A Multi-Class Homogenized Hyperbolic Model of Traffic Flow

P. Bagnerini and M. Rascle

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Abstract

We introduce a new homogenized hyperbolic (multi-class) traffic flow model which allows to take into account the behaviors of different type of vehicles (cars, trucks, buses, etc.) and drivers. We discretize the starting Lagrangian system introduced below with a Godunov scheme, and we let the mesh size h in (x, t) go to 0: the typical length (of a vehicle) and time vanish. Therefore, the variables - here (w, a) - which describe the heterogeneity of the reactions of the different car-driver pairs in the traffic, develop large oscillations when $h \to 0$. These (known) oscillations in (w, a) persist in time, and we describe the *homogenized* relations between velocity and density. We show that the velocity is the *unique* solution "à la Kružkov" of a scalar conservation law, with variable coefficients, discontinuous in x. Finally, we prove that the same macroscopic homogenized model is also the *hydrodynamic limit* of the corresponding multi-class Follow-the-Leader model.

Keywords : Hyperbolic systems of conservation laws, Traffic Flow, Multi-Class models, Homogenization, Discontinuous Flux, Uniqueness, Hydrodynamic limit

1 Introduction

In [1], Aw et Rascle developed a new macroscopic model of traffic flow which allows to avoid some inconsistencies of the models inspired by the gas dynamic system. In [2], a connection is established between the microscopic "Follow-the-Leader model", and (a semi-discretization of) the macroscopic model introduced in [1], which is its hydrodynamic limit.
In this paper, we introduce, still for a single lane traffic, a new macroscopic homogenized hyperbolic model for the multi-class traffic flow, described by a non linear hyperbolic system of three conservation laws. For references in multi-class (or multi-population) models see e. g. [6, 13, 18, 49], see also [44] for inhomogeneous road conditions and uniqueness results.

The starting system, written in Lagrangian mass coordinates, is the following:

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t w = 0, \\ \partial_t a = 0, \\ (\tau, w, a)(x, 0) = (\tau_0, w_0, a_0)(x), \end{cases} \tag{1.1}$$

where $\tau = 1/\rho$ is the specific volume, i.e the inverse of the density of vehicles (i.e. of the fraction of space occupied by the cars), v the velocity, and $a \in [0, 1]$ is a dimensionless coefficient which allows to take into account the behaviors of different types of vehicles and/or of drivers. Here, (up to a constant) w describes the difference between the velocity and some equilibrium velocity

$$w = v - V(\tau, a), \tag{1.2}$$

where (up to a constant) $V(\tau, a)$ is an "equilibrium" velocity for a given τ and for a given class a. For simplicity, *all the results are stated with* $w = v - V(\tau, a) := v + aP(\tau)$, *but they remain valid* in the more general case described below.
The last two equations express the fact that w and a are characteristic of each vehicle and therefore do not depend on time in Lagrangian coordinates.
In some sense, the model introduced in [1] was already multi-class, each class being described by its distance to the *same* equilibrium curve $v = V(\tau)$. Here we add (at least) a second parameter a, to let each equilibrium curve $v = V(\tau, a)$ depend on each class. Of course, we could add more parameters, with the same results. We could also take into account the length of cars by considering a non-uniform mesh in x.

We discretize the model (1.1) with the Godunov scheme: let $U_h(\cdot,\cdot)$ be the approximate solution for a discretization $(\Delta x, \Delta t)$ and initial piecewise-constant data U_h^0. On one hand, h is the step-size of the discretization, but on the other hand, h can be viewed as a scaling parameter $(x,t) \to (x',t') = (hx, ht)$. Assuming typically that in each cell there is a unique vehicle, we thus consider a large number of vehicles on a long stretch of the road, and the length of the vehicles vanishes as $h \to 0$.
Practically, the distribution of the different type of car-driver pairs can be highly oscillatory. We are thus led to studying an *homogenized* system: we consider a *sequence* of initial data $(v_h^0, w_h^0, a_h^0, \tau_h^0)$, with $w_h^0 = v_h^0 - V(\tau_h^0, a_h^0)$, and $h \to 0$. We assume that the sequence (v_h^0) converges boundedly almost everywhere to some v_0^*, whereas (w_h^0, a_h^0) and therefore τ_h^0 only converge *weakly* to (w_0^*, a_0^*) and τ_0^*.

Let us also emphasize the fact that this model, like the Follow-the- Leader models, is in principle a *single lane* model : no car can pass another car, and therefore the velocities cannot be wildly oscillating, although some differences and even discontinuities (braking etc...) are permitted. Therefore it is natural to assume that v_0 is a BV function.

In contrast, w_0 and a_0 can definitely oscillate : for instance, in rescaled coordinates (x', t'), a good prototype would be $w_h^0(x') := W_0(x', \frac{x'}{h})$, for some given function W_0 (or A^0) of (x', θ), oscillating (but not necessarily periodic)) in θ. In the sequel we will drop the primes and write (x,t) instead of (x',t').
We then study the weak-star limit (v^*, w^*, a^*, τ^*) of the solution (v_h, w_h, a_h, τ_h) of *the discretization of* system (1.1)-(1.2) as $h \to 0$. Using the notion of Young measure [43] and the compensated compactness theory, we study the propagation of these initial oscillations: we show that the Young measure $\nu_{x,t}$ associated to the variables (v, w, a) is a.e. a tensor product

$$\nu_{x,t} = \delta(v - v^*(x,t)) \otimes \mu_x(w, a),$$

where δ is the Dirac mass at the origin and $\mu_x(w,a)$ a probability measure defined a.e. in $\mathbb{R}_x$.
We prove that the approximate solutions U_h constructed by the Godunov scheme satisfy a discrete Lax entropy inequality, for any entropy convex *with respect to* τ, and under the CFL condition, converge as $h = (\Delta t, \Delta x) \to 0$ to an entropy solution of the homogenized system described below, in fact to some measure-valued solution [19]. The existence of

such a mv-solution is thus trivial.
In the case where the above-mentioned functions W_0 and A_0 are periodic in the second variable (and more regular in x) we recover the results on the homogenization of the corresponding Hamilton-Jacobi equation, see e.g. [31], [34], [22] and the references therein. Still in the periodic case, see also [20] for the construction of corrector terms to the next order.
We then study the *uniqueness* of the solution. The main difficulty is that w_h and a_h only converge weakly as $h \to 0$, so that the Young measure $\nu_{x,t}$ is not a δ-function in (w, a). Due to its special form, the system can be rewritten as the two trivial equations $\partial_t w^* = \partial_t a^* = 0$, coupled with a scalar equation where the flux explicitly depends on x, with a low regularity in x, so that we cannot use directly the uniqueness result of Kružkov.
However, since the flux is strictly increasing in τ, we can *exchange* the roles of x and t (and τ and v), so as to obtain an entropy inequality in conservative form, without any additional term involving the x-derivative of the flux. Therefore, we do not need stronger assumptions on the regularity of the flux with respect to x and we show the uniqueness by a variant of the Kružkov "doubling of variables" argument, in which we *first* "let y tend to x", and *then* "let s tend to t", as in [4].
Finally, *last but not least*, we consider the corresponding microscopic multi-class "Follow-the-Leader model":

$$\begin{cases} \dot{\tau}_j = \frac{v_{j+1} - v_j}{\Delta x}, \\ \dot{w}_j = \dot{v}_j - \frac{\partial V}{\partial \tau_j}(\tau_j, a_j) = 0, \ \dot{a}_j = 0, \\ \tau_j(0) = \tau_j^0, \ w_j(0) = w_j^0, \ a_j(0) = a_j^0. \end{cases} \tag{1.3}$$

where v_j is the speed of the vehicles at time t, Δx the length of the vehicle, $\tau_j = 1/\rho_j$ the local "specific volume around vehicle j" and ρ_j the local density, whereas a_j is a coefficient which depends on each type of vehicles.
The function V is the same as in (1.2) and w has the same meaning. The two last equations are due to the assumption that the coefficients (w_j, a_j) characterize each vehicle and therefore do not change in time.
We can easily see, at least formally, that the system (1.3) is a semi- discretization in space of the continuum model (1.1) in Lagrangian coordinates. In fact, see also [2], we establish *rigorously* in Section 6 that the solution of (1.3) converges as $\Delta x \to 0$ to the unique entropy solution of the homogenized macroscopic model.

The outline of the paper is as follows. In Section 2, we describe the model and the Riemann Problem. We also describe the scaling, and show in Remark 2.1 a *prototype* of measure μ_x for a "practical" applications. In Section 3, we study the Godunov scheme and the corresponding a priori estimates. In Section 4, we first show in Theorem 4.2 - at least for a subsequence - the convergence to a (variant of) measure-valued solution when $(\Delta x, \Delta t) \to (0, 0)$, with a fixed ratio and the CFL condition. In Theorem 4.3, we then reformulate the limit system. The homogenized relation between τ and v is now given by (4.6). The limit v is a solution "à la Kružkov" of the first equation of the limit system, i.e. of the scalar equation (4.14) (with variable non smooth coefficients) coupled with (4.6) and the two trivial equations (4.5) for w and a. Finally, in Section 5 we prove the *uniqueness* of this solution and in Section 6, we show that we recover (1.3) when $\Delta t \to 0$ and Δx is fixed, and then we show that the solution of (1.3) converges to the *unique* solution of the *same* homogenized model (4.14), (4.5), (4.6), which is therefore the *hydrodynamic limit* of (1.3).

2 Description of the model

In this Section, we describe the properties of system (1.1), that we first write in Eulerian coordinates. We then study the Riemann Problem, before describing more precisely in Section 3 the approximate solutions constructed by the Godunov scheme and the corresponding a priori estimates for all h. We recall that h is a scaling parameter which tends to 0, see Section 2.4 below, so that the sequence (w_h^0, a_h^0, τ_h^0) only converges weak-star in the L^∞ space when $h \to 0$.

2.1 The macroscopic model

Aw et Rascle [1] have introduced a macroscopic model of traffic flow which allows to avoid the severe inconsistencies of the so-called "second order" models, whose prototype is the Payne-Whitham model [35, 48]. Then, in [2], see also [24], a *rigorous* connection is established between the microscopic "Follow-the-Leader model", and a semi-discretization of the macroscopic model introduced in [1], see also [50]: namely, the macroscopic system can be viewed as the limit of the microscopic Follow-the-Leader (ODE) system (resp. of its explicit first order Euler (time) discretization) when $\Delta x \to 0$ (resp. when $(\Delta x, \Delta t) \to 0$ with a fixed ratio $\Delta t/\Delta x$ satisfying the CFL condition).
In this Section, we extend this macroscopic model to describe a multi-class traffic flow, in order to take into account the behaviors of different type of vehicles (cars, trucks, buses, etc.) and drivers (slow, aggressive, etc.). In conservative form, the model writes in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(v\rho) = 0, \\ \partial_t(\rho w) + \partial_x(v\rho w) = 0, \\ \partial_t(\rho a) + \partial_x(v\rho a) = 0, \end{cases} \tag{2.1}$$

where ρ denotes the (normalized) dimensionless density of vehicles, $a \in [0,1]$ a dimensionless coefficient which characterizes each type of vehicle-driver pair and w is defined by (1.2) with $\tau := 1/\rho$: in other words, up to some constant, w describes the difference between the actual velocity and the equilibrium one. In Lagrangian "mass" coordinates (X, T) [15], the system (2.1)-(1.2) can be rewritten under the form (1.1)-(1.2). We recall that

$$\partial_x X = \rho, \qquad \partial_t X = -\rho v, \qquad T = t, \qquad \tau := 1/\rho,$$

and that - even for weak (entropy) solution, see [47] - systems (2.1) and (1.1) are equivalent.
In our case, see [2], since ρ is dimensionless, $X = \int^x \rho(y,t)dy$ is not a mass, but describes the total length occupied by cars up to point x.

Now, when $h \to 0$, system (1.1) is again preserved in the rescaled variables $(X', T') := (hX, hT)$, see Section 2.4 below. We then drop the primes and even rewrite (x, t) instead of (X', T').
Finally, in these rescaled Lagrangian variables, we consider the system (1.1), with $\tau := 1/\rho$ and w given by (1.2).
We assume that for all a, $V(\cdot, a)$ is strictly
increasing and strictly concave. A good prototype of function V could be, up to a constant

$$V(\tau, a) = -[(1-a)P_1(\tau) + aP_2(\tau)], \;\; 0 \leq a \leq 1,$$

or even, more simply

$$V(\tau, a) = -\, aP(\tau), \;\; 0 < a_{min} \leq a \leq a_{max} \leq 1. \tag{2.2}$$

Here, P (or P_1, P_2) satisfies the same assumptions as $-\,V(\cdot, a)$, i.e.

$$P'(\tau) \leq c_1 < 0, \; P''(\tau) \geq c_2 > 0. \tag{2.3}$$

Again, up to a constant, w describes a distance to equilibrium. Practically, $a = a_{min}$ (resp. a_{max}) would correspond to the slowest (resp. fastest) car-driver pairs and P_1 and P_2 to a minimum and maximum equilibrium velocity for a given τ. Since V is invertible in τ, we write

$$\tau := \mathcal{T}(v, w, a) := (V(\cdot, a))^{-1}(v - w) = P^{-1}((w - v)/a). \tag{2.4}$$

As we already said, *all results below will be stated* in the particular case (2.2), but they remain valid in the more general situation (1.2). For concreteness, see [2], a practical example of function $P(\tau)$ is, up to a constant:

$$P(\tau) := \begin{cases} \frac{v_{ref}}{\gamma} \frac{1}{\tau^\gamma}, & \gamma > 0, \\ -v_{ref} \ln(\tau), & \gamma = 0, \end{cases} \tag{2.5}$$

where v_{ref} is a given reference velocity for all classes of vehicles (for instance 90 km/h). Here the parameter γ has no physical meaning, but similar power laws appear e.g. in the (strongly related) microscopic models [23, 25], see Section 6.
In the sequel, unless explicitly stated, we work in Lagrangian coordinates.

2.2 The Riemann Problem

In order to introduce the Godunov method, we first describe the solution of the Riemann Problem for (1.1), i.e. of the Initial Value Problem (IVP), with particular data $(\tau_0, w_0, a_0)(x) := (\tau_\pm, w_\pm, a_\pm)$ for $\pm x > 0$.
The eigenvalues of the system (1.1) are

$$\lambda_1 = -\,\frac{\partial V}{\partial \tau}(\tau, a) = aP'(\tau) < 0 = \lambda_2 = \lambda_3.$$

The system is non strictly hyperbolic: the associated matrix is diagonalizable and its diagonal form is:

$$\begin{cases} \partial_t v + aP'(\tau)\partial_x v = 0, \\ \partial_t w = 0, \\ \partial_t a = 0, \end{cases} \tag{2.6}$$

where v and (w, a) are the strict Riemann invariants, respectively associated to λ_1 and $\lambda_2 = \lambda_3 = 0$. The eigenvalue λ_1 is genuinely nonlinear (GNL) i.e. $\forall U$, $\nabla_U \lambda_1 \cdot r^1(U) = -a\, P''(\tau) < 0$, and $\lambda_2 = \lambda_3$ is linearly degenerate (LD), i.e. $\forall U$, $\nabla_U \lambda_k \cdot r^k(U) \equiv 0$, $k = 2, 3$.
We show that the solution of the Riemann Problem associated to the system (1.1), consists of two waves: a rarefaction or a shock wave associated to λ_1, followed by a contact discontinuity associated to $\lambda_2 = \lambda_3$ (see Figure 1).
The equivalent systems (1.1) and (2.1) are called Temple systems (in the extended sense), see [26]: their shock curves and rarefaction curves coincide. Therefore, in the space of the Riemann invariants v, w, a, the wave curves are straight lines.

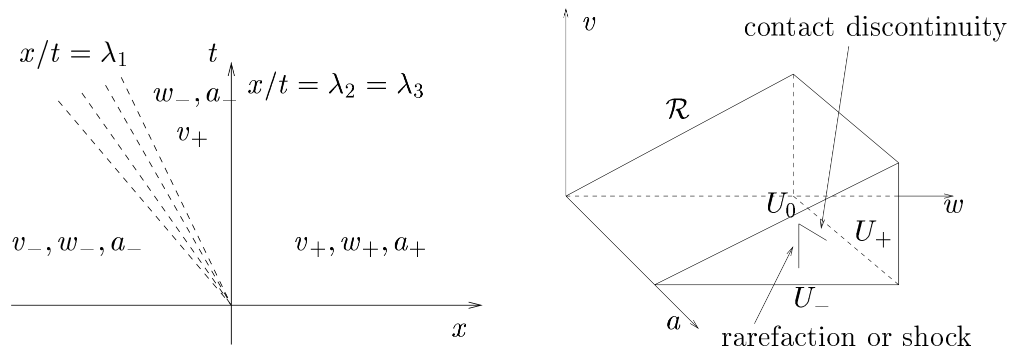

Figure 1: Riemann Problem (here in the case of a 1-rarefaction wave) and invariant region

Proposition 2.1. *We consider the Riemann Problem:*

$$\begin{cases} \partial_t \begin{pmatrix} \tau \\ w \\ a \end{pmatrix} - \partial_x \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ (\tau, w, a)(x, 0) = (\tau_\pm, w_\pm, a_\pm) := U_\pm, \ \textit{for} \ \pm x \geq 0. \end{cases} \tag{2.7}$$

The solution $U(x,t) := (\tau, w, a)(x,t)$ is as follows:

(i) we connect U_- to an intermediate constant state $U_0 = (\tau_0, w_0, a_0)$ such that $v_0 = v_+$, $w_0 = w_-$, $a_0 = a_-$, by a 1-rarefaction if $v_+ > v_-$ or by a 1-shock if $v_+ < v_-$. We connect then U_0 through U_+ by a 2-3-contact discontinuity of velocity 0, with $v_0 = v_+$.

(ii) Moreover, w and a only take the values (w_-, a_-) and (w_+, a_+), and remain constant in time for each x. Now, v is a monotone function of x/t, with $\min(v_-, v_+) \leq v(x,t) \leq \max(v_-, v_+)$.

(iii) In a 1-wave, the specific volume τ varies in the same direction as v, i.e. is a monotone function of x/t. Finally, for all x, τ is monotone with respect to t, and $\forall t \geq 0$,

$$\min(\tau_-, \tau_0) \leq \tau(x,t) \leq \max(\tau_0, \tau_+).$$

(iv) Therefore, $U(x,t)$ and $v(x,t)$ remain in an invariant bounded region $\mathcal{R}$, away from the vacuum:

$$\mathcal{R} := \{(\tau, w, a); (v, w, a) \in [v_{min}, v_{max}] \times [w_{min}, w_{max}] \times [a_{min}, a_{max}]\}, \qquad (2.8)$$

with v_{min} (e.g. $v_{min} = 0$), $v_{max}, w_{min}, w_{max}, a_{min}, a_{max} \geq 0$ some constants and $max\{\tau, (\tau, w, a) \in \mathcal{R}\} < \infty$ (in the case (2.2) we assume $a_{min}, a_{max} > 0$).

Proof. The proof is classical. The monotonicity of v is due to the fact that v only take the values v_-, v_+ and the wave curves are straight lines in the (v, w, a) space. Finally, since (w, a) is constant in a 1-wave and P monotone, $\tau = P^{-1}((w - v)/a)$ varies in the same sense as v. □

The qualitative properties of the solution are as expected: braking corresponds to a shock, accelerating to a rarefaction, no information travels faster than the velocity of cars, the velocity and the density remain nonnegative and bounded, etc.

2.3 Entropies of the system

In the sequel, since $v = w + V(\tau, a)$, we will sometimes denote the entropies

$$\eta(v, w, a) = \eta(w + V(\tau, a), w, a) \overset{e.g.}{=} \eta(w - aP(\tau), w, a), \qquad (2.9)$$

by $\tilde{\eta}(U) = \tilde{\eta}(\tau, w, a)$ or even - incorrectly - by $\eta(\tau, w, a)$, when this notation is not ambiguous. In such a case $\partial_\tau \eta$ means $\partial_\tau \tilde{\eta}$, so that e.g. $\partial_\tau \eta(\tau, w, a) = \partial_v \eta(v, w, a)(-aP'(\tau))$. There is no such problem for the entropy fluxes q, which only depend on v, see below. Now we study the entropy-flux pairs (η, q) of the system (1.1). In variables (v, w, a), multiplying formally the left-hand side of (2.6) by $(\partial_v \eta,\ \partial_w \eta,\ \partial_a \eta)$, we obtain

$$\begin{cases} aP'(\tau)\partial_v \eta = \partial_v q, \\ 0 = \partial_w q, \\ 0 = \partial_a q. \end{cases} \qquad (2.10)$$

Therefore, the flux q associated to η only depends on the variable v, i.e. $q \equiv q(v)$ and *all* entropy-flux pairs are given by

$$\eta(v, w, a) = \int_0^v \frac{q'(s)}{aP'(\tau(s, w, a))} ds + \eta_0(w, a),\ q = q(v), \qquad (2.11)$$

for any function $\eta_0 = \eta_0(w,a) := \eta(w,a,0)$ and $q = q(v)$. By (2.9) and the first equation (2.10), we obtain

$$\partial_\tau \eta(\tau,w,a) = \partial_v \eta(v,w,a)\, \partial_\tau v(\tau,w,a) = \frac{1}{aP'(\tau)} q'(v)(-aP'(\tau)) = -q'(v). \qquad (2.12)$$

Proposition 2.2. *For all entropy $\eta = \eta(\tau,w,a)$, associated to the flux $q \equiv q(v)$, we have:*

(i) η is convex with respect to τ *if and only if $q \equiv q(v)$ is concave: $q''(v) \le 0$.*

(ii) Let U be the solution to the Riemann Problem (2.7). Then, for all entropy η satisfying (i), we have

$$\partial_t \eta + \partial_x q \le 0 \ \text{in}\ \mathcal{M}(\mathbb{R} \times (0,\infty)). \qquad (2.13)$$

Proof. *(i)* Differentiating twice η with respect to τ (with fixed (w,a)), and using (2.12), we obtain

$$\partial_\tau^2 \eta(\tau,w,a) = \partial_\tau(-q'(v)) = \partial_v(-q'(v))\, \partial_\tau v = q''(v)\, a\, P'(\tau).$$

Therefore, η is convex if and only if q is concave, since $a > 0$ and $P' < 0$.
(ii) Through a 1-rarefaction wave and a 2-3 contact discontinuity, for $x > 0$, (2.13) is an equality. Through a 1-shock wave, therefore for $x < 0$, we have $(w,a) = (w_-,a_-)$. Using the Rankine-Hugoniot relations between U_- and $U = U_0$, the entropy condition is equivalent to proving that

$$S(v) := S_{v_-}(v) = \frac{v_- - v}{\tau - \tau_-}(\eta(\tau,w,a) - \eta(\tau_-,w,a)) - (q(v) - q(v_-)) \ge 0,$$

i.e. that $S(\cdot)$ is decreasing, since $v < v_-$ in a 1-shock and $S_{v_-}(v_-) = 0$. Since $\tau = \mathcal{T}(v,w,a) := P^{-1}((w-v)/a)$, differentiating S with respect to v, we obtain

$$S'(v) = \frac{-(\tau - \tau_-) - (v_- - v)\frac{\partial \mathcal{T}}{\partial v}}{(\tau - \tau_-)^2}(\eta(\tau,w,a) - \eta(\tau_-,w,a)) + \frac{v_- - v}{\tau - \tau_-}\frac{\partial \eta}{\partial \tau}\frac{\partial \mathcal{T}}{\partial v} - q'(v).$$

Using (2.12) and recombining the terms, we obtain

$$S'(v) = -\frac{\eta(\tau_-,w,a) - \eta(\tau,w,a) - (\tau_- - \tau)\frac{\partial \eta}{\partial \tau}}{(\tau - \tau_-)^2}(\mathcal{T}(v_-,w,a) - \mathcal{T}(v,w,a) - (v_- - v)\frac{\partial \mathcal{T}}{\partial v}),$$

which is nonpositive since η is convex *with respect to* τ and $\mathcal{T}$ strictly convex with respect to v.
We emphasize that we *only* need the convexity of η in τ, since (w,a) is constant through a 1-shock, i.e. here for $x < 0$.

□

2.4 Scaling. An example of $\mu_{\mathbf{x}}$

The macroscopic models are only valid if we consider a large number of vehicles on a long stretch of the road. Therefore, we introduce a scaling (zoom) such that the size of the domain and the number of vehicles is going to infinity, whereas the length of the vehicles tends to 0 (see also [2]). Given the Lagrangian coordinates x, t, we consider the new rescaled coordinates

$$x' := hx, \quad t' := ht,$$

where h is a small parameter, proportional to the inverse of the maximal possible number of cars per new unit length. In the new coordinates x', t', the variables ρ, τ and the Riemann invariants v, w, a are unchanged, whereas the length of a vehicle becomes $\Delta x' = h\Delta x$, which tends to 0 as $h \to 0$: in the new coordinates, the convergence of the Godunov scheme to the entropy solution of (4.4) can be viewed as the convergence of microscopic system to the macroscopic model, when the size of the road and the number of vehicles are going to infinity.
For instance, assume that the initial units are meter and second and the new one, 1500 m and 60 seconds, with $\Delta x = 5m$. In the rescaled coordinates, $x' := x/1500$, $t' := t/60$, a reference velocity of 90 km/h, i.e. 25 m/s becomes 1 in the new units and the length of the car $\Delta x' = 1/300$, which is a "reasonable" step-size.
In particular, a typical sequence of oscillating initial data $(w_0^h, a_0^h)(x) := (W_0, A_0)(x, hx)$ (and its limit in L^∞ weak*), could be in the new coordinates:

$$(w_h^0, a_h^0)(x') = (W_0, A_0)\left(\frac{x'}{h}, x'\right) \underset{h\to 0}{\rightharpoonup} (w_0^*, a_0^*)(x') = (<\mu_x, w>, <\mu_x, a>)\ L^\infty \text{ weak*},$$

where the Young measure μ_x will be introduced in Section 4.1.

Remark 2.1. *Typically, given a finite number N of classes a_i, associated to $w_i = v_i - V(\tau_i, a_i)$, a possible choice of μ_x is given by*

$$\mu_x(w, a) = \sum_{i=1}^{N} \mu_i(x)\delta_{w_i, a_i}(w, a), \quad \textit{with} \ \sum_{i=1}^{N} \mu_i(x) \equiv 1,$$

where the nonnegative coefficients $\mu_i(x)$ are the local proportion (possibly 0) of each class of car-driver pair and δ the Dirac measure. When we then compute an approximation of the average *velocity v^* by the Godunov scheme, not only we know the average specific volume $\tau^*(x,t) = \sum_{i=1}^{N} \mu_i(x)\mathcal{T}(v^*(x,t), w_i, a_i)$ introduced in equation (4.6), but also the specific volume τ_i (and then the density) of* every class *a_i.*

3 The Godunov scheme

Now we discretize the system (1.1), written in general form as

$$\partial_t U + \partial_x G(U) = 0,$$

with the Godunov scheme. We introduce a grid in time and space, with step-size Δx and Δt (related to a parameter h as indicated below) and grid point $x_{j-1/2}$ and t_n. Let

I_j be the *open* interval $(x_{j-1/2}, x_{j+1/2})$ and, $\forall(\Delta x, \Delta t)$, with $\Delta t/\Delta x = C$, the sequence $U_h := (\tau_h, w_h, a_h)$ is the approximation by the Godunov scheme, given for all $n \geq 0$ by

$$\begin{cases} \partial_t \tau_h - \partial_x v_h = 0, & \text{in } \mathbb{R} \times (t_n, t_{n+1}), \\ \partial_t w_h = 0, \ \ \partial_t a_h = 0, & \end{cases} \tag{3.1}$$

with $v_h := w_h - a_h P(\tau_h)$ and with piecewise-constant initial data

$$U_h(x, t_n^+) := (\tau_h, w_h, a_h)(x, t_n^+) := \sum_{j \in \mathbb{Z}} (\tau_j^n, w_j^n, a_j^n) \chi_j(x),$$

where $\chi_j(y) = 1$ on $(x_{j-1/2}, x_{j+1/2})$ and 0 elsewhere. Let $U_j^n := (\tau_j^n, w_j^n, a_j^n)$ denote the average value of the function $U_h(x,t)$ in the interval I_j, i.e.

$$U_j^n := (1/\Delta x) \int_{I_j} U_h(x, t_n^-) dx.$$

In every cell $I_j \times]t_n, t_{n+1}[$, we compute the solution of the local Riemann Problems centered in the grid points $x_{j\pm 1/2}$, with initial data (U_j^n, U_{j+1}^n). Let $G_{j+1/2}^n := G(U_j^n, U_{j+1}^n) = (-v_{j+1}, 0, 0)$ be the flux at point $x_{j+1/2}$. The solution U_j^{n+1} of the Godunov scheme is given by

$$U_j^{n+1} := U_j^n - \frac{\Delta t}{\Delta x}(G_{j+1/2}^n - G_{j-1/2}^n) = U_j^n + \frac{\Delta t}{\Delta x}(v_{j+1}^n - v_j^n, 0, 0). \tag{3.2}$$

with $v_j^n := w_j^n - a_j^n P(\tau_j^n)$. Now, we assume that (forall h), $\forall x \in \mathbb{R}$, $U_h(x, 0) = U_h^0(x) = (\tau_h^0, w_h^0, a_h^0) \in \mathcal{R}$, where $\mathcal{R}$ is defined in (2.8), and for simplicity, see (4.1), we also assume that the sequence v_h^0 is bounded in $BV(\mathbb{R})$, i.e. $\sum_{j \in \mathbb{Z}} |v_{j+1}^0 - v_j^0| \leq C_0 < +\infty$. This assumption is sufficient in practical life (just think of a road where the velocity of cars is *not* in $BV(\mathbb{R})$!), but it is not necessary since λ_1 is "GNL". We do *not* assume that the sequences w_h^0 and a_h^0 are bounded in BV.
The following Theorem shows that the region $\mathcal{R}$ defined in (2.8) is also invariant for the Godunov scheme and gives estimates on the total variation of some components of the solution. Under the above assumptions,

Theorem 3.1. *Let Δx and Δt satisfy the CFL condition. Then*

(i) The region $\mathcal{R}$ remains invariant for the Godunov scheme.

(ii) $\forall n \geq 0$, the total variation (in x) of $v_h(\cdot, t)$ is non increasing in time, and the total variation in t of $\tilde{v}_h(\cdot, \cdot)$ is bounded on $\mathbb{R} \times [0, T]$:

$$TV_x(v_h^n; \mathbb{R}) := \sum_{j \in \mathbb{Z}} |v_j^n - v_{j+1}^n| \leq TV(v_h^0; \mathbb{R}) \ , \tag{3.3}$$

$$\sup_h \sup_{t \geq 0} TV_x(v_h(\cdot, t); \mathbb{R}) \leq \sup_h TV(v_h^0(\cdot); \mathbb{R}) := C_0, \tag{3.4}$$

$$\sup_h TV_t(\tilde{v}_h(\cdot, \cdot); \mathbb{R} \times [0, T]) \leq C \max(T, \Delta t) \ C_0 \ , \tag{3.5}$$

where $v_h^0 := v_h(\cdot, 0^+)$ *is the piecewise-constant approximation of the initial datum* $v_0^*(\cdot)$*, and*

$$\tilde{v}_h(x,t) \equiv v_j^n + (t - t_n)(v_j^{n+1} - v_j^n)/\Delta t \text{ on } I_j \times (t_n, t_{n+1}), \tag{3.6}$$

and similarly for $\tilde{\tau}_h$*.*

(iii) The total variation (in x*) of* $\tau_h(\cdot, t)$ *on* $\cup_{j\in\mathbb{Z}} I_j$ *and the total variation in* t *of* $\tilde{\tau}_h(\cdot,\cdot)$ *on* $\mathbb{R} \times [0, \infty[$ *are bounded, uniformly in* h*:*

$$\sup_h \sup_{t\geq 0} \sum_{j\in\mathbb{Z}} TV_x(\tau_h(\cdot, t); \cup_{j\in\mathbb{Z}} I_j) \leq C' \, C_0, \tag{3.7}$$

$$\forall 0 \leq t \leq t', \;\; \sup_h TV_t(\tilde{\tau}_h(\cdot,\cdot); \mathbb{R} \times [t, t']) \leq C' \max(|t' - t|, \Delta t) \, C_0. \tag{3.8}$$

(iv) $\forall x \in \mathbb{R}$*,* $(w_h, a_h)(x,t) \equiv (w_h^0, a_h^0)(x)$*.*

Proof. *(i)* follows easily from the solution of the Riemann Problem in Proposition 2.1.
(ii) Adding and subtracting v_{j+1}^n and recombining the terms in the sum, we obtain

$$\begin{aligned} TV_x(v_h^{n+1}; \mathbb{R}) &= \sum_{j\in\mathbb{Z}} |v_j^{n+1} - v_{j+1}^{n+1}| \leq \sum_{j\in\mathbb{Z}} (|v_j^{n+1} - v_{j+1}^n| + |v_{j+1}^n - v_{j+1}^{n+1}|) = \\ &= \sum_{j\in\mathbb{Z}} (|v_j^{n+1} - v_{j+1}^n| + |v_j^n - v_j^{n+1}|). \end{aligned}$$

Now, using the monotonicity property of v in the Riemann Problem (see Proposition 2.1), we see that the average value v_j^n of $v(\cdot, t_{n+1})$ on I_j in the projection step belongs to the interval $I(v_j^n, v_{j+1}^n)$ (see Figure 1) and thus

$$|v_j^{n+1} - v_{j+1}^n| + |v_j^n - v_j^{n+1}| = |v_{j+1}^n - v_j^n|.$$

Therefore, we obtain

$$TV_x(v_h^{n+1}; \mathbb{R}) \leq \sum_{j\in\mathbb{Z}} |v_{j+1}^n - v_j^n| \leq \cdots \leq \sum_{j\in\mathbb{Z}} |v_{j+1}^0 - v_j^0| = TV(v_h^0; \mathbb{R}),$$

and for the same reason we obtain (3.4).
Concerning (3.5), since $\tilde{v}_h$ is piecewise-linear in time, $v_j^{n+1} \in I(v_j^n, v_{j+1}^n)$ for $t \in [t_n, t_{n+1})$, with $I(a,b) = [\min(a,b), \max(a,b)]$. Therefore,

$$\begin{aligned} TV_t(\tilde{v}_h(x,\cdot); [0,T]) &= \sum_{j\in\mathbb{Z}} \sum_{n\leq \frac{T}{\Delta t}} |v_j^{n+1} - v_j^n| \Delta x \leq \sum_{j\in\mathbb{Z}} \sum_{n\leq \frac{T}{\Delta t}} |v_{j+1}^n - v_j^n| \Delta x \leq \\ &\leq \frac{T}{\Delta t} \Delta x TV(v_h^0; \mathbb{R}) \leq C \;\max(T, \Delta t)\; C_0 \end{aligned} \tag{3.9}$$

where $C = \Delta x/\Delta t$. See an alternate proof below with a uniform constant C.
(iii) Since in each cell I_j, $\tau_h(x,t) = \mathcal{T}(v_h(x,t), w_j, a_j) = P^{-1}\left(\frac{w_j - v_h(x,t)}{a_j}\right)$, with P^{-1} Lipschitz continuous, we have

$$TV_x(\tau_h(\cdot,t), \cup_{j\in\mathbb{Z}} I_j) = \sum_{j\in\mathbb{Z}} TV_x(\tau_h(\cdot,t), I_j) \leq \|(P^{-1})'\|_{L^\infty} (a_{min})^{-1} \, C_0$$

and similarly for (3.8).
(iv) is obvious. $\square$

Remark 3.1. *We first note that in general, the Godunov approximate solutions do* not *satisfy such a BV estimate. The result is true here since in each cell (w, a) is* constant. *We also note that the total variation in space of τ_h on $\cup_{j\in\mathbb{Z}} I_j$ is bounded, whereas its total variation on $\mathbb{R}$ can be infinite, due to the jumps in $x = x_{j+1/2}$, $\forall j \in \mathbb{Z}$. Finally, as to the total variation in time, we could proceed differently and use (3.4) and the first equation in (3.2) to show first (3.8) and then (3.9) with in each case a constant C independent of $\Delta x/\Delta t$, see Section 6.*

Proposition 3.1. *On any time interval $[t_n, t_{n+1})$, the solution $U_h = (\tau_h, w_h, a_h)$ satisfies the* discrete entropy inequality *in the sense of Lax: for any entropy $\eta(U_h)$ convex with respect to τ_h, associated to the entropy flux $q(U_h)$, for any n and j,*

$$\eta(U_j^{n+1}) \leq \eta(U_j^n) - (\Delta t/\Delta x)(q(U_{j+1}^n) - q(U_j^n)). \tag{3.10}$$

Proof. Classically, on any time interval $[t_n, t_{n+1})$, U_h is the solution of the Riemann Problem (2.7) and satisfies the entropy inequality (2.13) in every cell I_j. The Jensen inequality allows to conclude. We remark that, since (w, a) is *constant* in each cell, we only need the convexity of η *with respect to τ*. $\square$

Finally, the sequence (U_h) is a sequence of *approximate* solutions to (1.1), associated for each h to the initial data U_h^0. More precisely,

Proposition 3.2. *(i) $\forall h > 0$, $\forall \varphi \in \mathcal{D}(\mathbb{R}_x \times \mathbb{R}_+) = C_0^\infty((\mathbb{R}_x \times \mathbb{R}_+)$,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (\tau_h \partial_t \varphi - v_h \partial_x \varphi)(x,t)\,dxdt + \int_{\mathbb{R}} \tau_h(x,0)\varphi(x,0)\,dx = \\ &= -\sum_{n\geq 1}\sum_{j\in\mathbb{Z}} \int_{I_j} (\tau_j^n - \tau_h(x,t_n^-))\varphi(x,t_n)\,dx := < L_h, \varphi > . \end{aligned} \tag{3.11}$$

(ii) If the support of φ is compact in $\{0 \leq t \leq T\}$, then

$$| < L_h, \varphi > | \leq (\Delta x)^2 \|\partial_x \varphi\|_{L^\infty} \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} TV_x(\tau_h(\cdot,t); I_j), \tag{3.12}$$

$$| < L_h, \varphi > | \leq \; C\,T\,\|\varphi\|_{L^\infty} \sup_{n\leq \frac{T}{\Delta t}} \sum_j TV_x(\tau_h(\cdot,t_n^-); I_j). \tag{3.13}$$

(iii) Therefore, $L_h \to 0$ as $h \to 0$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R})$ and for all $T > 0$, the sequence (L_h) is bounded in $\mathcal{M}(\mathbb{R}\times[0,T])$.

(iv) Concerning the entropy production, $\forall \eta$ convex with respect to τ, *associated to the flux q, $\forall \varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$, $\varphi \geq 0$, we have:*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (\eta(U_h)\partial_t\varphi + q(U_h)\partial_x\varphi)(x,t)\,dxdt + \int_{\mathbb{R}} \eta(U_h(x,0))\varphi(x,0)\,dx \geq \\ &\geq -\sum_{n\geq 1}\sum_{j\in\mathbb{Z}} \int_{I_j} (\eta(U_j^n) - \eta(U_h(x,t_n^-))\varphi(x,t_n)\,dx \geq 0. \end{aligned} \tag{3.14}$$

Consequently, for all C^2 nonnegative entropy η associated to q, such that $x \to \eta(U_h^0(x))$ is integrable on $\mathbb{R}$, $\partial_t\eta+\partial_x q$ is a bounded measure on $\mathbb{R}\times\mathbb{R}_+$, nonpositive if η is convex with respect to τ.

Proof. The proof of *(i),(ii),(iii)* uses the classical arguments provided by the BV estimates in the Theorem 3.1. In particular, the BV property of τ_h on $\cup_{j\in\mathbb{Z}} I_j$ (and not on $\mathbb{R}$) is sufficient for (3.12). On the other hand (3.13) shows that the functional L_h is a bounded measure.
The proof of *(iv)* classically combines the entropy inequality for the Riemann Problem (Proposition 2.2), a discrete integration by parts (see for instance [30]), as well as the obvious remark that any C^2 function is the difference of two C^2 convex functions. □

We are now ready to pass to the limit as $h \to 0$.

4 The homogenized model: existence of a solution

4.1 The Young measure

In order to introduce in the model oscillations describing the heterogeneity of the reaction of each car-driver pairs in the traffic, we consider a sequence of oscillating initial data (w_h^0, a_h^0, τ_h^0) bounded in $L^\infty(\mathbb{R})$. We study the evolution in time of these initial oscillations for the approximate solution (w_h, a_h, τ_h) constructed by the Godunov method. For a few references in the study of large amplitude oscillations in nonlinear hyperbolic systems of conservation laws, we refer e.g. to [19, 42, 37, 16, 10, 14, 36] and numerous other reference.
Here, we simply consider the measure-valued solution [19] associated to this sequence of approximate solutions U_h. We consider typically that there is a unique vehicle in every cell I_j and we pass to the limit in the system (3.1) as $h \to 0$, in order to obtain a homogenized model. To describe the limit as $h \to 0$ of τ_h, which is a non linear function of the three variables v_h, w_h, a_h, we need the concept of Young measures.
Let us briefly recall the concept of Young measures associated to a sequence u^n, see e.g. [5, 43, 17, 32]. For any sequence $u^n : \mathbb{R}^N \to \mathbb{R}^p$ of measurable functions, with values in a fixed compact set $K \subset \mathbb{R}^p$, and such that $u^n \overset{*}{\rightharpoonup} u^*$ in $L^\infty(\mathbb{R}^N)^p$, there exists a subsequence of u^n, still denoted by u^n, and a family of probability measures $\{\nu_x\}_{x\in\mathbb{R}^N}$, called *Young measures*, uniformly supported in K such that for *any* $f \in C(\mathbb{R}^p;\mathbb{R}^q)$, and for almost all x in $\mathbb{R}^N$,

$$f(u^n) \overset{*}{\rightharpoonup} f^* \neq f(u^*) \text{ in } L^\infty(\mathbb{R}^N)^q, \text{ with } f^*(x) = \int_{\mathbb{R}^N} f(s)d\nu_x(s) =< \nu_x(\cdot), f(\cdot) > .$$

From now on, we assume that the sequence of initial data, with values in $\mathcal{R}$, satisfies :

$$\begin{gathered}\text{The } \textit{whole} \text{ sequence } (w_h^0, a_h^0, \tau_h^0) \text{ converges in } L^\infty \text{ weak* to a unique limit}\\ (w_0^*, a_0^*, \tau_0^*) \text{ whereas } v_h^0 \to v_0^* \text{ boundedly a.e. as } h \to 0, \text{ with } \sup_h TV(v_h^0) \le C_0.\end{gathered} \tag{4.1}$$

So, we assume that for all h, the total variation of the sequence v_h^0 is bounded from above. As we already said, this assumption is not necessary, since λ_1 is GNL, but it is more than sufficient for practical applications.

Here by Theorem 3.1, the sequence (v_h, w_h, a_h) remains in the invariant region $\mathcal{R}$. Consequently, for any continuous function f, at least for a subsequence, the associated Young measure $\nu_{x,t}$ satisfies:

$$f(v_h, w_h, a_h)(x,t) \stackrel{*}{\rightharpoonup} f^* := < \nu_{x,t}(v,w,a), f(v,w,a) >,$$

where (v,w,a) are (dummy) integration variables and $\stackrel{*}{\rightharpoonup}$ denotes the convergence in $L^\infty(\mathbb{R} \times \mathbb{R}_+)$ weak*. In particular, since $\tau_h = \mathcal{T}(v_h, w_h, a_h) = P^{-1}((w_h - v_h)/a_h)$,

$$\tau_h \stackrel{*}{\rightharpoonup} \tau^*(x,t) = < \nu_{x,t}(w,a), \mathcal{T}(v,w,a) > = < \nu_{x,t}(w,a), P^{-1}((w-v)/a) > . \quad (4.2)$$

In the following Theorem, we show that $\nu_{x,t}$ is a tensor product of a Dirac measure associated to v and a probability measure μ_x (depending only on x) associated to (w,a).

Theorem 4.1. *Let $U_h = (v_h, w_h, a_h)$ be constructed by the Godunov scheme. We assume that U_h^0 is bounded in $L^\infty(\mathbb{R})$ and (v_h^0) bounded in $BV(\mathbb{R})$. Then, under the assumptions* (2.3), (4.1),

(i) Under the CFL condition, with $\Delta t/\Delta x = constant$, as $h \to 0$, there exists a subsequence, still denoted by U_h, such that

$$(w_h, a_h)(x,t) \equiv (w_h^0, a_h^0)(x) \stackrel{*}{\rightharpoonup} (w_0^*, a_0^*), \;\; \tau_h(\cdot,\cdot) \stackrel{*}{\rightharpoonup} \tau^*(\cdot,\cdot),$$
$$v_h(\cdot,\cdot) \to v^*(\cdot,\cdot) \text{ in } L^1_{loc}(\mathbb{R} \times \mathbb{R}_+) \text{ strong and in } L^\infty(\mathbb{R} \times \mathbb{R}_+) \text{ weak*}.$$

(ii) In the variables (v,w,a) $\nu_{x,t}$ is a tensor product. More precisely, since λ_1 is genuinely non linear (GNL), even if v were not initially in $BV(\mathbb{R})$, we have:

$$\nu_{x,t} = \gamma_{x,t}(v) \otimes \beta_{x,t}(w,a) := \delta(v - v^*(x,t)) \otimes \mu_x(w,a), \quad (4.3)$$

where δ is the Dirac measure at 0 *and μ_x does not depend on time.*

(iii) In particular,

$$\tau_h \stackrel{*}{\rightharpoonup} \tau^*(x,t) = < \mu_x(w,a), P^{-1}((w - v^*(x,t))/a) > .$$

Proof. *(i)* follows directly from the L^∞ and BV estimates on U_h and v_h.
Here is a first proof of *(ii)*. The sequence $(\tilde{v}_h)$ has a uniformly bounded total variation and is equicontinuous in time in the L^1 space. Therefore it is strongly convergent (at least for a subsequence) to some $v^*(x,t)$, and then $\nu_{x,t}$ satisfies (4.3).
A second proof, with no BV assumptions, is the following: using Murat's Lemma [33] and Proposition 3.2, we apply the div-curl Lemma [43] to entropy-flux pairs $(\eta_1 = \eta_1(w,a), q_1 \equiv 0)$ and $(\eta_2 = \int^v q'(s)/(aP'(\tau(s,w,a)))ds, q_2)$, with arbitrary η_1, q_2. Therefore, $\forall \eta_1, q_2$,

$$< \nu_{x,t}, \eta_1(w,a) q_2(v) > = < \nu_{x,t}, \eta_1(w,a) > < \nu_{x,t}, q_2(v) > .$$

Therefore $\nu_{x,t} = \gamma_{x,t}(v) \otimes \mu_x(w,a)$. The associated measure μ_x only depends on x, since w and a do not depend on t. Now, let $[v_1, v_2]$ be the convex hull of the support of $\gamma_{x,t}$, with $v_1 < v_2$. We want to prove that $\gamma_{x,t}$ is a Dirac measure, i.e. that $v_1 = v_2$.

We apply again the div-curl Theorem to entropy-flux pairs of "east-west type" $(\eta_1^\epsilon, q_1^\epsilon)$ and $(\eta_2^\epsilon, q_2^\epsilon)$ (see [40]): e.g. we choose a smooth q_1^ϵ such that its support is contained in $(-\infty, \bar{v}_1]$, where $\bar{v}_1 = v_1 + \epsilon(v_2 - v_1)$, and such that $q_1^\epsilon(v) \to -H(v_1 - v)$ as $\epsilon \to 0$, with H the Heaviside function. Similarly, $q_2^\epsilon(v) \to H(v - v_2)$.
Let η_1^ϵ,η_2^ϵ be the associated entropies. Applying the div-curl Theorem and passing to the limit as $\epsilon \to 0$, we obtain

$$< \mu_x(w,a), \frac{1}{aP'(\mathcal{T}(v_1,w,a))} >=< \mu_x(w,a), \frac{1}{aP'(\mathcal{T}(v_2,w,a))} >,$$

which implies $v_1 = v_2$, since $P'(\mathcal{T}(v,w,a))$ is strictly monotone with respect to v.
(iii) is then obvious. □

4.2 Existence of a weak entropy solution

In this Section, we prove the convergence of the Godunov scheme to a weak entropy solution to the initial value Problem for the homogenized system introduced below, as $(\Delta t, \Delta x) \to (0,0)$ with a fixed ratio satisfying the CFL condition.

Theorem 4.2. *Under the same assumptions as in Theorem 4.1,*

(i) At least for a subsequence, in fact for the whole sequence, the sequence (U_h) constructed by the Godunov scheme converges to a weak solution of the system

$$\begin{cases} \partial_t \tau^* - \partial_x v^* = 0, \\ \partial_t w^* = 0, \ \ \partial_t a^* = 0, \end{cases} \tag{4.4}$$

with initial data

$$\begin{cases} \tau_0^* =< \mu_x(w,a), \mathcal{T}(v_0^*(x), w, a) >, \\ w_0^* =< \mu_x(w,a), w >, \ a_0^* =< \mu_x(w,a), a >, \end{cases} \tag{4.5}$$

and

$$\tau^*(x,t) =< \mu_x(w,a), \mathcal{T}(v^*(x,t), w, a) >:= \mathcal{T}^*(x, v^*(x,t)), \tag{4.6}$$

where $\mathcal{T}$ is defined by (2.4)*.*

(ii) The family of probability measures $\nu_{x,t} = \delta(v - v^*(x,t)) \otimes \mu_x(w,a)$ is a measure-valued (mv) solution in the following sense: for any entropy-flux pair (η, q), defined by (2.11)*, with*

$$\eta(v,w,a) := \tilde{\eta}(\tau, w, a)\big|_{\tau=\mathcal{T}(v,w,a)}, \tag{4.7}$$

and $\tilde{\eta}$ convex with respect to *τ, i.e. q concave with respect to v (by Proposition 2.2 (i)), and for any test-function $\varphi \geq 0$, we have*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} (< \nu_{x,t}(v,w,a), \eta(v,w,a) > \partial_t \varphi + < \nu_{x,t}, q(v) > \partial_x \varphi) \ dxdt \\ &+ \int_{\mathbb{R}} < \nu_{x,0}(v,w,a), \eta(v,w,a) > \ \varphi(x,0) \, dx \geq 0. \end{aligned} \tag{4.8}$$

(iii) Applying (4.8) *to arbitrary entropy flux pairs* $(\eta(w,a),0)$*, we recover that* $\mu_{x,t}(w,a) \equiv \mu_x(w,a)$ *satisfies* (4.5)*. In addition,* $\forall \varphi \geq 0$*,* $\forall k \in \mathbb{R}$*, we have*

$$\int_0^\infty \int_{\mathbb{R}} (< \mu_x(w,a), |\mathcal{T}(v^*(x,t),w,a) - \mathcal{T}(k,w,a)| > \partial_t \varphi - |v^*(x,t) - k| \partial_x \varphi) dx dt$$
$$+ \int_{\mathbb{R}} < \mu_x(w,a), |\mathcal{T}(v_0^*(x),w,a) - \mathcal{T}(k,w,a)| > \varphi(x,0) dx \geq 0. \tag{4.9}$$

Proof. We first prove this Theorem for a subsequence. The uniqueness Theorem 5.1 will then imply the same results for the whole sequence.
(i) We write the first equation of (4.4) in the weak form:

$$< L, \varphi > := \int_0^\infty \int_{\mathbb{R}} (\tau^* \partial_t \varphi - v^* \partial_x \varphi) dx dt + \int_{\mathbb{R}} \tau_0^*(x) \varphi(x,0) dx = 0. \tag{4.10}$$

We add and subtract in (4.10) the term $< L_h, \varphi >$ of (3.11), to obtain

$$< L, \varphi > = \int_0^\infty \int_{\mathbb{R}} (\tau^* - \tau_h) \partial_t \varphi \, dx dt - \int_0^\infty \int_{\mathbb{R}} (v^* - v_h) \partial_t \varphi \, dx dt$$
$$+ \int_{\mathbb{R}} (\tau^*(x,0) - \tau_h(x,0)) \varphi(x,0) dx + < L_h, \varphi > .$$

Of course the first three integrals converge to 0 in $L^\infty(\mathbb{R} \times \mathbb{R}_+)$ weak*, due to Theorem 4.1 and by (3.12), the term $< L_h, \varphi >$ tends to 0 as $h \to 0$.
(ii) The inequality (4.8) is obtained by passing to the weak* limit in (3.14), using the L^1 equicontinuity in time of $\tilde{v}_h$ or $\tilde{\tau}_h$, see Proposition 5.1.
(iii) Now, we choose in (4.8) the entropy $\eta(v,w,a) = |\tau - P^{-1}((w-k)/a)| = |\mathcal{T}(v,w,a) - \mathcal{T}(k,w,a)|$, associated to the concave flux $(-|v-k|)$. Using the information (4.3) on the structure of $\nu_{x,t}$, we obtain (4.9). $\square$

The above Theorem describes the point of view of *measure-valued solutions*, see again [19, 42, 37], etc. We note that the uniqueness results of Di Perna and Szepessy are not directly applicable here, see Remark 5.2.
Now, the structure of $\mu_{x,t} \equiv \mu_x$ is obvious, whereas the evolution of $\delta(v - v^*(x,t))$ turns out to be governed by a scalar conservation law, whose flux depends on x in a non smooth way through μ_x. The striking fact is that, due to the strict monotonicity of function $\mathcal{T}(\cdot,w,a)$, we can do two things:

(i) the map $\{v^* \to \mathcal{T}^*(x,v^*) := \tau^* =< \mu_x, \mathcal{T}(v^*,w,a) >$ is strictly increasing, and (therefore) invertible. We define $\mathcal{V}(x,\tau^*) = v^*$.

(ii) On the other hand, in (4.8), we can "take the absolute value out of the integral with respect to μ_x". Therefore (4.8) looks like the Kružkov entropy inequality for the scalar equation (4.14) below. We consider *all* the entropy-flux pairs of the form $H(x,v^*) = \tilde{H}(x,\tau^*)$, $Q(v^*)$, "conservative" in the sense that for any smooth solution of (4.14),
$$\partial_t H(x,v^*) + \partial_x Q(v^*) = 0,$$

without any additional term. Moreover, we can show as in Section 2.3, that

$$\frac{\partial \tilde{H}}{\partial \tau}(x,\tau^*) = \frac{\partial H}{\partial v}(x,v^*)\frac{1}{\frac{\partial \mathcal{T}^*}{\partial v}(x,v^*)} = -Q'(v^*), \tag{4.11}$$

and that $\tilde{H}(x,\tau^*)$ is convex in τ^* if and only if $Q(v^*)$ is concave. Therefore, *all* these entropy-flux pairs are given by

$$H(x,v^*) = H_0(x) - \int_0^{v^*} Q'(s)\frac{\partial \mathcal{T}^*}{\partial v}(x,s)ds = \tilde{H}(x,\tau^*),\ \ Q \equiv Q(v^*), \tag{4.12}$$

with arbitrary functions $Q(v)$ and $H_0(x) = H(x,0)$.

Theorem 4.3. *(i) The first equation of system* (4.4) *can be rewritten as a scalar equation with a flux depending explicitly on* x,

$$\partial_t \tau^* + \partial_x \mathcal{V}(x,\tau^*) = 0, \tag{4.13}$$

or in the more convenient form:

$$\partial_t \mathcal{T}^*(x,v^*) - \partial_x v^* = 0. \tag{4.14}$$

Therefore, the weak solution of system (4.4) *satisfies* (4.14), (4.6).

(ii) Using the monotonicity in v^* *of* $\mathcal{T}^*(x,v^*)$, v^* *is also a weak solution "à la Kružkov" of* (4.14)*: for any* k *in* $\mathbb{R}$, (4.9) *is equivalent to*

$$\iint(|\mathcal{T}^*(x,v^*)-\mathcal{T}^*(x,k)|\partial_t\varphi-|v^*-k|\partial_x\varphi)dxdt+\int_{\mathbb{R}}|\mathcal{T}^*(x,v_0^*)-\mathcal{T}^*(x,k)|\varphi(x,0)dx \geq 0. \tag{4.15}$$

(iii) For any such entropy $H(x,v^*) = \tilde{H}(x,\tau^*)$ *convex in* τ^**, associated to a concave* $Q(v^*)$ *by* (4.12)*, let* η *be given by* (2.11)*, with the* same $q \equiv Q$*. Then, for any test-function* $\varphi \geq 0$*, we have*

$$H(x,v^*) =< \nu_{x,t}(v,w,a), \eta(v,w,a) >=< \mu_x(w,a), \eta(v^*(x,t),w,a) >, \tag{4.16}$$

$$\int_0^\infty \int_{\mathbb{R}}(H(x,v^*)\ \partial_t\varphi - Q(v^*)\ \partial_x\varphi)dxdt + \int_{\mathbb{R}} H(x,v_0^*)\varphi(x,0)dx \geq 0. \tag{4.17}$$

Proof. *(i)* is obvious.
(ii) Using the fact that $\mathcal{T}$ is increasing in v^*, we see that $\forall(w,a)$,

$$\text{sign}(\mathcal{T}(v^*,w,a) - \mathcal{T}(k,w,a)) = \text{sign}(v^* - k)$$

does not depend on (w,a), so that

$$< \mu_x(w,a), |\mathcal{T}(v^*,w,a) - \mathcal{T}(k,w,a)| >=$$
$$= \text{sign}(v^*-k) < \mu_x(w,a), \mathcal{T}(v^*,w,a) - \mathcal{T}(k,w,a) >= |\mathcal{T}^*(x,v^*) - \mathcal{T}^*(x,k)|.$$

(iii) Let η be defined by (2.11). Using (4.11) and differentiating in v in the integral, one can show that necessarily:

$$H(x,v^*) - H_0(x) = \int_0^{v^*} \frac{\partial H}{\partial v}(x,s)ds = \int_0^{v^*} -Q'(s)\frac{\partial \mathcal{T}^*}{\partial v}(x,s)ds =$$
$$= \int_0^{v^*} -Q'(s)\frac{\partial}{\partial v} < \mu_x(w,a), \mathcal{T}(s,w,a) > ds =< \mu_x(w,a), \int_0^{v^*} -Q'(s)\frac{\partial \mathcal{T}}{\partial v}(s,w,a)ds >=$$
$$=< \mu_x(w,a), \eta(v^*,w,a) > .$$

Substituting the expression of (4.16) in (4.8), we obtain (4.17). $\square$

Remark 4.1. *Therefore we have proved* (4.8) *and* (4.17)*, respectively for any entropy* $\eta(v,w,a)$ *and for the homogenized entropy* $H(x,v^*)$ convex *in* τ*. Thus these two quantities satisfy the corresponding entropy inequalities, but there is a priori no obvious relation between them, since* $\eta(v,w,a) = \tilde{\eta}(\tau,w,a)$ *is only convex* in τ*, with* no convexity assumption in (w,a)*:* (4.16) *holds true, but in general*

$$H(x,v^*) =< \mu_x(w,a), \eta(v^*,w,a) > \neq \eta(< \nu_{x,t}, (v,w,a) >) = \eta(v^*,w^*,a^*).$$

5 Uniqueness of the solution

We are interested in the homogenized model (4.4)-(4.6). The existence of an entropy solution of (4.4) follows directly from the convergence of the Godunov method in the Theorem 4.2.
Concerning the uniqueness, knowing the measure μ_x, we have written the first equation of system (4.4)-(4.6) as the scalar equation (4.13) with a flux depending explicitly on x. The low regularity in x of the flux does not allow to use directly the uniqueness result of Kružkov [29].
For references on the uniqueness of the solution for scalar conservation laws with a flux discontinuous in x, see e.g. [4, 28, 27, 45, 39], and concerning Temple systems, e. g. [12, 3, 11, 8, 9], but they are not applicable in this homogenized case.
Since in (4.14) the function $\mathcal{T}^*(x,v^*)$ (explicitly depending on x) appears in the derivative with respect to t, we have exchanged the roles of x and t so that inequality (4.15) is in conservative form, even though the flux depends on x. Therefore, assumptions on the x-regularity of the flux of (4.13) are not required.
The following proposition gives the L^1 - continuity in time of τ^*, which will be useful for the uniqueness of the solution.

Proposition 5.1. *Let* $(\tilde{\tau}_h, \tilde{v}_h)$ *be the approximate solution defined in* (3.1), (3.6)*, of system* (1.1)*, with* $\tau_0^* \in L^\infty(\mathbb{R})$ *and* $v_0^* \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$*. Then, at least for a subsequence,*

$$\tilde{v}_h \to v^* \text{ in } C^0([0,T]; L^1_{loc}(\mathbb{R})), \tag{5.1}$$

$$\|\tau^*(\cdot,t) - \tau_0^*(\cdot)\|_{L^1_{loc}(\mathbb{R})} \leq C \max(t, \Delta t)\, VT_x(v_0^*(\cdot);\mathbb{R}), \tag{5.2}$$

and then

$$\|\tau^*(\cdot,t) - \tau_0^*(\cdot)\|_{L^1_{loc}(\mathbb{R})} \to 0 \text{ as } t \to 0.$$

Proof. Let $X = BV(\mathbb{R})$ and $B = L^1_{loc}(\mathbb{R})$. The sequence $(\tilde{v}_h)$ is bounded in $L^\infty(0,T;X)$ by (3.4). On the other hand, for all compact subset K of $\mathbb{R}$, $v_h \in C([0,T];L^1(K))$ and by (3.8),

$$\begin{aligned} \|\tilde{v}_h(\cdot,t') - \tilde{v}_h(\cdot,t)\|_{L^1(K)} \le \int_K \int_t^{t'} |\partial_s \tilde{v}_h(x,s)| ds dx = \|\partial_t \tilde{v}_h\|_{L^1(t,t';L^1_{loc}(\mathbb{R}))} = \\ = TV_t(\tilde{v}_h(x,\cdot);[t,t']) \le \; C' \max(|t'-t|, \Delta t). \end{aligned} \tag{5.3}$$

where C' depends on $TV(v_0^*(\cdot);\mathbb{R})$. By a Theorem of Simon ([41], Thm. 3), the sequence $\tilde{v}_h(\cdot,t)$ is relatively compact in $C([0,T];L^1_{loc}(\mathbb{R}))$, which implies (5.1). Passing to the limit as $h \to 0$ in (5.3), with $t' = 0_+$, since $\mathcal{T}$ (or P^{-1}) is L-Lipschitz continuous and $\tau^*(\cdot,0_+) = \tau_0^*(\cdot)$, we obtain

$$\|\tau^*(\cdot,t) - \tau_0^*(\cdot)\|_{L^1_{loc}(\mathbb{R})} \le L\|v^*(x,t) - v_0^*(\cdot)\|_{L^1_{loc}(\mathbb{R})} \le L\, C'\, t \to 0 \text{ as } t \to 0.$$

□

Theorem 5.1. *We consider the scalar equation* (4.14), (4.6) *(or the equivalent equation* (4.13), (4.6), *due to the monotonicity in v^* of $\mathcal{T}^*$), with initial data* (4.5) *and under the assumption that V, defined by* (1.2), *is strictly increasing and strictly concave We also assume that $(\tau_0^*, w_0^*, a_0^*) \in L^\infty(\mathbb{R})$, $v_0^* \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and* (4.1). *Then, there is a unique weak entropy solution "à la Kružkov" v^* of problem* (4.14), (4.6), (4.5).
Therefore the whole sequence (τ_h, w_h, a_h, v_h) *converges to the unique limit of the system described in Theorem 4.2.*

In the proof, we will need the following Lemma *in which we write v instead of v^*:*

Lemma 5.1. *Let $\mathcal{T}^*(X, v(Y,t)) :=< \mu_X(w,a), \mathcal{T}(v(Y,t),w,a) >$ with $\mathcal{T}$ Lipschitz with respect to $v \in L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$. Then we have, for almost all X in $\mathbb{R}$ and t in* $(0,+\infty)$,

$$\lim_{h\to 0} \frac{1}{h} \int_{|Y|\le h} |\mathcal{T}^*(X \pm Y, v(X \pm Y, t)) - \mathcal{T}^*(X, v(X,t))| \, dY = 0,$$

where the two signs $\pm$ can be chosen independently.

Proof. The result would be *wrong* for a general function of two variables. It holds true here since, roughly speaking, $\mathcal{T}^*(X, v(Y,t))$ "looks like" a product $f(X)g(Y,t)$.
Here, we write λ instead of (w,a). For almost all (fixed) t, we first choose the *precise representing function* of $v(\cdot,t)$ ([21] p. 46), precisely defined at every Lebesgue point for $\{y \mapsto v(y,t)\}$, and identically equal to 0 on the null set $N_1 := N_1(t)$ of points y which are not Lebesgue points for this function. Similarly, see e.g. [5], we "remove" a null set $N_2 := N_2(t)$ of points x such that either μ_x is *not* a probability measure or x is *not* a Lebesgue point for the L^1_{loc} function:

$$\{x \mapsto F_v(x,t) := \mathcal{T}^*(x, v(x,t)) :=< \mu_x(\lambda), \mathcal{T}(v(x,t),\lambda) >\}, \tag{5.4}$$

with the same (fixed) t. We see that the set $\{(x,x), x \in N_1 \cup N_2\}$ is a (one dimensional) null set. Therefore, each $x \notin N_1(t) \cup N_2(t)$ is *simultaneously* a Lebesgue point for the two functions of *one* variable (5.4) and $\{y \mapsto G_v(x,y,t) :=< \mu_x(\lambda), \mathcal{T}(v(y,t),\lambda) >\}$. In particular, $\forall x \notin N_1(t) \cup N_2(t)$, $G_v(x,x,t) = F_v(x,t)$. The same result would be true if $\mathcal{T}$ were only *continuous*, see e.g. [7] for the related notion of Caratheodory functions. Therefore, we have for instance:

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{h} \quad \int_{|Y|\le h} |\mathcal{T}^*(X+Y, v(X-Y,t)) - \mathcal{T}^*(X, v(X,t))| \, dY = \\
&= \lim_{h\to 0}\frac{1}{h} \quad \int |< \mu_{X+Y}(\lambda), \mathcal{T}(v(X-Y,t),\lambda) > - < \mu_X(\lambda), \mathcal{T}(v(X,t),\lambda) > \ dY \le \\
&\le \lim_{h\to 0}\frac{1}{h} \quad \int \{ |< \mu_{X+Y}(\lambda), \mathcal{T}(v(X-Y,t),\lambda) > - < \mu_{X+Y}(\lambda), \mathcal{T}(v(X,t),\lambda) > | \\
&\qquad\qquad + |< \mu_{X+Y}(\lambda), \mathcal{T}(v(X,t),\lambda) > - < \mu_{X+Y}(\lambda), \mathcal{T}(v(X+Y,t),\lambda) > | \\
&\qquad\qquad + |< \mu_{X+Y}(\lambda), \mathcal{T}(v(X+Y,t),\lambda) > - < \mu_X(\lambda), \mathcal{T}(v(X,t),\lambda) > | \} \, dY \le \\
&\le \lim_{h\to 0}\frac{1}{h} \quad \int \{ < \mu_{X+Y}(\lambda), L|v(X-Y,t) - v(X,t)| + L|v(X,t) - v(X+Y,t)| > \\
&\qquad\qquad + |< \mu_{X+Y}(\lambda), \mathcal{T}(v(X+Y,t),\lambda) > - < \mu_X(\lambda), \mathcal{T}(v(X,t),\lambda) > | \} \, dY = \\
&\le \lim_{h\to 0}\frac{1}{h} \quad \int \{ L|v(X-Y,t) - v(X,t)| + L|v(X,t) - v(X+Y,t)| \\
&\qquad\qquad + |< \mu_{X+Y}(\lambda), \mathcal{T}(v(X+Y,t),\lambda) > - < \mu_X(\lambda), \mathcal{T}(v(X,t),\lambda) > | \} \, dY.
\end{aligned}
\tag{5.5}
$$

Therefore, almost everywhere in t, for any $X \notin N_1(t) \cup N_2(t)$, the integrals in (5.5) converge to 0 as $h \to 0$. $\square$

Remark 5.1. *Integrating in X the result of the Lemma 5.1 and applying Lebesgue Theorem, the corresponding double integrals in (X,Y) converge to 0 as h tends to 0. This new result does not explicitly involve the above Lebesgue points and could be also proved [46] as follows: approximate the function $\{v : y \mapsto v(y,t)\}$ by a sequence of smooth functions (v_n), for which there is no ambiguity: a.e. in x, $\forall n, F_{v_n}(x,t) = G_{v_n}(x,y,t)|_{y=x}$. Then, justify the result for each v_n, and pass to the limit as $n \to \infty$.*

Proof of Theorem 5.1. We consider two weak entropy solutions σ and τ of (4.13) (or two solutions u and v of (4.14)) which satisfy the entropy inequality (4.15). In this proof, we write $\sigma, \tau, u, v, \mathcal{T}, \ldots$ instead of $\sigma^*, \tau^*, u^*, v^*, \mathcal{T}^*, \ldots$. We have:

$$\int_{\mathbb{R}} \int_0^{\infty} |\mathcal{T}(x, u(x,t)) - \mathcal{T}(x,k)| \phi_t(x,t) - |u(x,t) - k| \phi_x(x,t) \, dx \, dt \ge 0, \tag{5.6}$$

$$\int_{\mathbb{R}} \int_0^{\infty} |\mathcal{T}(y, v(y,s)) - \mathcal{T}(y,l)| \phi_s(y,s) - |v(y,s) - l| \phi_y(y,s) \, dy \, ds \ge 0, \tag{5.7}$$

where $\phi \ge 0$ is a test-function $\in C_0^{\infty}(\mathbb{R} \times (0,+\infty))$. Following Kružkov [29], we obtain classically:

$$\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty \{|\mathcal{T}(x,u(x,t))-\mathcal{T}(x,v(y,s))|\phi_t+|\mathcal{T}(y,v(y,s))-\mathcal{T}(y,u(x,t))|\phi_s$$
$$-|u(x,t)-v(y,s)|(\phi_x+\phi_y)\}\,dx\,dt\,dy\,ds\geq 0. \tag{5.8}$$

We choose

$$\phi(x,t,y,s)=\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_k\left(\frac{t-s}{2}\right),$$

for any function ψ and where $\{\delta_h\}_{h\geq 0}$ et $\{\delta_k\}_{k\geq 0}$ are the usual regularizing sequences, with bounded support in $(-h,h)$, with $0\leq\delta_1(\cdot)\leq 1$. Denoting by

$$X=\frac{x+y}{2},\quad T=\frac{t+s}{2},\quad Y=\frac{x-y}{2},\quad S=\frac{t-s}{2}, \tag{5.9}$$

we rewrite (5.8):

$$\frac{1}{2}\int\{|\mathcal{T}(x,u(x,t))-\mathcal{T}(x,v(y,s))|+|\mathcal{T}(y,v(y,s))-\mathcal{T}(y,u(x,t))|\}\psi_T\;\delta_h(Y)\;\delta_k(S)$$
$$+\{|\mathcal{T}(x,u(x,t))-\mathcal{T}(x,v(y,s))|-|\mathcal{T}(y,v(y,s))-\mathcal{T}(y,u(x,t))|\}\;\psi\;\delta_h(Y)\;\delta_k'(S)$$
$$-|u(x,t)-v(y,s)|\;\psi_X(X,T)\;\delta_h(Y)\;\delta_k(S)\;dx\,dt\,dy\,ds:=(I_1+I_2)+(I_3-I_4)-I_5\geq 0.$$

We now proceed in the same spirit as [4]: we let h and k tend to 0 separately. We *only* show the convergence of $(I_3-I_4):=(I_3^{k,h}-I_4^{k,h})$, for which *it is crucial to first let h tend to* 0, since this term involves $\delta_k'(S)$. The proof for the other terms is similar. Writing (I_3-I_4) in variables (X,Y,t,s), adding and subtracting the term

$$I_6:=\int|\mathcal{T}(X,u(X,t))-\mathcal{T}(X,v(X,s))|\psi(X,(t+s)/2)\delta_h(Y)\delta_k'((t-s)/2)dX\,dY\,dt\,ds,$$

we first obtain

$$|I_3-I_4|\leq|I_3-I_6|+|I_6-I_4|.$$

Then, using the relation $||a-b|-|c-d||\leq|a-b-(c-d)|\leq|a-c|+|b-d|$, we have

$$|I_3-I_6|\leq\frac{1}{h}\int_{|Y|\leq h}\{|\mathcal{T}(X+Y,u(X+Y,t))-\mathcal{T}(X,u(X,t))|$$
$$+|\mathcal{T}(X+Y,v(X-Y,s))-\mathcal{T}(X,v(X,s))|\}|\psi(X,(t+s)/2)|\delta_k'((t-s)/2)\;dY\;dX\;dt\;ds, \tag{5.10}$$

and similarly,

$$|I_6-I_4|\leq\frac{1}{h}\int_{|Y|\leq h}\{|\mathcal{T}(X-Y,v(X-Y,s))-\mathcal{T}(X,v(X,s))|$$
$$+|\mathcal{T}(X-Y,u(X+Y,t))-\mathcal{T}(X,u(X,t))|\}|\psi(X,(t+s)/2)|\delta_k'((t-s)/2)\;dY\;dX\;dt\;ds. \tag{5.11}$$

By Lemma 5.1, for any fixed $k > 0$, a.e. in (s, t, X), the integrals in Y corresponding to the above integrals tend to 0 as $h \to 0$. Applying then the Lebesgue dominated convergence Theorem, the corresponding integrals in (X, Y), and finally in (X, Y, t, s), also tend to 0 as $h \to 0$, for any fixed k. A fortiori,

$$\lim_{k\to 0}\,(\lim_{h\to 0}(I_3^{k,h} - I_4^{k,h})) = \lim_{k\to 0}(0) = 0.$$

Therefore, this singular term $(I_3 - I_4)$ vanishes at the limit, contrarily to the other terms for which we apply again the Lebesgue Theorem when k tends to 0.
Then, we have shown that for any $\psi \geq 0$ (we now write (x, t) instead of (X, T) and $\mathcal{T}^*$ instead of $\mathcal{T}$),

$$\int_{\mathbb{R}}\int_0^\infty |\mathcal{T}^*(x, u(x,t)) - \mathcal{T}^*(x, v(x,t)|\psi_t(x,t) - |u(x,t) - v(x,t)|\psi_x(x,t)\,dx dt \geq 0. \quad (5.12)$$

Now, we choose classically the test-function $\psi(x, t)$ in (5.12) as a regularization of the characteristic function of the set $\Omega = \{(T, X); t_1 \leq T \leq t_2, |X| \leq R - NT\}$, for any $R > 0$. Using the L^1-continuity in time of the solution at $t = 0$, see Proposition 5.1, we obtain the L^1 contraction property:

$$\int_{\mathbb{R}} |\mathcal{T}^*(x, u(x,t)) - \mathcal{T}^*(x, v(x,t)|\,dx \;\leq \int_{\mathbb{R}} |\mathcal{T}^*(x, u(x,0)) - \mathcal{T}^*(x, v(x,0)|\,dx, \text{ i.e.}$$

$$\int_{\mathbb{R}} |\sigma(x,t) - \tau(x,t)|\,dx \;\leq \int_{\mathbb{R}} |\sigma(x,0) - \tau(x,0)|\,dx,$$

and therefore the uniqueness of the solution, for a given $\tau(\cdot, 0)$. □

Remark 5.2. *(i) First, we have established here the L^1 contraction for equations* (4.14), (4.6), *with only L^∞ assumptions in μ_x and $v^*(x, t)$. We have essentially used the strict monotonicity of $\mathcal{T}(\cdot, w, a)$, which allows to exchange the role of x and therefore of τ and v. After this exchange, the stationary solutions $u^\varsigma(x)$ introduced in [4] to solve the equation*

$$\partial_t u + \partial_x F(x, u) := \partial_t u + \partial_x f(v(x), u) = 0,$$

arise much more naturally: compare (4.15) *with formula* (60) *in [4].*

(ii) On the other hand, the uniqueness results on measure-valued solutions of Di Perna [19] and Szepessy [42] are (at least) not directly applicable here, since we deal with a system: the Young measure $\nu_{x,t} = \delta(\cdot - v^*(x,t)) \otimes \mu_x(\cdot, \cdot)$ involves several variables, which do not play the same role. Here, we wanted to prove the uniqueness of v^* for a given μ_x. Note that for instance, convolving in (x, t) such a measure $\nu_{x,t}$ does not *preserve its tensor product structure.*

6 The microscopic model

6.1 Introduction of the model

Microscopic models of vehicular traffic are usually based on so-called "Follow-the Leader models" [23, 25], which usually consist, in Eulerian coordinates, of a system of second

order ordinary differential equations. The basic idea is that the acceleration at time t depends on the relative speeds of the vehicle and its leading vehicle at time t and the distance between the cars.
The system in (1.3) is a Follow-the Leader type of model in which the function V also depends on the coefficients a_j, characteristic of the different types of vehicles and their heterogeneous response to their leading vehicle.
At least formally, the system (1.3) is clearly a semi-discretization in space of the macroscopic model in Lagrangian coordinates (1.1). In the following Sections, we will give a rigorous justification that (1.3) converges to the entropy solution of (4.4)-(4.6) as $\Delta x \to 0$.

6.2 First order Euler time approximation

We consider the infinite system of ordinary differential equations (1.3), written in general form as

$$\begin{cases} \frac{dU(t)}{dt} = F(U(t)), \\ U(0) = U_0, \end{cases} \tag{6.1}$$

where $U := (U_1, U_2, \ldots, U_j, \ldots)$ and $F(U) = (F_1(U), F_2(U), \ldots, F_j(U), \ldots)$, with $U_j := (\tau_j, w_j, a_j)$ and

$$F_j(U) := \left(\frac{v_{j+1} - v_j}{\Delta x}, 0, 0\right) = \left(\frac{w_{j+1} - w_j}{\Delta x} - \frac{a_{j+1}P(\tau_{j+1}) - a_j P(\tau_j)}{\Delta x}, 0, 0\right).$$

We introduce the first order explicit Euler discretization in time:

$$\begin{cases} \tau_j^{n+1} = \tau_j^n + \frac{\Delta t}{\Delta x}\,(v_{j+1}^n - v_j^n), \\ w_j^{n+1} = w_j^n, \\ a_j^{n+1} = a_j^n. \end{cases} \tag{6.2}$$

In the following Theorem, we prove that, for any fixed Δx, the previous discretization is stable and consistent and therefore convergent as $\Delta t \to 0$ to $U_{\Delta x}(x,t) := \sum_{j\in\mathbb{Z}} U_j(t)\chi_j(x)$, where χ_j is the characteristic function of the interval I_j. We also show that the microscopic multi-class model (1.3) is the semi-discretization of the Lagrangian system (1.1).

Theorem 6.1. *We consider the system* (6.2)*, with initial data $U_j^0 = (\tau_j^0, w_j^0, a_j^0)$ in the invariant region $\mathcal{R}$ defined by* (2.8)*, away from vacuum. We assume that the initial data are constant for x large enough, so that there is a "first" vehicle. Then,*

(i) The operator F is Lispchitz-continuous in the l^∞ space. Therefore the Initial Value Problem (1.3) *has a unique solution $U(t)$, globally defined in time.*

(ii) The first order approximation (6.2) *is stable and consistent in l^∞. Therefore the sequences $U_j^n := (\tau_j^n, w_j^n, a_j^n)$ and v_j^n converge as $\Delta t \to 0$, for any fixed Δx. Moreover, their limits $U_{\Delta x}(x,t)$ and $v_{\Delta x}(x,t)$ stay in the region $\mathcal{R}$ and satisfy the uniform L^∞ and BV estimates inherited from the Godunov scheme. The microscopic model* (1.3) *is then the semi-discretization of the macroscopic system* (1.1)*,* (1.2)*.*

(iii) $U_j(t)$ *satisfies a semi-discrete entropy inequality, i.e. for any entropy* η convex with respect to τ_j, *associated to the entropy flux* q *and for any* j,

$$\frac{d\eta(U_j(t))}{dt} + (1/\Delta x)(q(U_{j+1}(t))) - q(U_j(t)))) \leq 0. \tag{6.3}$$

Proof. *(i)* The proof is obvious, by Cauchy-Lipschitz Theorem.
(ii) Since F is Lipschitz continuous, adapting classical results, see e.g. [38], we can show that the Euler approximation (6.2) is stable and consistent and therefore convergent when $\Delta t \to 0$.
On the other hand, since the first order approximation (6.2) coincides with the Godunov scheme (3.2), U_j^n and v_j^n stay in the same bounded invariant region $\mathcal{R}$ and satisfy the L^∞ and BV estimates in x for v (resp. in t for τ and v) as in Theorem 3.1. In the latter case, we slightly modify the proof of (3.4) and (3.8) to obtain constants C independent of the ratio $\Delta x/\Delta t$, as indicated in Remark 3.1.
(iii) Since for each j the sequence v_j^n converges as $\Delta t \to 0$ to $v_j(t)$ for $t = n\Delta t$, $q(v_j^n)$ is also convergent. On the other hand, since w_j, a_j are constant in the cell I_j, we have for each j

$$\eta(U_j^n) = \eta(\tau_j^n, w_j^n, a_j^n) = \eta(\mathcal{T}(v_j^n, w_j^n, a_j^n), w_j^n, a_j^n) \to \eta(\tau_j(t), w_j(t), a_j(t))$$

Finally *(iii)* is obtained passing to the limit in the fully discrete entropy inequality (3.10). □

6.3 Hydrodynamic limit of the microscopic multi-class model

We rewrite (6.1) in the form

$$\begin{cases} \frac{dU_j(t)}{dt} + \frac{G(U_{j+1}(t)) - G(U_j(t))}{\Delta x} = 0, \\ U_j(0) = U_j^0, \end{cases} \tag{6.4}$$

with $G(U_j) := (-v_j, 0, 0) = (-(w_j - a_j P(\tau_j)), 0, 0)$. We now show that the entropy solution of the system (4.4) is the limit as $\Delta x \to 0$ (i.e. when the number of vehicles goes to infinity), of the solution of the infinite-dimensional system of ordinary differential equations (6.4).

Theorem 6.2. *Under the same assumptions as in of Theorem 6.1, when* $\Delta x \to 0$, *the* whole *sequence* $U_{\Delta x}$ *converges in* L^∞ *weak* (and almost everywhere for the velocity) to the* unique *entropy weak solution of the macroscopic system* (4.4)-(4.6).

Proof. Multiplying (6.4) by an arbitrary test-function $\varphi(x,t)$, making a discrete integration by parts (see e.g. [30]), we have:

$$\begin{aligned} I_{\Delta x} := \int_0^\infty \sum_j \int_{I_j} U_j(t) \partial_t \varphi dx dt - \int_0^\infty \sum_j G(U_j(t)) \left(\int_{I_j} \frac{\varphi(x,t) - \varphi(x - \Delta x, t)}{\Delta x} dx \right) dt \\ + \sum_j \int_{I_j} U_j^0 \varphi(x, 0)\, dx = 0 \end{aligned}$$

Due to the uniform L^∞ estimates, see Theorem 6.1, at least for a subsequence, $U_{\Delta x}(x,t)$ converges in $L^\infty(\mathbb{R}\times\mathbb{R}_+)$ weak* to some function $U^{**}(x,t)$ when $\Delta x \to 0$. Moreover, by compactness, $G(U_{\Delta x}(.,.)) = (-v_{\Delta x}(.,.),0,0) \to (-v^{**},0,0)$ in L^1_{loc} strongly when $\Delta x \to 0$. Therefore we obtain at the limit :

$$\int_0^\infty \int_{\mathbb{R}} U^{**}(x,t)\partial_t\varphi \, dx\, dt + \int_0^\infty \int_{\mathbb{R}} G(U^{**}(x,t))\partial_x\varphi \, dx\, dt + \int_{\mathbb{R}} U_0^*(x)\varphi(x,0)\, dx = 0,$$

which shows that U^{**} is a weak solution of (4.4).
Due to the uniform BV estimates on $v_{\Delta x}$, the Young measure associated to the sequence $(v_{\Delta x}, w_{\Delta x}, a_{\Delta x})$ is still a tensor product $\gamma_{x,t}(v)\otimes\beta_{x,t}(w,a) = \delta(v-v^{**}(x,t))\otimes\beta_{x,t}(w,a)$ as in (4.3), with the *same* initial data (v_0^*, w_0^*, a_0^*) as in Section 4.2, since we have assumed in (4.1) that *all* the sequence (v_h^0, w_h^0, a_h^0) converges to $(v_0^*, <\mu_x, w>, <\mu_x, a>)$ as $h \to 0$. Therefore $\beta_{x,t}(w,a) \equiv \beta_x(w,a) \equiv \mu_x(w,a)$. Now integrate by parts in the semi-discrete entropy inequality (6.3), for all $\varphi \geq 0$:

$$\int_0^\infty \sum_j \int_{I_j} \eta(U_j(t))\partial_t\varphi \, dx\, dt + \int_0^\infty \sum_j q(U_j(t)) \left(\int_{I_j} \frac{\varphi(x,t)-\varphi(x-\Delta x,t)}{\Delta x}\, dx \right) dt$$
$$+ \sum_j \int_{I_j} \eta(U_j^0)\varphi(x,0)\, dx \geq 0,$$

and pass to the limit as $\Delta x \to 0$. Note that $q(U_{\Delta x}(\cdot,\cdot)) \to q(U*(\cdot,\cdot))$ strongly, whereas, as in Section 4, $\eta(U_{\Delta x}(\cdot,\cdot))$ converges weakly to $<\mu_x, \eta(\mathcal{T}(v**(x,t),w,a)>$. Finally, as in Proposition 5.1, the L^1 equicontinuity in time is preserved for the sequence $\eta(U_{\Delta x})$ when Δx tends to 0. Therefore, the limit v^{**} satisfies (4.8) and the (Kružkov) entropy condition (4.15). Consequently, by the uniqueness result of Theorem 5.1, $v^{**} = v^*$ almost everywhere in (x,t), which also implies that the *whole* sequence converges to the same limit. $\square$

In conclusion, starting from the fully discrete system (3.1), we obtain the *same* limit (i.e. the macroscopic system (4.4)), either by letting $(\Delta x, \Delta t) \to 0$ with a fixed ratio and the CFL condition, or by first letting $\Delta t \to 0$ with a fixed Δx, and then letting $\Delta x \to 0$. This last limit process says that the homogenized model (4.4) is the hydrodynamic limit of the microscopic "Follow-the-Leader" system.

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