# Correction of Second Order Hyperbolic Model. Links with the "Follow the Leader" Models


**Michel Rascle**

Laboratoire J.A. Dieudonné UMR-CNRS n.6621
Université de Nice-Sophia Antipolis

Email: rascle@math.unice.fr
http://www-math.unice.fr/ rascle

# Introduction.

## 3 classes of models:

### Microscopic "Follow-the-leader" models :

(Large) ODE systems, see Gazis-Herman-Rothery (1961) and many others....
Example :

$$\begin{aligned} \dot{x}_i &= v_i, \qquad (1) \\ \dot{v}_i &= C(v_i)^m \frac{v_{i+1} - v_i}{(x_{i+1} - x_i)^{\gamma+1}} + \frac{A}{T_r}\left[ V_{eq}\left(\frac{\Delta X}{x_{i+1} - x_i}\right) - v_i \right] \end{aligned}$$

In principle, $\exists$ a delay in $2^{\text{nd}}$ equation, vanishing at the limit. We neglect it.
If $A > 0$, last term : (macroscopic ) relaxation term, with $T_r >> T_{reaction}$, Klar-Kühne-Wegener.
Here, we take $m = 0$ and in general, $A = 0$.

## Kinetic Models:

see Prigogine-Herman, Paveri-Fontana,
Klar-Wegener, Nelson, Illner, Sopasakis, Dolbeault ...

$$\partial_t f + v \cdot \partial_x f = C(f, f)$$

**Nice mathematical problems ...**

**Big particles** : small "Avogadro number": not much room for kinetic scale.

**Collision terms sometimes nonlocal in $x$.**

**Molecular chaos assumption questionable...**

However,

**Successful prediction** of multi-modal fundamental diagrams.

**Formal derivation** of our fluid model !...
Klar-Wegener, 2000.

## Macroscopic models :

### "First order" :

Lighthill-Whitham-Richards (LWR)

$$\partial_t \rho + \partial_x(\rho v) = 0 \,;\, v = V(\rho).$$

Equilibrium model, $V$ given, e.g. $V(\rho] = 1 - \rho$.
Scalar conservation law, often with a concave flux :
Shocks when braking, Rarefactions when accelerating.

### "Second order" :

#### I. Payne-Whitham (PW) :

Gas dynamics system, $P$ : "pressure" (but no momentum conservation !!):

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t v + v\partial_x v + \rho^{-1}\tilde{P}'(\rho)\partial_x \rho = 0 \end{cases}$$

Can add a diffusion and/or a relaxation term :

$$(\tau^{-1})\,(V_{eq}(\rho) - v) + \ \nu\partial_x^2 v$$

in RHS of second equation ( = anticipation equation).

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## II. Critics: C. Daganzo ('95): Requiem for $2^{\text{nd}}$ order ...

PW model is a crazy model ! Indeed,

**In some cases, cars are going backward !!**

**Part of the information travels faster than cars : !!**

(since $\lambda_2 := v + c > v$, $c$ : sound speed).

## III. Aw-Rascle (SIAP) : Resurrection ... ?

Correction of PW : New $2^{\text{nd}}$ order ("AR") model and

Study of Riemann Problem (see below).

**Question: what's wrong in PW ?**

Rewrite $2^{\text{nd}}$ equation :

$$\partial_t v + \ v\, \partial_x v = -(\rho)^{-1} p'(\rho) \partial_x \rho + \text{Diffusion} + \text{Relax.}$$

Question : **does a driver respond to a** space (PW) or time **derivative of the density ???**

Our correction: TIME derivative :

Replace $\partial_x \rho$ by $(\partial_t \rho + v \partial_x \rho)$ !!

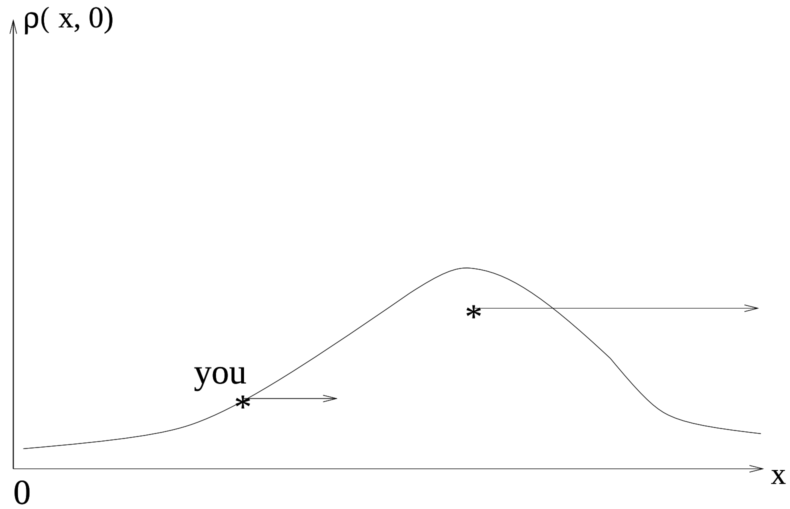


Figure 1: **In such a case, would you brake, or accelerate?**

With this simple fixing, no more paradox :

$$v \geq 0 \qquad\qquad\qquad\qquad \lambda_2 = v.$$

**Remark:** No diffusion !!, but relaxation is O.K. [ and $\Rightarrow$ **formal** diffusion (Chapman-Enskog)].

New $2^{\text{nd}}$ order ("AR") model and Study of Riemann Problem (see below). Related works include :

**J. Greenberg, SIAP 01 :**

Initial Value Problem (IVP) for same model, adding a special relaxation term (in "characteristic" case).

**Aw-Klar-Materne-Rascle (SIAP 02) : Derivation ... :**

Initial Value Problem (IVP) for the model with relaxation (in the general "(sub)characteristic" case), and rigorous derivation of this new model from Follow-the-Leader models.

**Independently, M. Zhang (Trans. Res., B, 02 ?):**

"A model devoid of ..." rediscovered same model and gave a formal derivation from Follow-the-Leader models : different communities, same ideas "in the air"...

**P. Bagnerini, M. Rascle, To appear SIMA**

Homogenization of same model for Multiclass Traffic Flow. Same links with multiclass Follow -the-Leader models.

**Incomplete list !!.**

see e.g. M. Zhang, Tong Li, R. Colombo, Benzoni-Colombo ...

## Summary.

- (1) : Microscopic Follow the Leader ODE system (1).
- (2) : our “AR” $2^{nd}$ order model (Eulerian) $\leftrightarrow$ (2’) (Lagrangian).
- (3) : its (Lagrangian) Godunov discretization.

Then :

- Solution of (3): uniform estimates in $L^\infty$ and $BV$.
- (3)$\equiv$ also ($1^{\text{rst}}$ order, explicit) time discretization of (1).
- When $\Delta X$ and $\Delta t \to 0$, fixed ratio, CFL, (3) $\to$ (2).

- When $\Delta t \to 0$, with fixed $\Delta X$, (3) converges to (1)$\to$ : same estimates for (1).
- Now, (1) $=$ 1/2-discretization of (2’). When $\Delta X \to 0$, (1)$\to$ (2’).

**In this sense, (2) or (2’) $=$ hydrodynamic limit of the Follow-the-Leader system.**

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## Outline.

- 1. Introduction.
- 2. Our model.
- 3. Riemann Problem. Comparison with PW.
- 4. The model with relaxation.
- 5. Homogenized model (with P. Bagnerini).

## Our model.

In Payne-Whitham (PW), in "anticipation equation" :

$$\partial_t v + \ v\, \partial_x v + (\rho)^{-1} \partial_x p(\rho) = \ 0,$$

replace $\partial_x p$ by convective derivative $(\partial_t + v\partial_x)$ of (new) $p$.

New model :

$$\partial_t \rho + \ \partial_x(\rho v) = \ 0,$$

$$\partial_t (v + p(\rho)) + \ v\, \partial_x (v + p(\rho)) = \ 0.$$

(Pseudo)-pressure = $p(\rho) = -V(\rho) + \text{constant}$, $V$ : equilibrium velocity : $V'(\rho) \leq 0$.

In conservative form , $2^{\text{nd}}$ equation $\rightarrow$

$$\partial_t(\rho w) + \ \partial_x(v \rho w) = \ 0\ , \quad w := \ v + p(\rho).$$

## Riemann Problem : Examples.

Strictly hyperbolic system: (e.g.) in variables $U = (\rho, v)$

$$\partial_t U + A(U)\,\partial_x U = 0,$$

$$A(U) := \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix}.$$

Real eigenvalues :

$$\lambda_1 = v - \rho\, p'(\rho) \leq \lambda_2 = v,$$

distinct except at vacuum : $\rho = 0$ if e.g. $p(\rho) = \rho^\gamma$, $0 < \gamma \leq 1$.

Diagonalize : Riemann invariants :

$$w = v + p(\rho) = v - V(\rho) + \text{Const}\ ,\ z = v.$$

$w$ : distance to equilibrium. $v$ : velocity. For smooth solutions,

$$\partial_t w + \lambda_2\, \partial_x w = 0, \quad \lambda_2 = v,$$

$$\partial_t z + \lambda_1\, \partial_x z = \partial_t v + (v - \rho p'(\rho))\, \partial_x v = 0.$$

Eigenvalues :

$\lambda_1$ : genuinely nonlinear (GNL) : strictly monotone in a 1-wave.

$\lambda_2$ : linearly degenerate (LD) : constant in a 2-wave.

Classically, $\lambda_2$ is LD, $\to$ $z = v =$constant across a 2-wave,

and $w =$ constant across a 1-rarefaction wave.

Here $w$ also $=$ constant across a 1- shock : much more exceptional : Systems " à la Temple " : shock curves and rarefaction curves coincide.

Solution of Riemann Problem : Given $U_\pm = (\rho_\pm, v_\pm) \in \mathbb{R}^2$, draw the curves

$$w = w_- := w(U_-) \text{ and } z = v = v_+ := z(U_+),$$

Unique intersection : $U_0 \leftrightarrow (w_-, v_+)$.

Connect $U_- \leftrightarrow U_0$ by 1-wave : $w = w_-$ and
shock wave (braking) if $v_0 = v_+ < v_-$
or rarefaction wave (acceleration) if $v_- < v_0 = v_+$ .

and $U_0 \leftrightarrow U_+$ by $2-$ wave : contact discontinuity, rear cars follow leading car, same speed $v_0 = v_+$ .

## Application : comparison with PW.

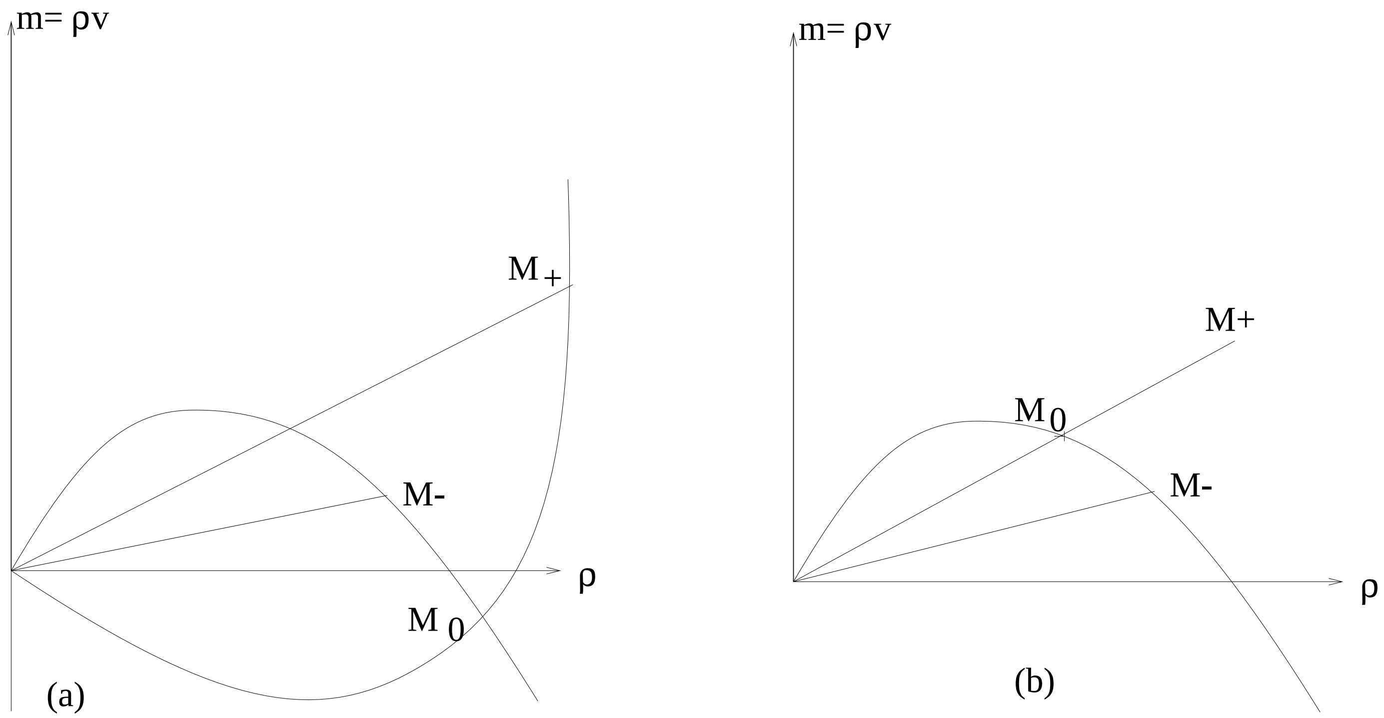


Figure 2: **Example of Riemann Problems for the PW model (a), and for our model (b). In (a) cars behind instantaneously reach negative speeds (!), although cars ahead travel *faster* !**

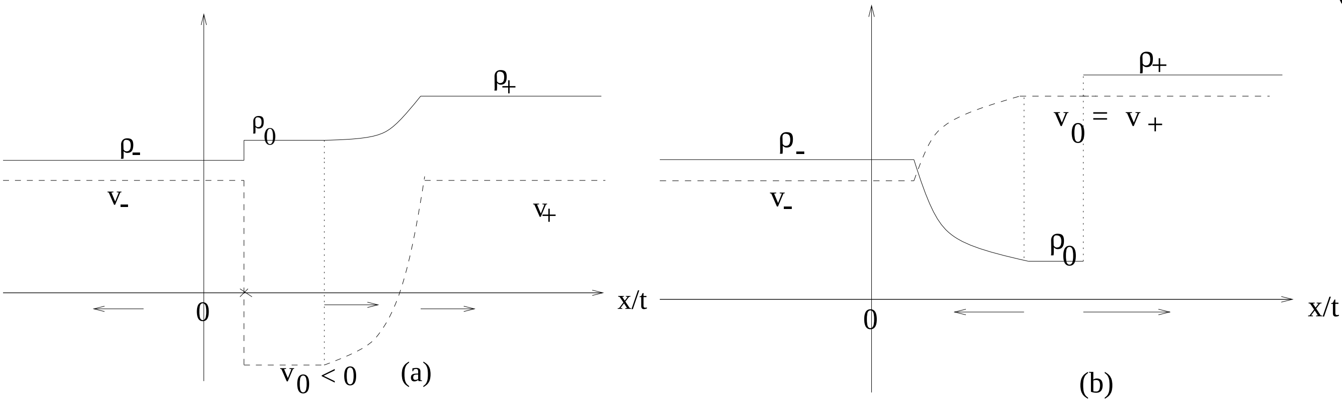


Figure 3: **Negative velocities appear with the PW model (a), not with our model (b).**

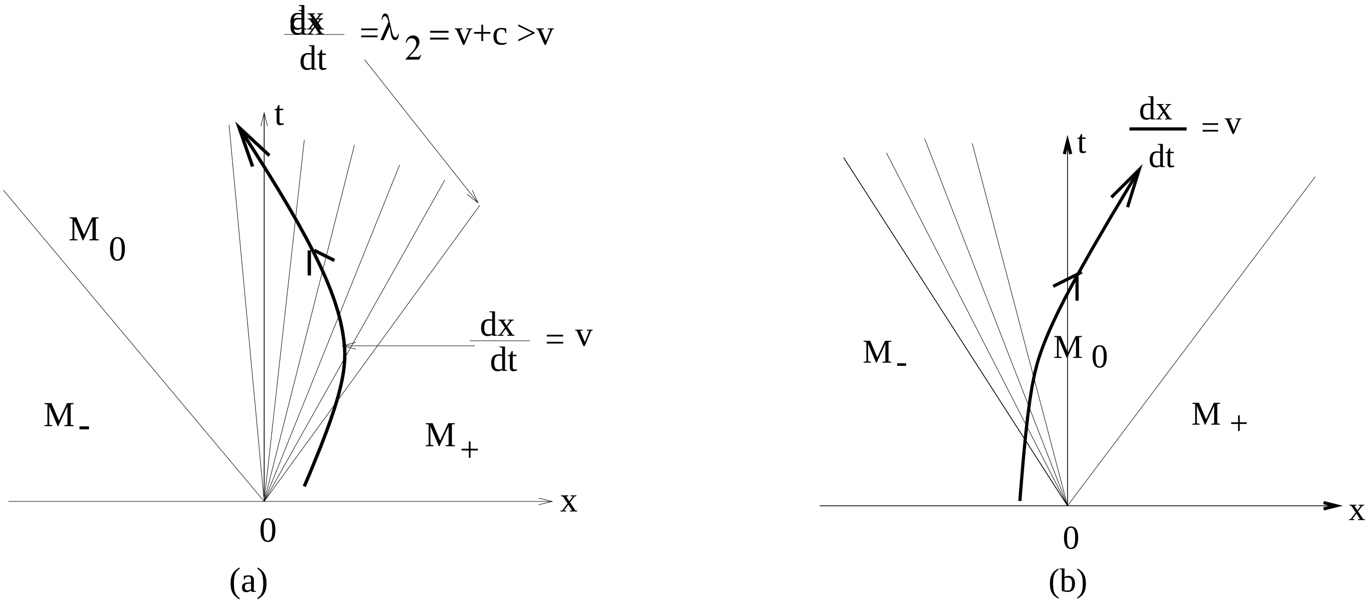


Figure 4: **The trajectories $\frac{dx}{dt} = v$ cross the 2 (resp 1)-rarefaction wave from the right to the left with PW model (a) from the left to the right with our model(b) : part of the information travels *faster* than cars ahead, and modifies their trajectories (!) in PW model (a), not in our model (b).**

## Remarks.

**All qualitative requirements now satisfied :**

**velocities and densities - remain $\geq 0$ and bounded (invariant region), except near $\rho = 0$ for $p(\rho) = ln\rho$.**

**No wave travels faster than the speed of the vehicles.**

**Model always predicts natural waves : shocks when braking, contacts when following, rarefactions, possibly with vacuum, when accelerating.**

**Only (milder, curable) drawbacks :**

**1. If $U_+$ at vacuum, maximal speed in rarefaction wave depends on $U_-$ : unrealistic ??. Fixing : either maximal speed (reachable by all cars) or add relaxation term.**

**2. Initial data must be in invariant region. If not, cars could hit each other. Fixing: relaxation (?) and/or add a singularity for $p$ at $\rho = \rho_m = 1$, cf Colombo ...**

## The model with relaxation.

$$\partial_t \rho + \partial_x(\rho v) = 0,$$

$$\partial_t(\rho(v + p(\rho))) + v\partial_x(\rho(v + p(\rho))) = \rho(T_{eq}^{-1} (V_{eq}(\rho) - v),$$

$T_{eq}$ : relaxation time, $V_{eq}(\rho)$ equilibrium velocity.

Weak entropy solutions :
Additional conservation laws of this system : entropy - entropy flux pairs of functions with scalar values :

$$(\eta(U),\ q(U)) := (\eta(\rho, v),\ q(\rho, v)).$$

For any convex entropy $\eta(U)$, impose

$$\partial_t \eta + \partial_x q \leq T_{eq}^{-1} \partial_v \eta \, (V_{eq}(\rho) - v).$$

Q :(strictly ??) convex , convex / w ??? ... Extension of Krushkov entropies : nonconvex, but entropy ineq. ... and minimal at equilibrium.... cf Aw's PhD

Remark : modeling needs a second heuristic choice function $V$.

Crucial assumtion: Subcharacteristic condition, Whitham, Chen-Liu-Levermore ... Here :

$$-p'(\rho) \leq V'(\rho) \leq 0.$$

Example of equilibrium velocity

$$V(\rho) = \alpha\ (p(\rho_m) - \ p(\rho))\ ,\ 0 \leq \alpha \leq 1,$$

$\rho_m = 1$, maximal (normalized) density, and

$0 < \alpha < 1$ : (strictly) subcharacteristic case.
$\alpha = 0$ or $1$ : characteristic case, cf J. Greenberg ($\alpha = 1$).

Role of subcharacteristic condition : $T_{eq} \to 0$,
Chapman-Enskog expansion... At leading order :

$$\partial_t \rho + \partial_x(\rho\ V(\rho)) = \ -T_{eq}\partial_x(V_1(\rho)) \sim \ dx(D(\rho)\ \partial_x \rho).$$

Nonlinear diffusion coefficient

$$D(\rho) := -T_{eq}^{-1}\ (V'(\rho) + p'(\rho))V'(\rho)$$

must be $\geq 0$. Also $\to$ the sum of TV of $v$ and $w$ controlled, cf Natalini ...

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Formal regularization without infinite speed of propagation and cars going backward...

Also essential condition for stability and convergence of numerical schemes and of zero relaxation limit, i.e. convergence of " AR" to Lighthill- Whitham-Richards (LWR) when $T_{eq} \to 0$.

Some related papers:
J. Greenberg (SIAP) , Aw (PhD), Schochet, Lattanzio-Marcati ...

Chen-Liu-Levermore, Natalini, Jin-Xin, Katsoulakis-Tzavaras ...

Same + Vasseur, Chen-Rascle : initial layer problem for non $BV$ case.

Case where (sub )characteristic condition not satisfied:
Greenberg'-Klar-Rascle: SIAP '02: model with hysteresis (cf Van der Waals fluids ) : bi-modal equilibrium function with $V_{eq}$ not monotone.

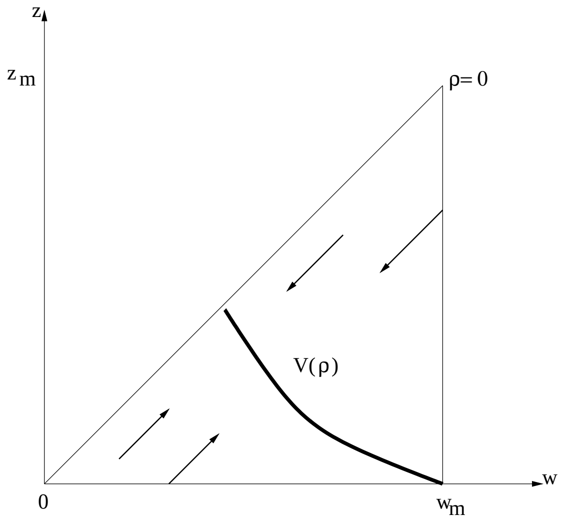



Figure 5: **Invariant region** $\mathbf{R}$ **and equilibrium curve** $v = V_{eq}(\rho)$ **for** $\gamma > 0$**, in the** $(w, z) = (w, v)$ **plane, in the subcharacteristic case (left), and in the characteristic case (right).**

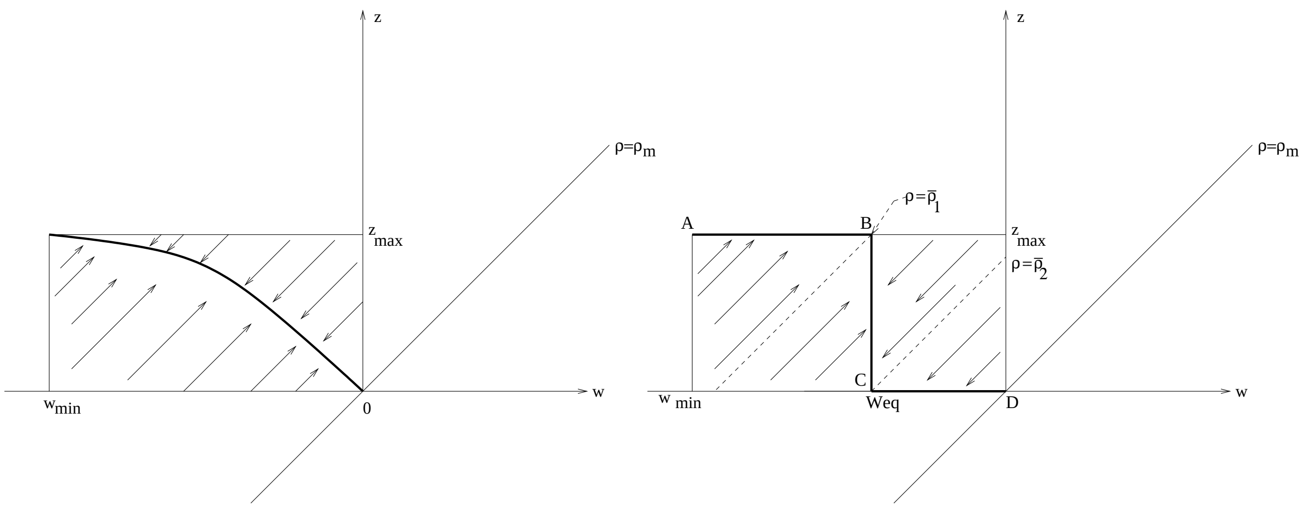



Figure 6: **Invariant region** $\mathbf{R}$ **and equilibrium curve** $v = V_{eq}(\rho)$ **for** $\gamma = 0$**, in the** $(w, v)$ **plane, in the subcharacteristic case (left), and in the characteristic case (right).**

# A Multiclass Homogenized Model for Traffic Flow.

**P. Bagnerini and Michel Rascle**

Laboratoire J.A. Dieudonné UMR-CNRS n.6621

Université de Nice-Sophia Antipolis

Email: `pbagneri or rascle@math.unice.fr`

`pbagneri@dm.unife.it`

## Introduction

**Goal: Macroscopic multiclass Model(cars, trucks etc.):**

Introduce two coefficients : $w, a$, which characterize each class.

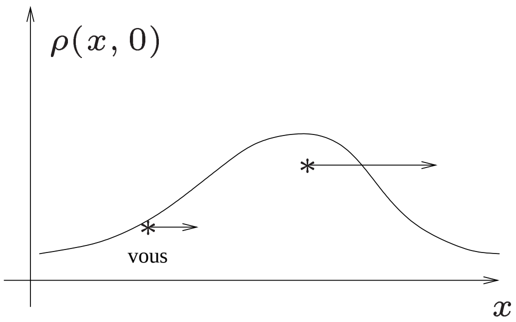
Long stretch of road: oscillating distribution of vehicle classes.
Average value : $< \mu_x(a), a >$

Prototype: $\mu_x = \sum_i \alpha_i(x)\delta_{(a_i,w_i)}$. Once $v$ is known, we know average density $\rho$, and density $\rho_i$ of each class.

Payne-Whitham model.(PW):

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t v + v\partial_x v + \rho^{-1}\tilde{P}'(\rho)\partial_x \rho = 0 \end{cases}$$

Daganzo, “Requiem for second-order fluid approximation to traffic flow” (’95)




Here, speed increasing $x$, decreasing in $t$ along your trajectory.
Q : brake ou accelerate? Aw-Rascle 2000:

$$\partial_x \tilde{P}(\rho) \longleftarrow (\partial_t + v\,\partial_x)\tilde{P}(\rho)$$
$$\partial_t(v + \tilde{P}(\rho)) + \partial_x(v + \tilde{P}(\rho)) = 0$$

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t \rho w + \partial_x v\rho w = 0 \end{cases} \qquad w = v + \tilde{P}(\rho) = v - \tilde{V}(\rho) + \text{Const}$$

$w$ : distance to equilibrium.

$\tilde{V}(\rho)$:= equilibrium speed : anticipation term.

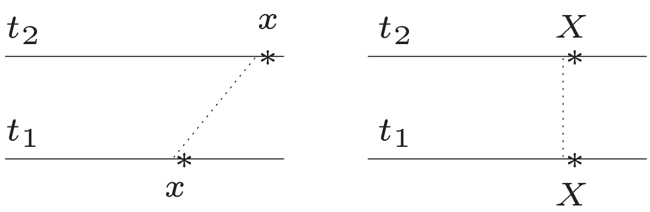
## Multiclass Model.

$$\begin{cases} \partial_t \rho + \partial_x v\rho = 0 \\ \partial_t \rho w + \partial_x v\rho w = 0 \\ \partial_t \rho a + \partial_x v\rho a = 0 \end{cases}$$

$$\begin{cases} \text{AR} \quad w = v + \tilde{P}(\rho) := \text{distance to equilibrium.} \\ \text{Here,} \quad w = v - \tilde{V}(\rho, a). \ \text{e.g.} \ w = v + a\tilde{P}(\rho) \end{cases}$$

Coefficients $a \in [0, 1]$ and $w$ ... characterize each class, e.g.

$$w = v - \tilde{V}(\rho, a) := v + a\tilde{P}(\rho) \ , \ a \in [1/2, 1].$$




Lagrangian mass coordinates $(X, t)$ :

$$T = t \qquad X = \int^x \rho(y, t)dy \qquad \tau = 1/\rho$$

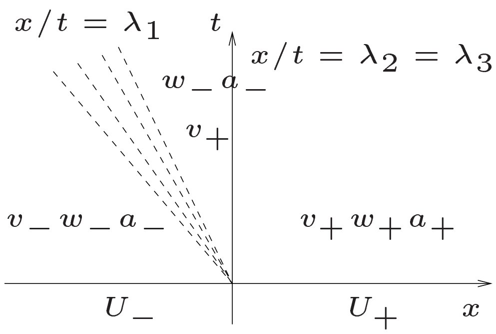
$$\begin{cases} \partial_t \tau - \partial_X v = 0 & v = w - aP(\tau) \\ \partial_t w = 0 \\ \partial_t a = 0. \end{cases}$$

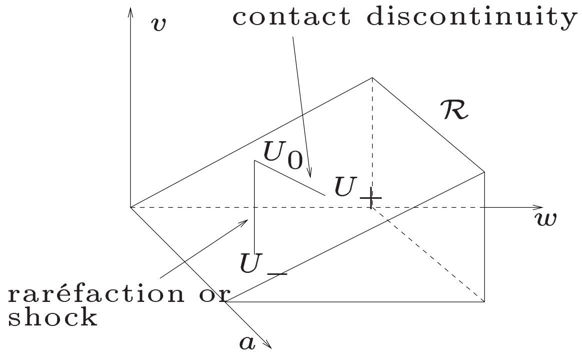
e.g. $V(\tau, a) = -aP(\tau) + \text{Const}\ ;\ P'(\tau) < 0, \qquad P''(\tau) > 0$

$$P(\tau) := \tilde{P}(1/\rho) = \begin{cases} \frac{v_{ref}}{\gamma}\frac{1}{\tau^\gamma}, & \gamma > 0, \\ -v_{ref}\ln(\tau), & \gamma = 0, \end{cases}$$

$$\text{VNL}\ \ \lambda_1 = aP'(\tau) < 0, \qquad \text{LD}\ \ \lambda_2 = \lambda_3 = 0$$

(Strict) Riemann Invariants: $v \leftrightarrow \lambda_1$, $(w, a) \leftrightarrow \lambda_2 = \lambda_3 = 0$.




Entropy-flux pairs:

$$\eta(v, w, a) = \int_0^v \frac{q'(s)}{aP'(\tau(s, w, a))} ds + \eta_0(w, a),\ q = q(v)$$

$\forall \eta$ convex / to $\tau$ ($\Leftrightarrow \forall q = q(v)$ concave / to $v$),

$$\partial_t \eta + \partial_x q \leq 0 \ \text{in} \mathcal{M}(\mathbb{R}\times]0, \infty[)$$

$$\partial_t \tau - \partial_x(w - aP(\tau)) = 0$$

## Godunov Scheme.

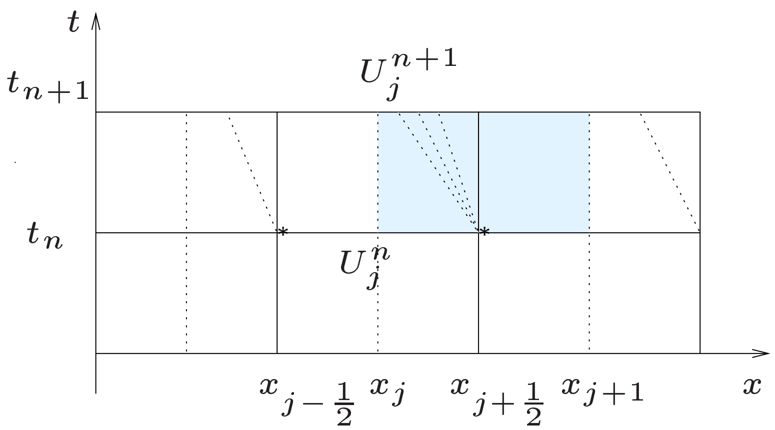
In Lagrangian coordinates $(x,t)$ : $\partial_t U + \partial_x F(U) = 0$

$$U_j^{n+1} := U_j^n - \frac{\Delta t}{\Delta x}[F_{God}(U_j^n, U_{j+1}^n)) - F_{God}(U_{j-1}^n, U_j^n)] =$$

$$= U_j^n + \frac{\Delta t}{\Delta x}(v_{j+1}^n - v_j^n, 0, 0)$$

$$\begin{cases} \partial_t \tau_h - \partial_x v_h = 0 & \text{in } \mathbb{R}\times]t_n, t_{n+1}[ \\ \partial_t w_h = 0 \\ \partial_t a_h = 0 \end{cases}$$

$$\begin{cases} U_h(x,t) \qquad h \leftrightarrow (\Delta x, \Delta t) : \text{scaling parameter} : h \to 0. \\ \Delta x : \text{ car length.} \end{cases}$$



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Limit $h \to 0$.

$$\begin{cases} \partial_t \tau_h - \partial_x v_h \sim 0 \quad (\equiv 0 \quad \text{in } \mathbb{R}\times]t_n, t_{n+1}[\,) \\ \partial_t w_h = 0 \\ \partial_t a_h = 0 \end{cases}$$

$$(v_h w_h, a_h) \overset{*}{\rightharpoonup} (v^*, w^*, a^*) L^\infty(\mathbb{R} \times \mathbb{R}_+)$$

$$f(v_h, w_h, a_h)(x,t) \overset{*}{\rightharpoonup} f^* \neq f(v^*, w^*, a^*) \text{ dans } L^\infty$$

Young Measure: $\exists (\nu_{x,t})_{x,t}$ probab. measures : $\forall f$ continuous,

$$f(v_h, w_h, a_h)(x,t) \overset{*}{\rightharpoonup} f^* := < \nu_{x,t}(v,w,a), f(v,w,a) >$$

Initial data : $(v_h^0, w_h^0, a_h^0, \tau_h^0)$ : piecewise constant, $BV$ for $v$, oscillating for $(w, a, \tau)$, with $w_h^0 = v_h^0 + a_h^0 P(\tau_h^0)$.
Moreover, assume that $(\nu_{x,0})_x$ is associated to the whole sequence $(v_h^0, w_h^0, a_h^0)$.

Convergence strong for $(v_h)$. Therefore $\nu_{x,t}$ = tensor product

$$\nu_{x,t} = \delta(v - v^*(x,t)) \otimes \mu_x(w,a),$$

$\mu_x$ well defined by the whole sequence $(w_h^0, a_h^0)$.

Limits :

$$w = v + aP(\tau) \qquad \tau_h = \mathcal{T}(v_h, w_h, a_h) := P^{-1}((w_h - v_h)/a_h)$$

$$\tau_h \overset{*}{\rightharpoonup} \tau^*(x,t) =< \nu_{x,t}(v,w,a), P^{-1}((w-v)/a) >=$$

$$=< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) >$$

Theorem 1.

$U_h := (v_h, w_h, a_h) \quad ; \quad \nu_{x,0} := \delta(v - v_*^0(x)) \otimes \mu_x(w,a)$, associated to the whole sequence $U_h^0$. Then, when $\Delta t$ and $\Delta x \to 0$, with $\Delta t/\Delta x = C$ and CFL, the whole sequence CV in $L^\infty$ weak-* to the unique weak "entropy" solution of homogenized system :

$$\begin{cases} \partial_t \tau^* - \partial_x v^* = 0 \\ \partial_t w^* = \partial_t a^* = 0 \end{cases}$$

with:

$$\tau^*(x,t) =< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) > .$$

Initial data :

$$\begin{cases} \tau_*^0 =< \mu_x(w,a), P^{-1}((w - v_0^*)/a) >, \\ (w_*^0,\ a_*^0) \ = \ < \mu_x(w,a), (w,\ a) > . \end{cases}$$

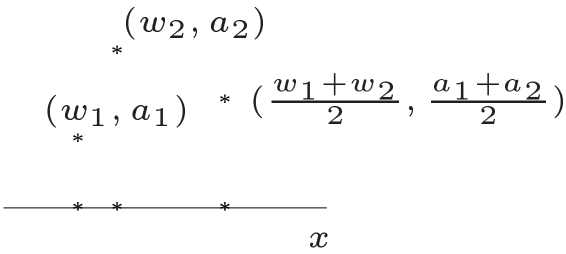
"Entropy" : $\forall k$,

$$\partial_t < \mu_x(w,a), |\mathcal{T}(v^*(x,t), w, a, ) - \mathcal{T}(k, w, a, )| > - \partial_x |v^*(x,t) - k| \leq 0$$

In fact, $| < \mu_x(w,a), \mathcal{T}(...) > \ - \ < \mu_x(w,a), \mathcal{T}(...) > |$, since $\mathcal{T}(., w, a)$ **increasing.**

**Example :** $\mu_x(w,a) = \sum_i \alpha_i(x)\delta_{w_i,a_i}$ :

$\tau^*(x,t) =< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) :=$
$\sum_i \alpha_i(x) P^{-1}((w_i - v^*(x,t))/a_i)$.




**Once $v^*$ is known (by Thm 1), we know average $\tau^*$, and**
**each $\tau_i^*$ corresponding to same macroscopic speed $v^*$ ($= BV$ function) : $\tau^*$ depends on $v^*$.**

## Sketch of proof...)

In the limit (subsequence...) :

$$\partial_t \tau^* - \partial_x v^* = 0$$

$$\tau^*(x,t) == < \mu_x(w,a), \mathcal{T}(w,a,v^*(x,t)) > := \mathcal{T}^*(x, v^*(x,t))$$

Scalar Equation , coupled with $w_t = a_t = 0$.

$L^\infty$ estimates, $\tau \geq 1$, $v \geq 0$... : Invariant regions.

$BV$ estimates in $x$ and $t$ for $v$ , in $t$ for $\tau$.
$\rightarrow$ $L^1(dx)$ equicontinuity in $t$. Trace at $t = 0$...

Entropy inequalities hold for all pair $q = q_k(v)$ concave /to $v$ $\leftrightarrow$ $\eta_k$ (convex / to $\tau$), with

$$\eta_k = |\mathcal{T}(v,w,a) - \mathcal{T}(k,w,a)| = |\tau - \mathcal{T}(k,w,a)|.$$

Weak-* limit : measure-valued solution (DiPerna, Szepessy...)

$$\partial_t < \mu_x(w,a), |\mathcal{T}(v^*(x,t),w,a,) - \mathcal{T}(k,w,a,)| > -\partial_x |v^*(x,t) - k| \leq 0$$

In fact, $| < \mu_x(w,a), \mathcal{T}(...) > \; - \; < \mu_x(w,a), \mathcal{T}(...) > |$,
since $\mathcal{T}(.,w,a)$ **increasing.**

Knowing $\mu_x(w,a)$, $v^*$ is the entropy solution of:

$$\partial_t \mathcal{T}^*(x,v^*) - \partial_x v^* = 0 \;, \; \tau^* := \mathcal{T}^*(x,v^*) \;, \text{ i.e.}$$
$$\partial_t |\mathcal{T}^*(x,v^*) - \mathcal{T}^*(x,k)| - \partial_x |v^* - k| \leq 0, \quad \forall k \,.$$

Here, inversion of the roles of $x$ and $t$ (of $v$ and $\tau$)
$\rightarrow$ entropy condition in conservative form.

Uniqueness à la Krushkov (weak regulariy / to $x$, but $\partial_t$) !!

Doubling of variables: first $y \rightarrow x$, then $s \rightarrow t$.

... $L^1$ contraction $\rightarrow$ uniqueness of the solution....

Cf ...

- Scalar non autonomous equations : cf e.g. Baiti-Jensen, Klingenberg- Risebro, Towers, Seguin-Vovelle ...
- Systems: cf Bressan *et al*, Bianchini ....
- Uniqueness of $mv-$solutions: Arguments of DiPerna, Szepessy don't work directly.
- Homogenization of Hamilton-Jacobi eq: cf Lions-Papanicolaou-Varadhan, Namah-Roquejoffre, Evans *et al*, Fathi .... Periodic case, smooth in $x$...
- Corrector terms (periodic case ) : E-Serre ...

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## Link with Multiclass microscopic model.

Continuous model and semi-discrétisation :

$$\begin{cases} \partial_t \tau - \partial_x v = 0 \\ \partial_t w = 0 \\ \partial_t a = 0 \end{cases} \quad (1) \qquad \begin{cases} \dot{\tau}_j(t) = \frac{v_{j+1}-v_j}{\Delta x} \\ \dot{w}_j(t) = 0 \\ \dot{a}_j(t) = 0 \end{cases} \quad (2)$$

$$\tau_j = (x_{j+1} - x_j)/\Delta x \qquad \dot{\tau}_j = (v_{j+1} - v_j)/\Delta x$$

$$w \overset{e.g.}{=} v + aP(\tau) \;,\; v_j = v_{j-1/2} \;,\; \Delta x : \text{vehicle length.}$$

Multiclass "Follow-the-Leader" model : $\equiv$ (2)

$$\begin{cases} \dot{x}_j & = v_j, \\ \dot{v}_j & = -a_j \, P'(\tau_j) \, \frac{v_{j+1}-v_j}{\Delta x} \; = \; - a_j P'(\tau_j) \, \dot{\tau}_j \\ \dot{a}_j & = 0. \end{cases} \quad (2)$$

(Explicit) Discretization of (2) $\equiv$ Godunov scheme for (1):

$$\begin{cases} \tau_j^{n+1} : & = \tau_j^n + \frac{\Delta t}{\Delta x}(v_{j+1}^n - v_j^n), \\ w_j^{n+1} & = w_j^n, \\ a_j^{n+1} & = a_j^n. \end{cases} \quad (3)$$

## Homogenized Multiclass Model $(1_{\mathbf{hom}})$ :

Theorem 1.

$U_h := (v_h, w_h, a_h) \quad ; \quad \nu_{x,0} := \delta(v - v_*^0(x)) \otimes \mu_x(w, a)$, associated to the whole sequence $U_h^0$. Then, when $\Delta t$ and $\Delta x \to 0$, with $\Delta t/\Delta x = C$ and CFL, the whole sequence CV in $L^\infty$ weak-* to the unique weak "entropy" solution of homogenized system :

$$\begin{cases} \partial_t \tau^* - \partial_x v^* = 0 \\ \partial_t w^* = 0 \\ \partial_t a^* = 0 \end{cases}$$

with:

$$\begin{cases} \tau^*(x,t) =< \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) > \\ \overset{\text{e.g.}}{=} \ \sum_i \alpha_i(x) P^{-1}((w_i - v^*(x,t))/a_i). \end{cases}$$

Initial data :

$$\begin{cases} \tau_*^0 =< \mu_x(w,a), P^{-1}((w - v_0^*)/a) >, \\ (w_*^0,\ a_*^0) \ = \ < \mu_x(w,a), (w,\ a) > . \end{cases}$$

Theorem 2

(i) When $\Delta t \to 0$ with fixed $\Delta x$, (3) $\to$ (2) : the whole sequence

$(v_h, \tau_h, w_h, a_h) \overset{\text{weak-}*}{\rightharpoonup} (v_\Delta, \tau_\Delta, w_\Delta, a_\Delta)$

$\equiv \; (v_j, \tau_j, w_j, a_j)(t)$ for $x \in (x_j, x_{j+1}) \quad \equiv$ solution of (2).

(ii) Same estimates : $BV$ in $x$ and $t$ for $v_\Delta$, in $t$ for $\tau_\Delta$.

(iii) Entropy inequalities $\to$ : $\forall$ concave $q(v)$,

$\frac{d}{dt}\eta(v_j(t), w_j, a_j) + (q(v_{j+1})(t) - q(v_j)(t))/\Delta x \leq 0.$

Theorem 3

(i) When $\Delta x \to 0$ (2) $\rightharpoonup$ $(1_{hom})$ : the whole sequence

$(v_\Delta, \tau_\Delta, w_\Delta, a_\Delta) \overset{\text{weak-}*}{\rightharpoonup}$ to the unique entropy solution of homogenized model $(1_{\text{hom}})$.

Sketch of Proof.

Strong CV for $v_\Delta, q(v_\Delta)$, weak-* for $(\tau_\Delta, w_\Delta, a_\Delta) \; \to$ (another (?)) mv-solution.

Same $\mu_x$, entropy inequalities : uniqueness of $v^* \to$ same limit.

Result = Commutation of limits.

Homogenization of $\tau$ as a function of $v$, with oscillating $w, a$.