed $\varepsilon > 0$, we assume that $V^\varepsilon(t, X)$ is the solution of the Cauchy problem (4.4) and set $X = x/\varepsilon$. By uniqueness, we have

$$V^\varepsilon(t, X) = v^\varepsilon(t, x).$$

Then we have

$$\Delta(\varepsilon, t) := \frac{1}{\varepsilon} \int_0^\varepsilon |v^\varepsilon(t, x) - \sigma(t, x/\varepsilon)| dx. \tag{4.9}$$

Notice that, for any nonnegative periodic function $h(x)$ with period P,

$$\begin{aligned} \int_0^1 h(x/\varepsilon) dx &= \left(\int_0^\varepsilon + \int_\varepsilon^{2\varepsilon} + \cdots + \int_{([\frac{1}{\varepsilon}]-1)\varepsilon}^{[\frac{1}{\varepsilon}]\varepsilon} + \int_{[\frac{1}{\varepsilon}]\varepsilon}^1 \right) h(x/\varepsilon) dx \\ &\leq \frac{2}{\varepsilon} \int_0^\varepsilon h(x/\varepsilon) dx. \end{aligned}$$

Then we conclude from (4.8) that

$$\int_0^1 |u^\varepsilon(t, x) - \underline{u} - \varepsilon \sigma(t, x/\varepsilon)| dx = o(\varepsilon). \tag{4.10}$$

This validates the weakly nonlinear geometric optics.

2. Case $\alpha < 1$. Similarly, we consider the following Cauchy problem:

$$\partial_t V + a \varepsilon^{1-\alpha} \partial_X V^2 = \varepsilon^{2-\alpha} \partial_X R(V, \underline{u}, \varepsilon), \qquad V|_{t=0} = u_1(X). \tag{4.11}$$

Then we want to prove that the solution sequence $V^\varepsilon(t,X)$ of (4.11) is determined as $\varepsilon \to 0$ by the profile $\sigma(t,x)$ solution to

$$\partial_t \sigma = 0, \qquad \sigma|_{t=0} = u_1(X), \tag{4.12}$$

that is,

$$\sigma(t,X) = u_1(X). \tag{4.13}$$

Choose $\delta = \delta(\varepsilon)$ such that $\varepsilon^{1-\alpha}/\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Define

$$\sigma^\delta(t,X) = \sigma * \rho_\delta(X),$$

where ρ_δ is the standard symmetric mollifier. Then σ^δ is the solution of

$$\partial_t \sigma = 0, \qquad \sigma|_{t=0} = u_1^\delta(X).$$

Using Lemma 2.1, we conclude

$$\int_0^1 |V^\varepsilon - \sigma^\delta|(t,X)dX \le \int_0^1 |u_1(X) - u_1^\delta(X)|dX + C\varepsilon^{1-\alpha} t \int_0^1 |\partial_X u_1^\delta(X)|dX,$$

and hence

$$\begin{aligned} &\int_0^1 |V^\varepsilon - \sigma|(t,X)dX \\ &\le \int_0^1 |\sigma^{\delta(\varepsilon)}(X) - \sigma(X)|dX + \int_0^1 |u_1^{\delta(\varepsilon)}(X) - u_1(X)|dX + o_\varepsilon(1) \le o_\varepsilon(1). \end{aligned} \tag{4.14}$$

Now we return to our problem for v^ε of (4.2) with the aid of the result (4.14) for (4.11)–(4.13). For fixed $\varepsilon > 0$, assume that $V^\varepsilon(t,X)$ is the solution of the Cauchy problem (4.11) and set $X = x/\varepsilon^\alpha$. Then the same argument as in the case $\alpha = 1$ yields

$$\int_0^1 |u^\varepsilon(t,x) - \underline{u} - \varepsilon u_1(x/\varepsilon^\alpha)|dx = o(\varepsilon) \qquad \text{as } \varepsilon \to 0, \tag{4.15}$$

where we used Lemmas 3.1. This means that the initial oscillation propagates.

3. Case $\alpha > 1$. For this case, consider the following Cauchy problem:

$$\partial_t V + a\partial_X V^2 = \varepsilon \partial_X R(V, \underline{u}, \varepsilon), \quad V|_{t=0} = u_1(\frac{X}{\varepsilon^{\alpha-1}}) \overset{*}{\rightharpoonup} 0 = \langle u_0 \rangle \quad \text{in } L^\infty. \tag{4.16}$$

We want to prove that the solution sequence V^ε of (4.16) is determined as $\varepsilon \to 0$ by the profile $\sigma = \sigma(t,X)$ governed by

$$\partial_t \sigma + a\partial_X \sigma^2 = 0, \qquad \sigma|_{t=0} = 0, \tag{4.17}$$

that is,

$$\sigma(t,X) = 0. \tag{4.18}$$

Using Lemma 2.6, we conclude

$$V^\varepsilon(t,X) \to \sigma(t,X) \equiv 0 \qquad \text{as } \varepsilon \to 0. \tag{4.19}$$

Now we return to our problem for v^ε of (4.2) with the aid of the result (4.19) for (4.16)–(4.18). For fixed $\varepsilon > 0$, assume that $V^\varepsilon(t,X)$ is the solution of the Cauchy problem (4.16) and set $X = x/\varepsilon$. Then, combining the same argument as in the case $\alpha = 1$ with Lemmas 3.1 and 2.7 yields

$$\int_0^1 |u^\varepsilon(t,x) - \underline{u}|dx = o(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{4.20}$$

This means that the initial oscillation is canceled.

5. Validity of Nonlinear Geometric Optics in L^∞: Multi-D Case

In this section we turn to the multidimensional case to analyze further the nonlinear geometric optics.

Let $u := u^\varepsilon$ be the Krushkov solution of (1.1)–(1.2). Consider the geometric optics asymptotic expansion of the solution (1.4). Then the new approach in Section 4 for one-dimensional conservation laws requires further refinement for solving the general nonlinear geometric optics for multidimensional scalar conservation laws. We need a general scaling of variables to recover all the numerous cases. We will perform that with a "quasi" LU factorization depending on the magnitude of all frequencies. We will also use Lemma 3.1 to preserve the L^1_{loc}-convergence after a triangular scaling of variables which depends on ε.

We first recall that, under the one-to-one *constant* linear change of coordinates $\Phi := J\mathbf{x} = (\phi_1, \cdots, \phi_n)$ determined by (1.3), equation (1.1) can be rewritten in the weak form in the variable (t, Φ) as

$$\partial_t \widetilde{u}^\varepsilon + \operatorname{div}_\Phi(J\mathbf{F}(\widetilde{u}^\varepsilon)) = 0, \tag{5.1}$$

where the Jacobian matrix $J = \frac{D\Phi}{D\mathbf{x}}$ is constant. We also assume that the nonlinear flux matrix

$$N := \begin{pmatrix} F_1^{(2)}(\underline{u})/2! & \cdots & & F_1^{(n+1)}(\underline{u})/(n+1)! \\ \vdots & & F_i^{(j+1)}(\underline{u})/(j+1)! & \vdots \\ F_n^{(2)}(\underline{u})/2! & \cdots & & F_n^{(n+1)}(\underline{u})/(n+1)! \end{pmatrix} \tag{5.2}$$

is invertible. The invertibility of matrix $J = \frac{D\Phi}{D\mathbf{x}}$ expresses the linear independence of the phases $(\phi_i)_{1\le i\le n}$, while the invertibility of matrix N in (5.2) is an assumption of genuine nonlinearity and "genuine multidimensionality".

The possibility that the initial oscillations with high frequency propagate for (1.1) depends on the magnitude indices $\alpha := (\alpha_1, \cdots, \alpha_n)$ of ε in (1.2) and on the matrix:

$$M = JN. \tag{5.3}$$

With our choice, the matrix M expresses the flux in the Φ–coordinates: The $j^{\underline{th}}$ column corresponds to the term $\varepsilon^j v_\varepsilon^{j+1}$ in the Taylor expansion of $\mathbf{F}(u^\varepsilon) := \mathbf{F}(\underline{u} + \varepsilon v_\varepsilon)$; while the $i^{\underline{th}}$ row corresponds to the derivatives with respect to the $i^{\underline{th}}$ phase ϕ_i (see (5.6) below).

In this Section, we want to find the profile σ and to show the convergence of v_ε to σ after a suitable *triangularization* of the matrix M, replacing each variable ϕ_i with $\phi_i/\varepsilon^{\mu_i}$ for a suitable exponent μ_i. In order to preserve the L^1_{loc}-convergence of v_ε to σ in the rescaled coordinates and to go back in the original space coordinates, all the exponents μ_i must be *nonnegative*, see Lemma 3.1. Therefore, in this general multidimensional case, we need a triangularization (up to a permutation) of matrix M, combined with a suitable scaling. In the statement of Theorem 5.1 below, the profiles are defined in the "final" variables

$$\mathbf{X} := (SLE^\varepsilon)^{-1}\Phi,$$

where S is a permutation (substitution) matrix, L a lower triangular matrix (almost) as in the LU decomposition of any $n \times n$ matrix, and the rescaling diagonal matrix E^ε satisfies

$$E^\varepsilon_{ii} = \varepsilon^{\mu_i} := \varepsilon^{\min(\gamma_i, \beta_i)},$$

for which the precise notations are given below. Now there are two cases.

Case 1. Single phase in (1.2): $u_1 = u_1(\varepsilon^{-\alpha_1}\phi_1)$. In this case, consider the *first integer* $\gamma_1 \geq 1$ such that $\nabla_{\mathbf{x}}\phi_1 \cdot F^{(1+\gamma_1)}(\underline{u}) \neq 0$. As in the previous sections, we say that the corresponding oscillation is *linear* if $\gamma_1 > \alpha_1$ and is *nonlinear* if $\gamma_1 \leq \alpha_1$. In the latter case, if $\gamma_1 < \alpha_1$, the fast oscillations are canceled by the nonlinearity of the flux function, whereas the case $\gamma_1 = \alpha_1$ ("WNLGO") corresponds the particular case $\gamma_1 = \alpha_1 = 1$ to the classical weakly nonlinear geometrical optics ($\alpha_1 = 1$) for the Burgers equation ($\gamma_1 = 1$) since $\nabla_{\mathbf{x}}\phi_1 \cdot F^{(2)}(\underline{u}) \neq 0$.

Case 2. Multiple phases in (1.2): This situation is of course much more complicated. For instance, if $\gamma_1 < \alpha_1$, not only the corresponding oscillation is canceled, but also it can interact with the other oscillations, see examples below, just after Theorem 5.1.

The structure of the final form of the matrix in Lemma 5.1 below reflects the partition between these two different cases: the last rows ($m < i \leq n$) correspond to the *linear* oscillations, whereas the first rows ($1 \leq i \leq m$) correspond to the *nonlinear* oscillations. More precisely, as in the proof of Lemma 5.1, at each step k, there are three sets of indices E_k, F_k, and G_k. In the "final" coordinates $\mathbf{X}$, the set E_k corresponds (among the remaining coordinates) to the *nonlinear* oscillations, due to the term $\varepsilon^k v_\varepsilon^{k+1}$ in the Taylor expansion of $\mathbf{F}(u^\varepsilon) = \mathbf{F}(\underline{u} + \varepsilon v_\varepsilon)$; in contrast, the set G_k corresponds to the *linear* oscillations, whereas the set F_k correspond to the "fast" oscillations in the directions that are *orthogonal* to $\mathbf{F}^{(k+1)}(\underline{u})$, which therefore do not play a role in this step.

Thus, in order to extend the results of Section 4 to the general multidimensional case, we need the following variant of LU-type factorization.

Lemma 5.1 (LU-type factorization with respect to $\alpha = (\alpha_1, \cdots, \alpha_n)$). Let M be an invertible $n \times n$ real matrix and $\alpha \in [0, \infty)^n$. Then there exist an $n \times n$ permutation matrix S, a lower triangular invertible $n \times n$ matrix L, an $n \times n$ matrix U, and an integer $m \in \{0, 1, \cdots, n\}$ with the following properties:

(i). $M = SLU$;

(ii). For $\gamma_i := \min\{j : U_{ij} \neq 0\}$ with $1 \leq i \leq n$ and $\beta := S^{-1}\alpha$,

- (a). $i < j \leq m \Rightarrow \gamma_i < \gamma_j$ and $\beta_i \leq \beta_j$,
- (b). $U_{i\gamma_i} = 1$ for $1 \leq i \leq m$,
- (c). $\gamma_i \leq \beta_i$ for $i \leq m$,
- (d). $\gamma_i > \beta_i$ for $i > m$;

(iii). $L_{ij} = \delta_{ij}$ for $m < j \leq n$ and $1 \leq i \leq n$, where δ_{ij} is the Kronecker symbol.

Proof. The proof follows the classical proof of the LU decomposition. In fact, in general, it is an incomplete LU factorization depending on the magnitude of α. We give an algorithmic proof here.

Initialization: Set $M^0 := M$, $\alpha^0 := \alpha$, $m_0 := 0$, $n_0 := n$.

Loop: For $k = 1, \cdots, n$, let $i_k := 1 + m_{k-1}$ and $I_k := \{i_k, \cdots, n_{k-1}\}$.

If $I_k = \emptyset$, stop.

If $I_k \neq \emptyset$, then the algorithm is continued. We write I_k as the following disjoint union:

$$I_k = E_k \cup F_k \cup G_k,$$

where

$$E_k = \{\alpha_i^{k-1} \geq k \,:\, i \in I_k, M_{ik}^{k-1} \neq 0\}, \quad e_k := Cardinal(E_k),$$
$$F_k = \{\alpha_i^{k-1} \geq k \,:\, i \in I_k, M_{ik}^{k-1} = 0\}, \quad f_k := Cardinal(F_k),$$

and

$$G_k = \{\alpha_i^{k-1} < k \,:\, i \in I_k\}, \quad g_k := Cardinal(G_k).$$

We make a permutation on the row of matrix M^{k-1} with index in I_k such that

$$\widetilde{M}^k := S^k M^{k-1}, \qquad \widetilde{\alpha}^k := S^k \alpha^{k-1},$$

where $\widetilde{E}^k$ has the same definition as E_k replacing α and M by $\widetilde{\alpha}$ and $\widetilde{M}$. We do the same for $\widetilde{F}^k$ and $\widetilde{G}^k$. We require that $\widetilde{E}^k$ be ordered: $i, j \in \widetilde{E}^k$, $i \leq j \Rightarrow \widetilde{\alpha}_i^k \leq \widetilde{\alpha}_j^k$. We also require that I_k begins by $\widetilde{E}^k$, continues by $\widetilde{F}^k$, and finishes by $\widetilde{G}^k$, i.e.,

$$\{i_k, \cdots, i_k - 1 + e_k\} = \widetilde{E}^k, \quad \{i_k + e_k, \cdots, i_k - 1 + e_k + f_k\} = \widetilde{F}^k,$$
$$\{i_k + e_k + f_k, \cdots, n_k\} = \widetilde{F}^k.$$

Since $\widetilde{G}^k$ represents the low oscillations, we define $n_k := n_{k-1} - g_k$.

Now, there are two subcases:

(i) If $\widetilde{E}^k \neq \emptyset$, then we calculate one step on the classical Gauss elimination on the submatrix of $\widetilde{M}^k$ with indices in $\widetilde{E}^k \times \widetilde{E}^k$, with pivot entry $\widetilde{M}^k_{i_k,i_k} \neq 0$, $M^k := (L^k)^{-1}\widetilde{M}^k$, and $M^k_{i_k,i_k} = 1$. Therefore, we have $m_k := 1 + m_{k-1} = i_k$ and $\gamma_{i_k} := k$.

(ii) If $\widetilde{E}^k = \emptyset$, then we do nothing, i.e., $L^k := Id$, $M^k := \widetilde{M}^k$, and $m_k := m_{k-1}$.

End Loop. Then we have $m = m_n$ and $U := M^n$.

Indeed, we can easily prove by induction that $I_1 = \{1, \cdots, n\}$ and $I_{k+1} \subset I_k$. Moreover, I_k becomes empty for $k > n$ and $M^k_{i,j} = 0$ for $i < k$ and $j \in I_k$. All the statements of Lemma 5.1 follow by induction.

Therefore, this algorithm yields

$$M := \left[S^1 L^1 S^2 L^2 \cdots S^{n-1} L^{n-1}\right] U.$$

Finally, we define

$$\beta = \alpha^{n-1}, \qquad S := S^1 S^2 \cdots S^{n-1} = \Pi_{j=1}^{n-1} S^j,$$

and

$$L := \Pi_{k=1}^{n-1} \left(\Pi_{j=k+1}^{n-1} S^j\right)^{-1} L^k \left(\Pi_{j=k+1}^{n-1} S^j\right).$$

The result follows since the structure of each L^k matrix is invariant by any permutation of the labeling of coordinates of index $j > k$. This concludes the proof. □

We will give some examples after Theorem 5.1.

Remark 5.1.

(i). Generically, $\det\left((M_{ij})_{1 \leq i,j \leq k}\right) \neq 0$ for all k. Thus, if $\alpha_i \geq i$, then $S = Id$, $m = n$, $\gamma_i \equiv i$, $\beta = \alpha$, and we obtain the classical LU decomposition with an upper triangular matrix U.

(ii). If all $\alpha_i < 1$, then we have only $S = L = Id$, $M = U$, and $m = 0$.
(iii). If all $\alpha_i \geq n$, then we have the classical LU factorization with $m = n$ and $\gamma_i \equiv i$.
(iv). This factorization is not unique. In fact, L and U depend on S.

We now use the factorization in Lemma 5.1: $M = SLU$, β, and ψ.

Remark 5.2. For the diagonal matrices $A^\varepsilon, B^\varepsilon$, and E^ε such that

$$A^\varepsilon_{ii} := \varepsilon^{-\alpha_i}, \quad B^\varepsilon_{ii} := \varepsilon^{-\beta_i}, \quad E^\varepsilon_{ii} := \varepsilon^{\min(\gamma_i,\beta_i)}, \tag{5.4}$$

then $A^\varepsilon S = SB^\varepsilon$. Define $L^\varepsilon := B^\varepsilon L E^\varepsilon$ which is the lower triangular matrix. Note that its diagonal terms L^ε_{jj} are nonzero constants or go to ∞ if $j \leq m$. Indeed, the structure of L^ε is the same as T^ε described in Section 3.2. Furthermore, we find that, for $j > m$, $L^\varepsilon_{jj} = 1$ and $L^\varepsilon_{ij} = 0$ if $j \neq i$.

In fact, this can be seen by a careful examination of the entries $(L^\varepsilon)_{ij}$ for $j \leq i \leq m$. Indeed, L^ε is like the identity matrix, except its strictly triangular part, in which we have $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$ from Lemma 5.1 and

$$(L^\varepsilon)_{ij} = \varepsilon^{\mu_j - \beta_i} L_{ij} \qquad \text{with } \mu_j = \min(\gamma_j, \beta_j) = \beta_j.$$

Since $\mu_j - \beta_i \leq 0$, we arrive at the conclusion.

Therefore, we have the following main theorem.

Theorem 5.1. Assume that $\mathbf{F} \in C^{n+2}(\mathbb{R};\mathbb{R}^n)$, and $u_1 \in L^\infty(\mathbb{R}^n;\mathbb{R})$ is $P-$periodic. Let $u^\varepsilon(t,\mathbf{x})$ be the entropy solution in L^∞ of (1.1)–(1.2). Then

$$u^\epsilon(t,\mathbf{x}) = \underline{u} + \varepsilon v^\varepsilon(t,\mathbf{x})$$

with

$$v_\varepsilon(t,\mathbf{x}) - \sigma(t,(SLE^\varepsilon)^{-1}\Phi(t,\mathbf{x})) \to 0 \qquad \text{in } L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^n), \tag{5.5}$$

where SLE^ε are defined in Lemma 5.1, $\Phi(t,\mathbf{x})_i := \phi_i(x_1 - \lambda_1 t, \cdots, x_n - \lambda_n t)$, and the profile σ is the unique entropy solution to the Cauchy problem

$$\partial_t \sigma + \sum_{i=1}^m \partial_{X_i} \sigma^{1+\gamma_i} \;=\; 0, \qquad \sigma(0,\mathbf{X}) = w_1(\mathbf{X}),$$

where $w_1(\mathbf{X})$ is the weak-star limit of the whole sequence $\{u_1(SL^\varepsilon\mathbf{X})\}_{\varepsilon>0}$ in L^∞, that is,

$$u_1(SL^\varepsilon\mathbf{X}) \overset{*}{\rightharpoonup} w_1(\mathbf{X}) \qquad \text{in } L^\infty \quad \text{when } \varepsilon \to 0.$$

Furthermore, σ is the unique profile satisfying (5.5) (see Remark 5.3(i) for a precise statement).

Before proving Theorem 5.1, let us explain several examples of numerous cases for the two-dimensional conservation laws, written in the phase variables, included in Theorem 5.1. In this case, we have

$$M := \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \begin{cases} \partial_t u^\varepsilon + (a\partial_{\phi_1} + b\partial_{\phi_2})(u^\varepsilon)^2 + (c\partial_{\phi_1} + d\partial_{\phi_2}))(u^\varepsilon)^3 = 0, \\ u^\varepsilon(0,x_1,x_2) = \varepsilon u_1(\phi_1/\varepsilon^{\alpha_1}, \phi_2/\varepsilon^{\alpha_2}). \end{cases}$$

For each example, we give the factorization: $M = SLU$, L^ε, the integer m, the profile equation with initial data $w_1(X_1,X_2)$ as a weak oscillating limit, and an asymptotic expansion of $v^\varepsilon := (u^\varepsilon(t,x_1,x_2) - \bar{u})/\varepsilon$ ($\bar{u} = 0$ here). In each case, in each of the first m rows of matrix U, say in row i, the only important term is the first nonzero entry (often on the diagonal, and normalized to be 1), which defines the dominant term in the Taylor expansion of the nonlinear part of the flux in the corresponding direction X_i. Note that all the other entries in these rows do not play any role in the profile equations.

(i). Case $\alpha_1 = 1, \alpha_2 = 2, b = 0$:

$$S = Id := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ U = \begin{pmatrix} 1 & c/a \\ 0 & 1 \end{pmatrix},\ L^\varepsilon = L,\ m = 2,$$
$$\partial_t\sigma + \partial_{X_1}\sigma^2 + \partial_{X_2}\sigma^3 = 0, \qquad \sigma(0, X, Y) = w_1(X_1, X_2) = u_1(aX_1, dX_2),$$
$$v_\varepsilon(t, x_1, x_2) \simeq \sigma(t, \phi_1(x_1, x_2)/(a\varepsilon), \phi_2(x_1, x_2)/(d\varepsilon^2)).$$

(ii). Case $\alpha_1, \alpha_2 < 1$:

$$S = Id, \quad L = Id, \quad U = M, \quad L^\varepsilon = Id,\ m = 0,$$
$$\partial_t\sigma = 0, \qquad \sigma(0, X_1, X_2) = u_1(X_1, X_2),$$
$$v_\varepsilon(t, x_1, x_2) \simeq u_1(\phi(x_1, x_2)/\varepsilon^\alpha, \psi(x_1, x_2)/\varepsilon^\beta).$$

(iii). Case $\alpha_1 = 1, \alpha_2 = 1, a \neq 0$:

$$S = Id, \quad L = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & c/a \\ 0 & (ad - bc)/a \end{pmatrix}, \quad L^\varepsilon = L, m = 1$$
$$\partial_t\sigma + \partial_{X_1}\sigma^2 = 0, \qquad \sigma(0, X_1, X_2) = u_1(aX_1, bX_1 + X_2),$$
$$v_\varepsilon(t, x_1, x_2) \simeq \sigma\left(t, \phi_1(x_1, x_2)/(a\varepsilon), \left(\phi_2(x_1, x_2) - \frac{b}{a}\phi_1(x_1, x_2)\right)/\varepsilon\right).$$

(iv). Case $\alpha_1 < 1 < \alpha_2, b \neq 0$:

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, L = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & d/b \\ a & c \end{pmatrix}, L^\varepsilon = \begin{pmatrix} b\varepsilon^{1-\beta} & 0 \\ 0 & 1 \end{pmatrix},\ m = 1,$$
$$\partial_t\sigma + \partial_{X_1}\sigma^2 = 0, \quad \sigma(0, X_1, X_2) = w_1(X_1, X_2) = \int_0^1 u_1(X_2, \theta)\, d\theta := \overline{w}_1(X_2),$$

that is,

$$\sigma(t, X_1, X_2) = \overline{w}_1(X_2),$$

which implies

$$v_\varepsilon(t, x_1, x_2) \simeq \overline{w}_1(\phi_1(x_1, x_2)/(b\varepsilon)).$$

(v). Case $\alpha_1 = 1, \alpha_2 = 2, a \neq 0, b \neq 0$:

$$S = Id,\ L = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix},\ U = \begin{pmatrix} 1 & c/a \\ 0 & 1 \end{pmatrix},\ L^\varepsilon = \begin{pmatrix} a & 0 \\ b\varepsilon^{-1} & d \end{pmatrix},\ m = 2,$$
$$\partial_t\sigma + \partial_{X_1}\sigma^2 + \partial_{X_2}\sigma^3 = 0, \qquad \sigma(0, X_1, X_2) = \overline{w}_1(X_1) = \int_0^1 u_1(aX_1, \theta) d\theta,$$

that is, $\sigma(t, X_1, X_2) = \overline{\sigma}(t, X_1)$ is the unique solution of

$$\partial_t\overline{\sigma} + \partial_{X_1}\overline{\sigma}^2 = 0, \qquad \overline{\sigma}(0, X_1) = \overline{w}_1(X_1),$$

which implies

$$v_\varepsilon(t, x_1, x_2) \simeq \overline{\sigma}(t, \phi_1(x_1, x_2)/(a\varepsilon)).$$

(vi). Case $1 < \alpha_1 = \alpha_2 < 2, a \neq 0, b \neq 0, \dfrac{a}{b} \in \mathbb{Q}$ (rational numbers):

$$S = Id,\ L = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix},\ U = \begin{pmatrix} 1 & c/a \\ 0 & \det(M)/a \end{pmatrix},\ L^\varepsilon = \begin{pmatrix} \varepsilon^{1-\alpha}a & 0 \\ \varepsilon^{1-\alpha}b & 1 \end{pmatrix},\ m = 1,$$

$$\partial_t\sigma + \partial_{X_1}\sigma^2 = 0, \qquad \sigma(0, X_1, X_2) = \widetilde{w}_1(X_2) = \int_0^1 u_1(p\theta, q\theta + X_2)d\theta,$$

where (p, q) belongs in $\mathbb{Z}^2 - \{(0, 0)\}$ such that $qa - pb = 0$, which implies

$$\sigma(t, X_1, X_2) = \widetilde{w}_1(X_2),$$

and

$$v_\varepsilon(t, x_1, x_2) \simeq \widetilde{w}_1\left(\left(\phi_2(x_1, x_2) - \frac{p}{q}\phi_1(x_1, x_2)\right)/\varepsilon^\alpha\right).$$

(vii). Case $\alpha_1 = \alpha_2 = 2, \dfrac{a}{b} \in \mathbb{Q}$: This is an interesting case of nonlinear propagation of high oscillations with maximal frequency without orthogonality between the phase gradients and the second flux derivative. Details for this last case are left to the reader.

Proof of Theorem 5.1. We need several steps and linear scalings of variables to prove this general theorem.

Step 1: Get rid of the linear transport. With a linear change of variables, we may assume that the gradient of the flux vanishes at the constant state $\underline{u}$. That is, for any $i \in \{1, \cdots, n\}$,

$$y_i := x_i - \lambda_i t, \qquad G_i(u) := F_i(u) - \lambda_i u.$$

Then $\mathbf{G}'(\underline{u}) = (0, \cdots, 0)$ and $\mathbf{G}''(u) = \mathbf{F}''(u)$ for all u, and the problem becomes

$$\begin{aligned} &\partial_t u^\varepsilon + \operatorname{div}_{\mathbf{y}}(\mathbf{G}(u^\varepsilon)) = 0, \\ &u^\varepsilon(0, \mathbf{y}) = u_0^\varepsilon(\mathbf{y}) \equiv \underline{u} + \varepsilon\, u_1(A^\varepsilon\Phi). \end{aligned}$$

Step 2: Move to a periodic case. With a second constant change of variables $\Phi := J\mathbf{y}$, the solution and the data are periodic, and the problem becomes

$$\begin{aligned} &\partial_t u^\varepsilon + \operatorname{div}_\Phi(J\mathbf{G}(u^\varepsilon)) = 0, \\ &u^\varepsilon(0, \Phi) = u_0^\varepsilon(\Phi) \equiv \underline{u} + \varepsilon\, u_1(A^\varepsilon\Phi). \end{aligned}$$

This change does not affect the convergence in $L^1_{loc}(\mathbb{R}^n)$. We still denote by u^ε the same function after the change of variables.

Step 3: Make a Taylor expansion. Now, set $u^\varepsilon := \underline{u} + \varepsilon v_\varepsilon$. Performing a Taylor expansion and defining the vector

$$\mathcal{V}^\varepsilon := {}^t(\varepsilon^1 v_\varepsilon^2, \cdots, \varepsilon^i v_\varepsilon^{i+1}, \cdots, \varepsilon^n v_\varepsilon^{n+1}).$$

Then

$$\operatorname{div}_\Phi(\mathbf{F}(u^\varepsilon)) = \varepsilon\Big(\sum_{i,j\le n} M_{ij}\varepsilon^j\partial_{\phi_i}v_\varepsilon^{j+1} + \varepsilon^{n+1}\operatorname{div}_\Phi(R_0^\varepsilon)\Big), \tag{5.6}$$

and we can rewrite the problem as

$$\begin{aligned} &\partial_t v_\varepsilon + \operatorname{div}_\Phi(M\mathcal{V}^\varepsilon) = \varepsilon^{n+1}\operatorname{div}_\Phi(R_0^\varepsilon), \\ &v_\varepsilon(0, \Phi) \equiv u_1(A^\varepsilon\Phi), \end{aligned}$$

where R_0^ε is a vector function of v_ε, which is bounded in L^∞, thanks to the maximum principle.

Step 4: Factorize M. Now we can apply the change of variables in Lemma 5.1. Since $M = (SL)U$, we use the new variable Θ such that $\Phi = (SL)\Theta$. Then we find that the problem becomes

$$\begin{aligned} &\partial_t v_\varepsilon + \operatorname{div}_\Theta(U\mathcal{V}^\varepsilon) \; = \varepsilon^{n+1} \operatorname{div}_\Theta(R_1^\varepsilon), \\ &v_\varepsilon(0,\theta) = u_1(A^\varepsilon SL\Theta) = u_1(SB^\varepsilon L\Theta), \end{aligned}$$

where $(SL)R_1^\varepsilon = R_0^\varepsilon$.

Step 5: Rescale. With the new variable $\mathbf{X}$ given by $\Theta = E^\varepsilon \mathbf{X}$, we obtain

$$\begin{aligned} &\partial_t v_\varepsilon + \sum_{i=1}^m \partial_{X_i}(v_\varepsilon^{\gamma_i+1} + \varepsilon r_i^\varepsilon) \; = \varepsilon^\gamma \operatorname{div}_{\mathbf{X}_p}(R_2^\varepsilon), \\ &v_\varepsilon(0,\mathbf{X}) = u_1(SB^\varepsilon LE^\varepsilon \mathbf{X}) = u_1(SL^\varepsilon \mathbf{X}), \end{aligned}$$

where $R_2^\varepsilon, r_i^\varepsilon, 1 \le i \le m$, are functions of v_ε which are bounded in L^∞ and $\gamma > 0$.
Step 6: Smooth initial data. For $\gamma > 0$ chosen in Step 5, take ω^ε a standard mollifier such that $u_1^\varepsilon = u_1 * \omega^\varepsilon$ satisfies

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma TV(u_1^\varepsilon) = 0. \tag{5.7}$$

Step 7: Apply the L^1-stability with respect to small BV perturbations. Set $\mathbf{X} := (\mathbf{X}_c, \mathbf{X}_p)$ with $\mathbf{X}_c := (X_i)_{i\le m}$ and $\mathbf{X}_p := (X_i)_{i>m}$. Notice the important fact that the initial data has *no oscillation in* $\mathbf{X}_p$, which is due to the structure of matrix L^ε or, more precisely, to the **incomplete** LU factorization (see Remark 5.2). Therefore, we can control the remainder by Lemma 2.7 and Step 6 so that $v_\varepsilon - w_\varepsilon$ is small in L^1_{loc}, where w^ε is the solution of the following reduced equation:

$$\begin{aligned} &\partial_t w_\varepsilon + \sum_{i=1}^m \partial_{X_i}(w_\varepsilon^{\gamma_i+1} + \varepsilon r_i^\varepsilon(w_\varepsilon)) \; = \; 0, \\ &w_\varepsilon(0,\mathbf{X}) = u_1(SL^\varepsilon \mathbf{X}), \end{aligned}$$

in which $\mathbf{X}_p$ plays the role of a parameter. Notice that, for almost all fixed $\mathbf{X}_p$, thanks to Lemma 3.2, the initial data sequence converges weakly:

$$v_\varepsilon(0,\mathbf{z}) \rightharpoonup w_1(\mathbf{X}_c; \mathbf{X}_p).$$

Finally, in order to pass to the limit in the conservation law, we need the compactness with respect to the variable $\mathbf{X}_c$.

Step 8: Use compactness. Set

$$w_\varepsilon(\mathbf{X}) = w_\varepsilon(\mathbf{X}_c; \mathbf{X}_p),$$

where $\mathbf{X}_p$ plays only the role of a parameter. For any fixed $\mathbf{X}_p$, $w_\varepsilon(t, \mathbf{X}_c; \mathbf{X}_p)$ is the solution of the Cauchy problem for a genuinely nonlinear m-dimensional conservation law. Thus we can use the compactness argument for each $\mathbf{X}_p$ to get

$$w_\varepsilon(\cdot; \mathbf{X}_p) \to \sigma(\cdot; \mathbf{X}_p) \in L^1_{loc,t,\mathbf{X}_c}(\mathbb{R}_+ \times \mathbb{R}^m).$$

Now, since the sequence is bounded in L^∞, by Lebesgue's Theorem,

$$w_\varepsilon \to \sigma \qquad \text{in } \; L^1_{loc,t,\mathbf{X}}(\mathbb{R}_+ \times \mathbb{R}^n).$$

Noting $\Phi = (SLE^\varepsilon)\mathbf{X}$ and using Lemma 3.1 about the triangular change of variables, we conclude that the L^1_{loc}–convergence is preserved, which implies

$$v_\varepsilon(t,\Phi) - \sigma(t,(SLE^\varepsilon)^{-1}\Phi) \to 0 \qquad \text{in } L^1_{loc} \text{ when } \varepsilon \to 0.$$

Step 9: Use the uniqueness of the profile. First, with the notations in Section 3.2, we have

$$\langle |R^\varepsilon| \rangle \to 0,$$

where

$$R^\varepsilon := v_\varepsilon(t,\mathbf{x}) - \sigma(t,(SLE^\varepsilon)^{-1}\Phi(t,\mathbf{x})).$$

which implies the uniqueness of such a profile.

Indeed, Lemma 3.1 provides exactly such a convergence in L^1_{ap} and therefore here the L^1 convergence for 1-periodic functions. Furthermore, our change of variables satisfies the assumptions of Lemma 3.1. Therefore, the asymptotics in (5.5) is valid in L^1_{ap}, and the uniqueness follows, as stated below in Remark 5.3.

This completes the proof of Theorem 5.1. □

Remark 5.3. From Theorem 5.1, we have

(i). Although the factorization is far from being unique, the profile defined in Theorem 5.1 is unique modulo a linear change of variables, which is one-to-one on $span\{Sp[\sigma(0,.)]\}$, which is a subspace of dimension at most m. Along the lines of Lemma 3.3, another factorization could perhaps provide **formally** a different profile, but two different profiles are equal up to a linear change of variables, and the only differences between them involve fast oscillating variables, which are therefore "killed" by the nonlinearities.

(ii). If all $\alpha_i < 1$, then, at the limit, all the waves propagate linearly:

$$v^\varepsilon(t,\mathbf{x}) \simeq u_1(A^\varepsilon\Phi(t,\mathbf{x})).$$

(iii). If all $\alpha_i > n$, then, at the limit, all the initial oscillations are canceled by the nonlinearity:

$$v^\varepsilon(t,\mathbf{x}) \simeq \int_P u_1(\theta)d\theta.$$

(iv). If all α_i are between 1 and n, there is a large number of cases.

(v). If all $\alpha_i = 1$, we recover the classical case of weakly nonlinear geometric optic (WNLGO).

(vi). If $\alpha_i > 1$, plus a *generic* assumption, then again all the oscillations are canceled by the nonlinearity. An example of such an assumption is that *no* phase gradient is orthogonal to the vector $\mathbf{F}''(\underline{u})$ *and* all α_i are distinct.

(vii). More surprisingly, with a suitable phase choice with respect to the nonlinearity (for instance, choose J such that $M = JN$ becomes upper triangular), it is always possible to allow for the propagation of an oscillation with small amplitude ε and frequency $\varepsilon^{-\gamma}$ for all $\gamma \in (0,n]$. This is a new multidimensional feature! In contrast, if $\gamma > n$, the "true" nonlinearity always cancels this oscillation. Therefore, in dimension $n \geq 1$, the critical exponent is n, provided that the solution oscillates in very singular directions! For $n = 1$, we recover the classical geometric optics.

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References

[1] Bouchut, F. and Perthame, B., Kruskhov's estimate for scalar conservations laws revisited, *Trans. Amer. Math. Soc.* **350** (1998), 2847–2870.

[2] Chen, G.-Q. and Frid, H., Decay of entropy solutions of nonlinear conservation laws, *Arch. Rational Mech. Anal.* **146** (1999), 95–127.

[3] Chen, G.-Q. and Frid, H., Large-time behavior of entropy solutions in L^∞ for multidimensional conservation laws, In: *Nonlinear Partial Differential Equations*, edited by G.-Q. Chen et al, pp. 28-44, World Scientific: River Edge, NJ, 1998.

[4] Chen, G.-Q. and Rascle, M., Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **153** (2000), 205–220.

[5] Chen, G.-Q. and Lu, Y.-G., A study of approaches to applying the theory of compensated compactness, *Chinese Sci. Bull.* **34** (1989), 15–19.

[6] Cheverry, C., The modulation equations of nonlinear geometric optics, *Commun. Partial Diff. Eqs.* **21** (1996), 1119–1140.

[7] Cheverry, C., Justification de l'optique géométrique non linéaire pour un système de lois de conservation (in French) [Justification of nonlinear geometric optics for a system of conservation laws], *Duke Math. J.* **87** (1997), 213–263.

[8] Corduneanu, C., *Almost Periodic Functions*, Interscience: New York, (1968).

[9] Dafermos, C., *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: New York, 2000.

[10] DiPerna, R.-J., Measure-valued solutions to conservation laws, *Arch. Rational Mech. Anal.* **88** (1985), 223-270.

[11] DiPerna, R.-J. and Majda, A., The validity of nonlinear geometric optics for weak solutions of conservation laws, *Commun. Math. Phys.* **98** (1985), 313-347.

[12] Engquist, B. and E, W., Large time behavior and homogenization of solutions of two-dimensional conservation laws, *Comm. Pure Appl. Math.* **46** (1993), 1-26.

[13] Giusti, E., *Minimal Surfaces and Functions of Bounded Variation,* Birkhäuser: Basel, 1984.

[14] Glimm, J. and Lax, P., *Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws*, Mem. Amer. Math. Soc. **101**, AMS: Providence, 1970.

[15] Guès, O. Ondes multidimensionnelles ε-stratifiées et oscillations (in French) [Multidimensional ε-stratified waves and oscillations], *Duke Math. J.* **68** (1992), 401–446.

[16] Hunter, J. K., Majda, A., and Rosales, R., Resonantly interacting, weakly nonlinear hyperbolic waves II. Several space variables, *Stud. Appl. Math.* **75** (1986), 187–226.

[17] Joly, J. L., Metivier, G., and Rauch, J., Justification of resonant one-dimensional nonlinear geometric optics, *J. Functional Anal.* **114** (1) (1993), 106-231.

[18] Junca, S., Réflexion d'oscillations monodimensionnelles, *Commun. Partial Diff. Eqs.* **23** (1998), 727-759.

[19] Junca, S., A two-scale convergence result for a nonlinear conservation law in one space variable, *Asymptotic Analysis*, **17** (1998), 221-238.

[20] Kruskhov, S. N., First-order quasilinear equations in several independent variables, *Math. USSR Sb.* **10** (1970), 217–243.

[21] Lax, P., *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS. **11**, SIAM, 1973.

[22] Lions, P. L., Perthame, B., and Tadmor, E., A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.* **7** (1994), 169–192.

[23] Majda, A. and Rosales, R., Resonant one dimensional nonlinear geometric optics, *Stud. Appl. Math.* **71** (1984), 149-179.

[24] Murat, F., Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **5** (1978), 489-507.

[25] Perthame, B., *Kinetic Formulation of Conservation Laws*, Oxford University Press: Oxford, 2002.

[26] Serre, D., *Systems of Conservation Laws I, II*, Cambridge University Press: Cambridge, 1999,2000.
[27] Szepessy, A., An existence result for scalar conservation laws using measure valued solutions, *Commun. Partial Diff. Eqs.* **14** (1989), 1329-1350.
[28] Tartar, L., Compensated compactness and applications to partial differential equations, In: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, (ed.) R. J. Knops, vol. 4, pp. 136–212, Res. Notes in Math. **39**, Pitman, Boston, Mass.-London, 1979.
[29] Vasseur, A., Strong traces for solutions to multidimensional conservation laws, *Arch. Rational Mech. Anal.* **160** (2001), 181–193.
[30] Volpert, A.I., The spaces BV and quasilinear equations, *Math. USSR-Sb.* **73** (1967), 225-267.

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