# Measure Solutions to the Linear Multi-dimensional Transport Equation with Non-smooth Coefficients

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## 1 Introduction.

We are interested in the initial value problem

$$\frac{\partial}{\partial t}u + \operatorname{div}(a(t,x)\,u) = 0 \text{ in } (0,+\infty) \times \mathbb{R}^n \,, \tag{1.1}$$

$$u(0,x) = u_0(x) \text{ in } \mathbb{R}^n \,. \tag{1.2}$$

This is the familiar (multi-dimensional) linear transport equation, in conservation form. We are going to assume a very weak regularity on the coefficients

$$a \in L^1_{loc}(0,+\infty; L^\infty(\mathbb{R}^n))^n.$$

Obviously the characteristics $X(t,x)$, given by

$$\frac{dX}{dt} = a(t, X(t,x)) \,, \quad X(0,x) = x \,, \tag{1.3}$$

have to be defined in a generalized sense. We choose the construction of Filippov [10], which is classically used in the context of conservation laws, [6, 5].
Our definition requires the *uniqueness* of these generalized characteristics. To illustrate this point, let us consider the two following examples.
**Example 1.** Singular compressive case: $n = 1, a(t,x) = -sign(x)$. Clearly the characteristics are given by

$$X(t,x) = \begin{cases} (x+t).H(-x-t), & x < 0 \,, \\ (x-t).H(x-t), & x > 0 \,, \\ 0, & x = 0 \,, \end{cases} \tag{1.4}$$

and therefore are uniquely defined.
**Example 2.** Singular expansive case: $n = 1, a(t,x) = +sign(x)$. Now the characteristics are given by

$$X(t,x) = \begin{cases} (x-t), & x < 0 \,, \\ (x+t), & x > 0 \,, \\ \pm(t-t^*).H(t-t^*), & x = 0 \,, \end{cases} \tag{1.5}$$

where H is the Heaviside function and the switch time $t^*$ is arbitrary in $(0, +\infty)$. Clearly the latter example is much more difficult, and our theory does not handle this case, as we will see in the next Sections.
The main result of this paper is actually obvious. It essentially consists in showing that the natural definition, given in Section 3

$$u(t,.) := X(t,.)(u_0) \tag{1.6}$$

perfectly makes sense, as soon as the Filippov characteristics are uniquely defined. In other words, at time $t$ the total mass in any Borel set $B$ in $I\!R^n$ is

$$u(t)(B) := u_0((X(t,.))^{-1}(B). \tag{1.7}$$

This is exactly the physical meaning of the conservation law (1.1). Moreover this definition nicely preserves the total mass and the natural order on the space of measures, and therefore, thanks to an obvious extension of Crandall-Tartar's lemma, is naturally associated to a semi-group of contractions in the space $\mathcal{M}_1$ of bounded measures on $I\!R^n$.
We emphasize that there is no need here to define the product of distributions $a\ u$. *A priori,* this product is *not* defined in general. In particular, in the multi-dimensional case, the Hausdorff dimension of the set of singular points can be much too small to define this product $a\ u$ by a $BV$ theory *à la* Vol'pert, see [19], [11]. On the contrary, here, the measure solution is perfectly well defined, as soon as the Filippov characteristics are uniquely defined.
Let us also remark that the conservative equation (1.1), with discontinuous coefficients, naturally appears in nonlinear scalar equations or systems of conservation laws, for instance when deriving with respect to $x$ the non-conservative equation satisfied by the Riemann invariants.
As we will see in Section 5 for the particular case of Burgers equation, "normal" equations (or systems) never involve wildly compressive cases as Example 1. (see however in Section 5 the case of a shock with $u_+ < 0 < u_-$). In contrast, non-strictly hyperbolic systems, for instance perfect plasticity, pressure-less gases etc... very often involve singular solutions, typically delta-functions, see several one-dimensional examples in [16], [15], [11], [17], [18]. For other recent works in this direction, see also [8], [7] and the references therein. Let us also remark that the method of "sticky particles" used in [13, 4] is very similar to the notion introduced here.
By the way, several of the above-mentioned papers are related to this system of pressure-less gases equations, which can be formally - but not rigorously ! - solved in two steps, by first solving Burgers equation

$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}(v^2/2) = 0,$$

and then solving the linear transport equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) = 0,$$

with a given discontinuous coefficient $v$.
We also note that in one space dimension the above equation implies the existence of a distribution $\Phi$, (the Lagrangian mass coordinate, in this case) such that

$$\rho = \frac{\partial}{\partial x}\Phi, \quad \rho\, v = -\frac{\partial}{\partial t}\Phi,$$

which is intensely used in some of the above-mentioned papers.
We also have to compare the notion introduced here with other notions of solutions to the *linear* transport equation. First, so far we have only considered the *conservative* equation (1.1). However, our approach can be easily extended, on one hand to the more general case of a conservative equation with a zero-order term and a right-hand side, see Theorem 3.3 below, and on the other hand to the non-conservative case, see Definition 3.3 below. Moreover, there are several other relations with the non-conservative equation

$$\frac{\partial}{\partial t}u + a \cdot \nabla u = 0. \tag{1.8}$$

First, in one space-dimension, we can pass from (1.8) to (1.1) by derivation with respect to $x$. In addition, the adjoint problem to the initial-value problem (1.1), (1.2) is precisely the backward problem associated to (1.8).
In the one-dimensional case, Bouchut and James, [2], [3] finally recover the same solution as the measure solution introduced here, but with a different view-point : namely they solve the conservative problem by considering the above adjoint problem, where the characteristics can be wildly expansive as in Example 2, and by defining a notion of (unique) "reversible" Lipschitz solution to this last problem.
Let us also note that, contrarily to the *renormalized* solutions of [9], we have to *assume* the uniqueness of the (Filippov) characteristics, but these characteristics are defined *everywhere*, which is of course crucial if the initial datum is a measure. Besides, let us also mention a related functional approach in [14] for Vlasov equation, see also [12].
Finally, as we will see in Section 3, under suitable stability conditions there exists a representing function of the vector field $a$ such that

$$a(t, X(t;x)) = \frac{\partial}{\partial t}X(t;x) \quad \forall x, \text{ a.e. } t.$$

This will allow us to define *a posteriori* $a\ u$ in $\mathcal{D}'(I\!R^{1+n})$ by (see Section 3)

$$<< a\ u, \varphi >>= \int_{I\!R^n} \left( \int_{I\!R} \varphi(t, X(t;x)) a(t, X(t;x))\, dt \right) u_0(dx),$$

where $<< .,. >>$ denotes the duality with respect to $(t, x)$.
The outline of the paper is the following : after this introduction we first recall the essential results of [10] and a few other basic facts in Section 2. Then we give the definition of measure solutions and the main properties in Section 3. Next we give a few examples and comments in Section 4. Finally we revisit Burgers equation in Section 5.

## 2 Basic facts.

In this section we recall the definition and the basic properties of the generalized solutions of ordinary differential equations as introduced by Filippov in [10].
Let $a = a(t,x) \in I\!R^n$ be a given vector-field on $I\!R^n$. We assume:

$$a \in \left[L^1_{loc}(I\!R; L^\infty(I\!R^n))\right]^n. \tag{2.1}$$

For any scalar measurable function $\phi$ defined on $I\!R^n$, we define its essential minimum and maximum values at a point $x$ by:

$$\{m\,(\phi)\}(x) := \sup_{r>0} \text{ess} \quad \inf_{y\in B(x;r)} \phi(y), \tag{2.2}$$

$$\{M\,(\phi)\}(x) := \inf_{r>0} \text{ess} \quad \sup_{y\in B(x;r)} \phi(y), \tag{2.3}$$

where $B(x;r)$ denotes the ball of radius $r$ centered at $x \in I\!R^n$. More generally, let us define for almost all time $t$

$$\{Conv\,ess\,(a(t,.))\}(x) := \cap_{r>0} \quad \cap_{N,\ Meas(N)=0}\, Conv\,(a(t, B(x;r) \setminus N)) \tag{2.4}$$

where $Conv(E)$ denotes the smallest closed convex set containing the set $E$. We first recall the essential definitions and results of [10].

**Definition 2.1** *[10]*
*A generalized solution in the sense of Filippov or in short a Filippov characteristic $X(t;s,x)$ issued from $x \in I\!R^n$ at time $s \in I\!R$ is a continuous function $X(.;s,x) \in C^0(I\!R; I\!R^n)$ which satisfies*

$$X(s;s,x) = x$$

$$\frac{\partial}{\partial t} X(t;s,x) \in \{Conv\,ess\,(a)(t,.)\}(X(t;s,x)) \quad a.e. \quad t \in I\!R.$$

**Remark 2.1** *(i) The definition (2.4) is the definition of the "essential convex hull" of $a(t,x)$ given in [10]. In particular it is easy to see that the definition of characteristics is equivalent to asking that for any vector $v \in I\!R^n$ we have*

$$\frac{\partial}{\partial t} X(t;s,x).v \leq \{M\,(a(t,.).v)\}(X(t;s,x)) \quad a.e. \quad t \in I\!R. \tag{2.5}$$

*(ii) Throughout the paper, we will use the notation $X(t;x) := X(t;0,x)$.*
*(iii) In view of condition (2.1) the characteristics are uniformly bounded in $W^{1,1}_{loc}(I\!R)$ and therefore are equicontinuous.*

The property (iii) is the key-point to prove the following theorem.

**Theorem 2.1** *[10]*
*For every $s \in I\!R$ and $x \in I\!R^n$, there is at least one Filippov characteristic $X(t;s,x)$. Moreover if these characteristics are unique for $t \geq s$ then the flow, i.e. the map:*

$$(t,x) \to X(t;s,x)$$

*is continuous on $[s, +\infty[\times I\!R^n$.*

To prove the existence part of a solution, we mollify $a$ by a non negative function and pass to the limit. The continuity is implied by the equicontinuity of the characteristics since every uniform limit of characteristics is also a characteristic.
Filippov has given the following uniqueness criterion : let $L(t,u)$ be a function of $L^1(\mathbb{R}; L^\infty(\mathbb{R}))$ such that the only solution to

$$\frac{\partial}{\partial t}u(t) = L(t,u(t)) \text{ a.e. } \ t > s, \ u(s) = 0 \tag{2.6}$$

is identically equal to 0.

**Theorem 2.2** *[10]*
*If $a$ satisfies*

$$(a(t,x) - a(t,y)).(x-y) \leq |x-y| L(t,|x-y|) \, a.e. \ \ t > s, \ x,y \in \mathbb{R}^n, \tag{2.7}$$

*where $L$ satisfies the above uniqueness property, then the characteristics are unique for $t \geq s$.*

We point out that (2.7) allows discontinuities of compressive type. Moreover the criterion holds true for $n = 1$ if $a \ u = f(u)$ where $u$ is the (unique) admissible solution to a scalar conservation law.
For sake of completeness we recall the main ingredient of the proof of [10]. Let $X(t;s,x)$, $Y(t;s,y)$ be two Filippov characteristics. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(X(t;s,x) - Y(t;s,y)\right)^2 &= 2\frac{\partial}{\partial t}X(t;s,x).\left(X(t;s,x) - Y(t;s,y)\right) - \\ & 2\frac{\partial}{\partial t}Y(t;s,x).\left(X(t;s,x) - Y(t;s,y)\right) \quad \text{a.e. } t \in \mathbb{R}.\end{aligned}$$

In view of (2.5) this leads to

$$\begin{aligned}&\frac{\partial}{\partial t}\left(X(t;s,x) - Y(t;s,y)\right)^2 \\ &\quad \leq \{M\left(a(t,.).\left(X(t;s,x) - Y(t;s,y)\right)\right)\}\left(X(t;s,x)\right) \\ &\quad -2\{m\left(a(t,.).\left(X(t;s,x) - Y(t;s,y)\right)\right)\}\left(Y(t;s,y)\right).\end{aligned}$$

But for almost all time $t$, for arbitrary $\varepsilon > 0$ there exists $x_*$, $y_*$ such that

$$|X(t;s,x) - x_*| \leq \varepsilon, \quad |Y(t;s,y) - y_*| \leq \varepsilon,$$

$$(a(t,x_*) - a(t,y_*)).(x_* - y_*) \leq L(t,|x_* - y_*|)|x_* - y_*|,$$

$$\begin{aligned}&\{M(a(t,.).\left(X(t;s,x) - Y(t;s,y)\right))\}\left(X(t;s,x)\right) \\ &\quad \leq a(t,x_*).\left(X(t;s,x) - Y(t;s,y)\right)) + \varepsilon\end{aligned}$$

$$\begin{aligned}&\{m(a(t,.).\left(X(t;s,x) - Y(t;s,y)\right))\}\left(Y(t;s,y)\right) \\ &\quad \geq a(t,y_*).\left(X(t;s,x) - Y(t;s,y)\right)) - \varepsilon.\end{aligned}$$

We obtain:

$$\frac{\partial}{\partial t}(X(t;s,x)-Y(t;s,y))^2 \leq 2L(t,|x_*-y_*|)|x_*-y_*| + 4\varepsilon(1+\|a(t,.)\|_{L^\infty(I\!R^n)}).$$

So in the case where $L(t,.)$ is continuous, we take the limit $\varepsilon \to 0$ and obtain

$$\frac{\partial}{\partial t}(X(t;s,x)-Y(t;s,y))^2 \leq 2L(t,|X(t;s,x)-Y(t;s,y)|)|X(t;s,x)-Y(t;s,y)| \tag{2.8}$$

In the general case, see [10], this expression has to be replaced by

$$\frac{\partial}{\partial t}(X(t;s,x)-Y(t;s,y))^2 \\ \leq 2\{M(L(t,.))\}(|X(t;s,x)-Y(t;s,y)|)\ |X(t;s,x)-Y(t;s,y)|.$$

Then the uniqueness result is obtained when $x=y$ by comparing with (2.6).

We conclude this section with a result which shows that Filippov characteristics can be obtained as limit of smooth characteristics.

**Theorem 2.3** *[10]*
*Let $a$, $a_k$, $k=0,1,\ldots$ be vector fields which satisfy (2.1) and assume that*

$$(i)\quad a_k = b_k + r_k,$$

*with*

$$(ii)\quad r_k \to 0 \ \text{in} \ L^1(]0,T[;L^\infty(I\!R^n))^n,$$

*and*

$$(iii)\quad \text{For a sequence } \varepsilon_k \to 0, \text{ and for any vector } v \in I\!R^n, \text{ we have} \\ b_k(t,x).v \leq ess \quad sup_{y\in B(x;\varepsilon_k)} a(t,y).v \quad a.e. \ \ t \in ]0,T[, \ a.e. \ \ x \in I\!R^n.$$

*Then if $X_k(t;0,x)$ is a Filippov characteristic associated to $a_k$, up to a subsequence it converges in $C^0([0,T])$ towards a Filippov characteristic associated to $a$.*

# 3 Measure Solutions.

Throughout this section we *assume* the uniqueness of the characteristics $X(t;0,x)$ for $t \geq 0$. For instance this is the case if condition (2.7) is satisfied. It follows that the map $X(t;x) := X(t;0,x)$ belongs to $C^0([0,\infty[\times I\!R^n)$, so that for all $t$ the map $X(t) := x \to X(t;x)$ is continuous on $I\!R^n$.

## 3.1 Definition and first properties

In the sequel $\mathcal{M}_1(\Omega)$ denotes the space of bounded measures on the open set $\Omega$, $\mathrm{Lip}(\Omega)$ denotes the space of Lipschitz functions and $C_0^0(I\!R^n)$ (resp. $C_c^0(I\!R^n)$) the space of continuous functions which tend to zero at infinity (resp. which have a compact support). We start with a classical result on the problem

$$\frac{\partial}{\partial t}u + \mathrm{div}(au) = 0, \tag{3.1}$$

$$u(t=0,.) = u_0. \tag{3.2}$$

**Theorem 3.1** *Assume that $a$ is smooth:*

$$a \in L^1_{loc}(]0,\infty[; C^1(I\!R^n))^n. \tag{3.3}$$

*Then for every $u_0 \in \mathcal{M}_1(I\!R^n)$ there exists a unique distribution solution of (3.1), (3.2) in $C^0([0,\infty[; \mathcal{M}_1(I\!R^n) - weak*)$, given by*

$$u(t) = X(t)(u_0). \tag{3.4}$$

The above equation (3.4) simply means that $u(t)$ is the image of $u_0$ by the continuous map $X(t)$ :

$$\text{for any Borel set } B, \qquad u(t)(B) = u_0(X(t)^{-1}(B))$$

or equivalently :

$$\text{for any function } \varphi \in C_0^0(I\!R^n), \qquad < u(t,.), \varphi(.) >=< u_0(.), \varphi(X(t;.)) > .$$

Let us denote by $<< .,. >>$ the duality product between distributions and test-functions (or between measures and continuous functions) on $((0,+\infty) \times I\!R^n)$. Then for any $\varphi \in C_c^0(I\!R^{1+n})$ we can define:

$$<< u, \varphi >>=< u_0(.), \int_0^\infty \varphi(X(t;.))\ dt > .$$

**Proof of Theorem 3.1**
If $u$ is a solution then for every test function $\varphi \in \mathcal{D}(I\!R^{1+n})$:

$$\int_0^{+\infty} < u(t,x), (\frac{\partial}{\partial t}\varphi + a.\nabla_x\varphi)(t,x) > \ dt + < u_0(x), \varphi(0,x) >= 0. \tag{3.5}$$

A density argument shows that (3.5) remains true for any $\varphi$ in

$$\mathcal{V} = \{\varphi \in C_c^0(I\!R^{1+n}),\ \frac{\partial}{\partial t}\varphi + a.\nabla_x\varphi \in C_c^0(I\!R^{1+n})\}.$$

Indeed $\mathcal{D}(I\!R^{1+n})$ is dense in $\mathcal{V}$ for the topology of $L^1(I\!R; C_0^0(I\!R^n))$ if $a$ satisfies (3.3), see [1]. So let $\psi$ be any function of $C_c^1(I\!R^{1+n})$. We set:

$$\varphi(t,x) = \int_t^{+\infty} \psi(s, X(s;t,x))\ ds.$$

Since $X(s;t,X(t;x)) = X(s;x)$ we have

$$\frac{d}{dt}[\varphi(t,X(t;x))] = \psi(t,X(t;x)) = (\frac{\partial}{\partial t}\varphi + a.\nabla_x\varphi)(t,X(t;x))$$
$$\psi(t,x) = (\frac{\partial}{\partial t}\varphi + a.\nabla_x\varphi)(t,x), \quad \text{for any } t \in I\!R,\ x \in I\!R^n.$$

Therefore $\varphi \in \mathcal{V}$ and (3.5) yields

$$\int_0^{+\infty} < u(t,x), \psi(t,x) > \ dt$$
$$=< u_0, \int_0^{+\infty} \psi(t,X(t;x)) >= \int_0^{+\infty} < u_0, \psi(t,X(t;x)) > \ dt$$

which ends the proof.
The previous theorem shows that there is a natural extension to the concept of solution of (3.1) (3.2).

**Definition 3.2** *Let a be a vector field which satisfies (2.1) and such that the Filippov characteristics $X(t;x)$ are unique for $t \geq 0$. Then for every $u_0 \in \mathcal{M}_1(I\!R^n)$ the measure solution to (3.1), (3.2) is by definition*

$$u(t) := X(t)(u_0).$$

Now we study the properties of measure solutions. We start with

**Remark 3.2** *Under the same assumption as in Definition 3.2, the characteristics $X(t;s,x)$ are unique for $t \geq s \geq 0$ and the map $x \to X(t;s,x)$ is continuous and onto from $I\!R^n$ to $I\!R^n$.*

Using Theorem 2.1 we obtain the continuity of $X(t;s,.)$. The existence of Filippov backward characteristics trivially implies that $X(t;s,.)$ is onto.
The semi-group properties of measure solutions are listed below. They are immediately obtained from Definition 3.2 and Remark 3.2.

**Theorem 3.2** *Under the same assumptions as in Definition 3.2 the unique measure solution u satisfies the following properties.*
*(i)* $u(t) = X(t)(u_0) \in C^0([0,+\infty[; \mathcal{M}_1(I\!R^n) - weak*)$,
*(ii) Monotonicity:* $u_0 \geq v_0$ *implies* $X(t)(u_0) \geq X(t)(v_0)$.
*(iii) Mass conservation:* $X(t)(u_0)(I\!R^n) = u_0(I\!R^n)$.
*(iv) Contraction property:* $|X(t)(u_0)|(I\!R^n) \leq |u_0|(I\!R^n)$.
*(v) Semi-group property:* $X(t;s)(X(s)(u_0)) = X(t)(u_0)$ *for any* $t \geq s \geq 0$ *and any* $u_0$.

**Proof** The only property which is not obvious is the weak-star continuity of $u(t)$. We first prove that the measure $u(t)$ tends uniformly to 0 at infinity (compactness at infinity):

**Lemma 3.1** *Under the same assumption as in Definition 3.2, let*

$$K(T) = \int_0^T \|a(t,.)\|_{L^\infty(I\!R^n)}\ dt$$

*then for any $\varepsilon > 0$ there exists $R$ which depends only on $u_0$, $\varepsilon$ and $K(T)$ such that for any $t \in [0,T]$*

$$|X(t)(u_0)|(\mathbb{R}^n \setminus B_R) \leq \varepsilon$$

*where $B_R := B(0;R)$.*

**Proof of Lemma 3.1**
There exists $R_0$ such that $|u_0|(\mathbb{R}^n \setminus B_{R_0}) \leq \varepsilon$. We remark that

$$|X(t;x) - x| \leq \int_0^t \|a(s,.)\|_{L^\infty(\mathbb{R}^n)}\, ds \leq K(T) \quad \text{ for } t \in [0,T].$$

Consequently, $|X(t;x)| \geq R_0 + K(T) = R$ implies $|x| \geq R_0$. Therefore, using the obvious property (iii) in Proposition 3.2 below, we clearly have

$$|X(t)(u_0)|(\mathbb{R}^n \setminus B_R) \;\leq\; |u_0|(\mathbb{R}^n \setminus B_{R_0}) \;\leq\; \varepsilon.$$

We can now prove the weak-star continuity of $u(t)$.
Let $\eta \in \mathcal{D}(\mathbb{R}^{1+n})$, $0 \leq \eta \leq 1$, with $\eta = 1$ on $B_1$. We set $u_R(t)(.) = \eta(./R)u(t)(.)$. In view of the previous lemma $u_R(t) \to u(t)$ in $\mathcal{M}_1(\mathbb{R}^n)$ uniformly on $[0,T]$. So it is sufficient to prove the continuity of $u_R$. But $X(t;x) \to X(t_0;x)$ uniformly with respect to $x$ when $t \to 0$. So for any $\varphi \in C_0^0(\mathbb{R}^n)$, $\varphi(X(t;x))\eta(X(t;x)/R) \to \varphi(X(t_0;x))\eta(X(t_0;x)/R)$ uniformly with respect to $x$ when $t \to 0$. It follows that $< u_R(t), \varphi > \to < u_R(t_0), \varphi >$, which concludes the proof of theorem 3.2.

## 3.2 General transport equations.

Naturally, we can also consider the more general problem

$$\frac{\partial}{\partial t}u + \operatorname{div}(a(t,x)\, u) + b\, u = f \text{ in } (0,+\infty) \times \mathbb{R}^n\,, \tag{3.6}$$

$$u(0,x) = u_0(x) \text{ in } \mathbb{R}^n\,, \tag{3.7}$$

with the same assumptions on $a$ as in Definition 3.2 and for instance

$$b \in L^1(0,+\infty; C_b^0(\mathbb{R}^n)), \tag{3.8}$$

where $C_b^0$ denotes the space of bounded continuous functions, and

$$f \in L^1(0,+\infty; \mathcal{M}_1(\mathbb{R}^n)). \tag{3.9}$$

An obvious fixed point argument allows to show the following

**Theorem 3.3** *Under the same assumptions as in Definition 3.2 and the above assumptions (3.8), (3.9), for every $u_0 \in \mathcal{M}_1(\mathbb{R}^n)$, Problem (3.6), (3.7) has a unique measure solution*

$$u \in C^0([0,+\infty[; \mathcal{M}_1(\mathbb{R}^n) - weak*),$$

*implicitly given by Duhamel's formula*

$$u(t,.) = X(t)(u_0) + \int_0^t X(t-s)(f(s,.) - b(s,.)u(s,.))\, ds. \tag{3.10}$$

*This measure solution satisfies the same stability properties as in Theorem 3.6 and Proposition 3.1 below.*

It is remarkable that the concept of measure solutions for the conservative transport equation also allows to obtain a natural way to define the solution to the problem

$$\frac{\partial}{\partial t}u + a(t,x).\,\nabla_x u = 0, \tag{3.11}$$

$$u(0,x) = u_0(x). \tag{3.12}$$

Indeed, let $\rho(t)$ be the image by the flow $X(t)$ of the Lebesgue measure. In other words, $\rho$ is the measure solution to

$$\frac{\partial}{\partial t}\rho + \mathrm{div}\,(a(t,x)\rho) = 0,$$

$$\rho(0,x) = \rho_0(x).$$

Then, at least formally, we have:

$$\frac{\partial}{\partial t}\rho u + \mathrm{div}\,(a(t,x)\rho u) = 0,$$

$$\rho u(0,x) = u_0(x).$$

Therefore $\rho u(t)$ is given by $X(t)(u_0)$. It is easy to check that if $u_0 \in L^1_{loc}(I\!R^n)$, then $X(t)(u_0)$ is absolutely continuous with respect to $\rho(t)$. This motivates the following

**Definition 3.3** *Under the same assumption as in Definition 3.2, for any $u_0 \in L^1_{loc}(I\!R^n)$ the "measure solution" $u(t,.)$ to the non conservative problem (3.11), (3.12) is by definition the Radon-Nikodym derivative of $X(t)(u_0)$ with respect to $\rho(t)$. This measure is defined $d\rho(t)$ a.e. and is locally integrable with respect to $\rho$.*

### 3.3 Definition of the product $a \bullet u$

We now turn back to the conservative problem (1.1), (1.2). If $a$ is continuous we have for any time $t \geq 0$ and any $\varphi \in \mathcal{D}(I\!R^n)$

$$< a\ u(t,.), \varphi(.) >=< u_0(.), a(t, X(t;.))\varphi(X(t;.)) >$$

$$< a\ u, \varphi >=< u_0(x), \frac{\partial}{\partial t}X(t;x)\varphi(X(t;x)) > . \tag{3.13}$$

Therefore, for any $\varphi \in \mathcal{D}((0,+\infty)\times I\!R^n)$ we have

$$< \mathrm{div}\,(a\ u)(t,.), \varphi(t,.) >= - < u_0(.), \frac{\partial}{\partial t}X(t;.)\nabla_x\varphi(t, X(t;.)) >$$

$$\begin{aligned}&< \mathrm{div}\,(a\ u)(t,.), \varphi(t,.) >\\ &\quad = - < u_0(.), \tfrac{d}{dt}\varphi(t, X(t;.)) > + < u_0(.), \tfrac{\partial}{\partial t}\varphi(t, X(t;.)) > .\end{aligned}$$

Integrating the last equation with respect to time, we obtain

$$\begin{aligned}&<< \mathrm{div}\,(a\ u), \varphi >>\\ &\quad =<< u_0(x)\otimes 1(t), \tfrac{\partial}{\partial t}\varphi(t, X(t;x)) >>=<< u(t,x), \tfrac{\partial}{\partial t}\varphi(t,x) >> .\end{aligned}$$

Therefore $u$ is also a distribution solution. Starting from (3.13) we remark that in the general case we can always define the product $a\ u$ in $\mathcal{D}'((0,+\infty)\times I\!R^n)$ by

$$<< a \bullet u, \varphi >>=< u_0(x), \int_0^{+\infty} \frac{\partial}{\partial t}X(t;x)\ \varphi(t, X(t;x))\ dt > . \tag{3.14}$$

It is easy to check that the above duality product makes sense, since

$$\int_0^{+\infty} \frac{\partial}{\partial t}X(t;x)\varphi(t, X(t;x))\ dt$$

is continuous with respect to $x$. Indeed the functions $\frac{\partial}{\partial t}X(.;x)$ are uniformly bounded with respect to $x$ in $L^1([0,T])$. Now, since

$$X(.;x) \to X(.;x_0) \text{ in } C^0([0,T]) \text{ when } x \to x_0,$$

we have

$$\frac{\partial}{\partial t}X(.;x) \to \frac{\partial}{\partial t}X(.;x_0) \text{ in } \ \mathcal{M}_1([0,T])weak*$$

and

$$\varphi(t, X(t;x)) \to \varphi(t, X(t;x_0) \text{ in } C^0([0,T]).$$

Therefore the integral of the product with respect to $t$ is continuous.
We summarize these results in

**Theorem 3.4** *With the same assumptions as in Definition 3.2 let us define the product $a \bullet X(t)(u_0)$ by (3.14). Then $u = X(t)(u_0)$ satisfies (3.2) and*

$$\frac{\partial}{\partial t}u + div(a \bullet u) = 0$$

*in the distribution sense. Moreover if $a$ is continuous with respect to $x$ for almost all time $t$ then this definition coincides with the usual definition of the product $a\ u$ and $X(t)(u_0)$ is a distribution solution of (3.1).*

In the case where there the assumption (3.15) below is satisfied, we can be more precise.

**Theorem 3.5** *Let $a$ be a vector field in $[L^\infty((0,T)\times(I\!R^n))]^n$ such that there exists a function $\alpha$ in $L^1(0,T)$ for which:*

$$(a(t,x) - a(t,y)).(x-y) \leq \alpha(t)\ |x-y|^2 \quad a.e. \ \ t > s,\ x, y \in I\!R^n. \tag{3.15}$$

*then there exists a representing function of $a$ such that*

$$a(t, X(t;x)) = \frac{\partial}{\partial t}X(t;x) \quad \forall x \in I\!R^n \ a.e.\ t \in (0,T).$$

*For this choice of the representing function the product $a \bullet u$ given in (3.14) coincides with the usual product*

$$<< a \bullet u, \varphi >>=< u_0(x), \int_0^{+\infty} a(t, X(t;x))\varphi(t, X(t;x)) >$$

The proof of this result is postponed to the end of the next subsection.

## 3.4 Stability results.

We start with an easy consequence of Theorem 2.2.

**Proposition 3.1** *Let $a_k$ and $a$ be as in Theorem 2.2. We assume that the Filippov characteristics $X_k$ and $X$ associated to $a_k$ and $a$ are unique. Then for any $u_0 \in \mathcal{M}_1(I\!R^n)$ and any $T > 0$, the measures solutions $u_k(t) = X_k(t)(u_0)$ converges in $C^0([0,T]; \mathcal{M}_1(I\!R^n) weak - *)$ to the measure solution $u(t) = X(t)(u_0)$.*

**Proof**
Let $\varphi \in C_0^0(I\!R^n)$, then $\varphi(X_k(t,.)) \to \varphi(X(t,.))$ pointwise. So by Lebesgue theorem $u_k(t) \to u(t)$ in $\mathcal{M}_1(I\!R^n) weak - *$. By Ascoli theorem it remains to prove that the sequence $< u_k(t), \varphi >$ is equicontinuous on $[0,T]$. Now the conditions (i), (ii) and (iii) of Theorem 2.2 imply that the sequence $a_k$ is uniformly bounded in $L^1((0,T); L^\infty(I\!R^n))^n$. Therefore by Lemma 3.1 the sequence $u_k(t)$ is uniformly compact at infinity, so that it is sufficient to consider the case where $\varphi$ has a compact support. The result follows from the uniform continuity of $\varphi$ and from the equicontinuity of $X_k(t,.)$ with respect to time.

**Remark 3.3** *The above proposition shows that the measure solutions of Definition 3.2 can be obtained as limit of classical solutions in the distribution sense, for instance by using a sequence of mollifiers $a_k(t,.) = a(t,.) * \rho(k.)k^n$.*

Our main result is the following

**Theorem 3.6** *Let $a_k$ be a sequence of uniformly bounded vector fields in the space*

$$[L^1_{loc}(]0,\infty[; L^\infty(I\!R^n))]^n$$

*such that there exists a function $\alpha$ in $L^1(0,T)$ for which:*

$$(a_k(t,x) - a_k(t,y)).(x-y) \leq \alpha(t)\ |x-y|^2, \quad a.e.\ \ t > s,\ x,y \in I\!R^n. \tag{3.16}$$

$$|a_k(t,x)| \leq \alpha(t), \quad a.e.\ \ t > s,\ x,y \in I\!R^n. \tag{3.17}$$

*Then for any $u_0 \in \mathcal{M}_1(I\!R^n)$ and any $T > 0$, there exists a subsequence of measure solutions $u_k(t) = X_k(t)(u_0)$ which converges in $C^0([0,T]; \mathcal{M}_1(I\!R^n) weak - *)$ to the measure solution $u(t) = X(t)(u_0)$, where $X$ is the unique Filippov flow associated to some $a$ which satisfies (3.16) and (3.17).*

**Proof**
Using (3.16) and the same argument as in the proof of Theorem 2.2 we see that the characteristics are uniquely defined and satisfy:

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}|X_k(t;x) - X_k(t;y)|^2 \\ &\quad \leq \alpha(t)\ |X_k(t;x) - X_k(t;y)|^2, \text{ a.e. } t \in (0,T),\ \forall\ x,y \in I\!R^n,\end{aligned}$$

$$|X_k(t;x) - X_k(t;y)| \leq C_T\ |X_k(s;x) - X_k(s;y)|,\ \forall\ t \geq s \in (0,T),\ \forall\ x,y \in I\!R^n.$$

Then (choosing $s = 0$) $X_k$ is Lipschitz with respect to $x$ with a Lipschitz constant at most equal to $C_T$. Using (3.17) we also obtain

$$\left|\frac{\partial}{\partial t} X_k(t;x)\right| \leq \alpha(t), \quad \text{a.e. } t \in (0,T) \quad \forall\, x \in I\!R^n,$$

$$|X_k(t;x) - X_k(s;x)| \leq \int_s^t \alpha(\tau)\, d\tau, \quad \forall\, t \geq s \in (0,T),\ \forall\, x \in I\!R^n.$$

Therefore the family $(X_k)$ is equicontinuous with respect to $t$ and $x$, and bounded, since all the images $X_k(t,.)(B_R)$ are contained in $B_{R+L}$, with $L = \int_0^T \alpha(\tau)\, d\tau$. Therefore there exists a subsequence, still denoted by $X_k$, which converges (uniformly on every compact subset of $[0,T] \times I\!R^n$) to some continuous function $X$. This function $X$ satisfies:

$$X(0;x) = x, \qquad \forall t \in [0,T],\ X(t;.) \text{ is onto from } I\!R^n \text{ to } I\!R^n, \tag{3.18}$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}|X(t;x) - X(t;y)|^2 \\ &\quad \leq \alpha(t)\ |X(t;x) - X(t;y)|^2, \quad \text{a.e. } t \in (0,T) \quad \forall\, x,y \in I\!R^n, \end{aligned} \tag{3.19}$$

$$|X(t;x) - X(t;y)| \leq C_T\ |X(s;x) - X(s;y)|, \quad \forall\, t \geq s \in (0,T),\ \forall\, x,y \in I\!R^n, \tag{3.20}$$

$$\left|\frac{\partial}{\partial t} X(t;x)\right| \leq \alpha(t), \quad \text{a.e. } t \in (0,T) \quad \forall\, x \in I\!R^n, \tag{3.21}$$

$$|X(t;x) - X(s;x)| \leq \int_s^t \alpha(\tau)\, d\tau, \quad \forall\, t \geq s \in (0,T),\ \forall\, x \in I\!R^n. \tag{3.22}$$

Here, in order to see that $X(t,.)$ is onto, we have simply observed that for any $t \in [0,T]$ and any $y \in I\!R^n$ there is $x_k \in B(y;L)$ such that $X_k(t,x_k) = y$. Therefore, at least for a subsequence, $x_k \to x$ and $X(t,x) = y$. The estimate (3.19) is obtained by passing to the limit in the term $\frac{\partial}{\partial t}|X_k(t;x) - X_k(t;y)|^2$ in $\mathcal{D}'(I\!R^{1+n})$, which is easy, in view of the strong convergence of $X_k$.

Now with the same arguments as in Proposition 3.1 and Lemma 2.3, it is easy to see that $u_k(t) = X_k(t)(u_0)$ converges in $C^0([0,T];\mathcal{M}_1(I\!R^n) weak-*)$ to $u(t) = X(t)(u_0)$ . It is more difficult to show that $X$ is the Filippov flow corresponding to some vector field $a$.

In order to do that, we first remark that in view of (3.20) the forward characteristics $t \to X(t;x)$ are uniquely defined. Therefore, for any $y \in I\!R^n$ and any $s \in [0,T]$ we can define $X(t;s,y) = X(t;x)$ for $t \geq s$, where $x$ is a point such that $X(s;x) = y$. Obviously, in case of non-uniqueness, this definition does not depend on the choice of point $x$. Now for any $h > 0$ let us introduce :

$$a_h(s,y) := \frac{X(s+h;s,y) - y}{h}, \qquad \forall s \in [0,T-h], \forall y \in I\!R^n.$$

We observe that:

$$a_h(t, X(t;x)) = \frac{X(t+h;x) - X(t;x)}{h} = \int_0^1 \frac{\partial}{\partial t} X(t + h\tau, x)\ d\tau. \tag{3.23}$$

In view of (3.21), for all $x$, $a_h(., X(., x))$ converges in $L^1(0, T')$, $T' < T$ to $\frac{\partial}{\partial t} X(., x)$. Moreover, for every $T' < T$,

$$\int_0^{T'} |a_h(t, X(t;x)) - \frac{\partial}{\partial t} X(t;x)| dt \leq 2 \int_0^T \alpha(t) dt.$$

Therefore, by Lebesgue Theorem, $a_h(., X(., x))$ converges in $L^1_{loc}((0, T') \times I\!R^n)$, $T' < T$. and satisfies (at least for a subsequence $a_j := a_{h_j}$)

$$a_j(t, X(t;x)) \to \frac{\partial}{\partial t} X(t;x) \quad \forall (t, x) \in ([0, T'] \times I\!R^n) \setminus N$$

where $N$ is a null-set of $([0, T'] \times I\!R^n)$. We now use the following lemma, whose proof will be given below.

**Lemma 3.2** *Let $X : [0, T] \times I\!R^n \to I\!R^n$ be a continuous map such that $X(t, .)$ is Lipschitz for all $t \in (0, T)$. Then $(t, x) \to (t, X(t, x))$ maps null-sets of $[0, T] \times I\!R^n$ into null-sets.*

Now let $N'$ be the image of $N$ by the map $(t, x) \to (t, X(t, x))$. Then $N'$ is also a null-set. Moreover, since the previous map is onto we remark that $a_j(t, y)$ converges on $([0, T'] \times I\!R^n) \setminus N'$. Let us denote by $a$ this limit ($a$ is defined almost everywhere). We have:

$$\begin{aligned} a_j(t, y) &\to a(t, y) \quad \forall (t, y) \in ([0, T'] \times I\!R^n) \setminus N', \\ a(t, X(t, x)) &= \frac{\partial}{\partial t} X(t;x) \quad \forall (t, x) \in ([0, T'] \times I\!R^n) \setminus N. \end{aligned}$$

Clearly, $a$ satisfies (3.16) and (3.17), in view of (3.19) and (3.21). More precisely we get:

$$\begin{aligned} (a(t, y) - a(t, z)) . (y - z) \leq \alpha(t) |y - z|^2 \quad &\forall (t, y, z) \in (0, T') \times I\!R^{2n} \text{ such that} \\ &(t, y) \notin N',\ (t, z) \notin N', \text{ and } t \notin M \\ |a(t, y)| \leq \alpha(t) \quad \forall (t, y) \in (0, T') \times I\!R^n &\text{ such that } (t, y) \notin N', \text{ and } t \notin M \end{aligned}$$

where $M$ is a null-set of $(0, T')$. So let $Y$ be the unique Filippov flow associated to $a$. It remains to prove that $Y = X$. We have

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} |X(t;x) - Y(t;x)|^2 &= a(t; X(t;x)).(X(t;x) - Y(t;x)) \\ &- \tfrac{\partial}{\partial t} Y(t;x).(X(t;x) - Y(t;x)), \quad \forall (t, x) \in ([0, T'] \times I\!R^n) \setminus N \end{aligned}$$

But

$$\frac{\partial}{\partial t}Y(t;x).(X(t;x)-Y(t;x))$$
$$\geq \{m\,(a(t;.).(X(t;x)-Y(t;x))\}(Y(t;x)) \quad \text{a.e. } t\in(0,T').$$

We remark that $N'_t = \{y;\ (t,y)\in N'\}$ is a null-set of $I\!R^n$, for almost all time $t$. Therefore, almost everywhere in time, i.e. $\forall t\in(O,T')\setminus M'$ (where $M'$ is a null-set) and for any $\varepsilon>0$ there exists $y_*$ such that $(t,y_*)\notin N'$ and

$$|Y(t;s,y)-y_*|\leq\varepsilon,$$

$$m\{(a(t,.).(X(t;s,x)-Y(t;s,y)))\}(Y(t;s,y))$$
$$\geq a(t,y_*).\,(X(t;s,x)-Y(t;s,y)))-\varepsilon.$$

We thus obtain, $\forall(t,x)\in[0,T']\times I\!R^n\setminus N$, with $t\notin M'$ :

$$\frac{1}{2}\frac{\partial}{\partial t}|X(t;x)-Y(t;x)|^2\leq(a(t;X(t;x))-a(t,y_*))\,.(X(t;x)-y_*)$$
$$+(a(t;X(t;x))-a(t,y_*)).(y_*-Y(t;x))+\varepsilon,$$

For $t\notin M$, $t\notin M'$ and $(t,x)\notin N$ the above inequality implies

$$\frac{1}{2}\frac{\partial}{\partial t}|X(t;x)-Y(t;x)|^2\leq\alpha(t)|X(t;x)-Y(t;x)|^2+(1+2\alpha(t))\varepsilon.$$

We conclude that

$$\frac{1}{2}\frac{\partial}{\partial t}|X(t;x)-Y(t;x)|^2\leq\alpha(t)|X(t;x)-Y(t;x)|^2,\quad \text{a.e. } (t,x)\in[0,T']\times I\!R^n,$$

which shows that $X(t;.)=Y(t;.)$ in $L^2_{loc}$ for almost all $t$. Now since $X$ and $Y$ are continuous with respect to $t$ and $x$ we conclude that $X=Y$.

**Proof of Lemma 3.2**

For each time $t$ the map $x\to X(t;x)$ is Lipschitz and therefore maps null-sets of $I\!R^n$ into null-sets. Now if $N$ is a null-set of $(0,T)\times I\!R^n$ then $N_t=\{x;\ (t,x)\in N\}$ is also a null-set of $I\!R^n$ for almost all $t\in(0,T)$. Therefore $N'_t=\{X(t;x);\ (t,x)\in N\}$ is a null-set of $I\!R^n$ for almost all $t\in(0,T)$. Consequently $N'=\{(t,X(t;x));\ (t,x)\in N\}$ is a null-set of $(0,T)\times I\!R^n$.

**Proof of Theorem 3.5**

Since $a$ is bounded the characteristics are Lipschitz with respect to time. Therefore $a_h(t,X(t;x))$ defined by (3.23) converges for all $x$ almost everywhere in time. The limit of $a_h$ gives the correct representing function of $a$.

## 4 Examples and Comments.

We first come back to the examples of Section 1. We will essentially describe the solution in the case where the initial datum $u_0$ is a $L^1_{loc}$ function. Apart from

Example 2, which is precisely the case where our theory does not apply, the extension to the general case is obvious.
**Example 1:** $n=1, a(t,x)=-sign(x)$.
Here, the Filippov characteristics are defined by (1.4). Therefore, the solution $u$ is uniquely defined by (1.6) or (1.7). In particular, for any $t>0$ and $a<b$

$$u(t)([a,b[) = \begin{cases} \int_{a-t}^{b-t} u_0(x)\,dx & \text{if } a<b<0, \\ \int_{a+t}^{b+t} u_0(x)\,dx & \text{if } 0<a<b, \\ \int_{a-t}^{b+t} u_0(x)\,dx & \text{if } a<0<b. \end{cases} \tag{4.1}$$

Therefore if $u_0 \in L^1_{loc}(I\!R)$ then

$$u(t,x) = \ u_0(x-t).\ H(-x) + u_0(x+t).\ H(x) + (\int_{x-t}^{x+t} u_0(y)dy\ ).\ \delta(x). \tag{4.2}$$

This formula has an obvious extension to the case where $u_0$ is an arbitrary measure. In particular if $u_0 = \delta(x)$ then $u(t,x) = \delta(x)$ for all $t>0$.
**Example 2:** $n=1, a(t,x) = +sign(x)$.
Now the Filippov characteristics, given by (1.5) are no longer uniquely defined. Therefore our definition does not handle this case. For any $t>0$ any Borel set $B \subset [-t,+t]$ emanates from the same"father" $\{0\}$. For this reason, $u(t)$ defined by (1.7) is no longer additive. For instance at time $t>0$ the disjoint intervals $I=[-2t,t/3]$ and $J=[2t/3,2t]$ satisfy

$$u(t)(I)+u(t)(J) = \ u_0([-t,0]) + \ u_0([0,t]). \tag{4.3}$$

whereas

$$u(t)(I\cup J) = \ u_0([-t,+t]). \tag{4.4}$$

In other words, in (4.3) we count twice the initial mass at the origin in (4.3) and only once in (4.4). Clearly this difficulty vanishes if $u_0 \in L^1_{loc}(I\!R)$ or more generally if

$$u_0(\{0\}) = 0, \tag{4.5}$$

Let us also remark that, in this particular case, the unique"reversible" solution to the non-conservative equation

$$\frac{\partial}{\partial t}v + a\ \frac{\partial}{\partial x}v = 0,$$

introduced in [2], [3] would be constant in the fan $(tx); |x|/t \leq 1$. Therefore, its derivative $u$ with respect to $x$ would vanish in this fan, which exactly corresponds to the above case, and therefore to the unique solution to (1.1) which preserves the total mass.
**Example 3:** $n=1, a(t,x) = sign(x).\ sign(t-t^*),\ t^*>0$.
There is no hope to construct a unique solution, since by (4.2) $u(t^*,.)$ has a singular part at the origin and therefore cannot satisfy the analog of (4.5).
We now turn to a few multi-dimensional examples.

**Example 4:** $n=2, a(t,x)=(1,-sign(x_2))$.
Here the characteristics are uniquely defined by

$$\begin{array}{rcl} X_1(t,x_0) & = & x_{0,1}+t, \\ X_2(t,x_0) & = & \begin{cases} (x_{0,2}+t).H(-x_{0,2}-t) & , x_{0.2}<0, \\ (x_{0,2}-t).H(+x_{0,2}-t) & , x_{0.2}>0, \\ 0 & , x_{0.2}=0. \end{cases} \end{array} \tag{4.6}$$

Therefore the measure solution associated to any $u_0 \in L^1_{loc}(I\!R)$ is given by

$$\begin{array}{l} u(t,x_1,x_2)) = \; u_0(x_1-t,x_2-t).H(-x_2) + \; u_0(x_1-t,x_2+t).H(-x_2) \\ + \; (\int_{x_1-t}^{x_1} \int_{y_1-x_1}^{x_1-y_1} \; u_0(y_1,y_2)\; dy_1 dy_2) \; H(x_1).\; \delta(x_2). \end{array} \tag{4.7}$$

**Example 5:** $n$ is arbitrary, $a(t,x)=-|x|^{\alpha-1}x, |; \alpha < \; 1$.
The characteristics are uniquely defined by

$$X(t,x_0) = (|x_0|^{1-\alpha} - (1-\alpha)t)^{1/(1-\alpha)}.\; x_0/|x_0|. \tag{4.8}$$

They reach the origin at time $t$ if and only if

$$|x_0| = r(t) := ((1-\alpha)t)^{1/(1-\alpha)}$$

Therefore the unique measure valued solution associated to any locally integrable function $u_0$ can be easily computed. We leave that to the reader.
**Remark 4.1.**
(i) We could of course consider more complicate examples, for instance
Example 6: $n=2, a(t,x)=-|x|^{\alpha-1}x+x^{\perp},\ \alpha<1,\ x^{\perp}:=(-x_2,x_1)$.
Example 7: $n=3, a(t,x)=(-|x'|x',1),\ \alpha<1,\ x':=(x_1,x_2)$.
(ii) In the above examples, the solution typically contains a singular measure, concentrated on a set of points where the vector-field $a$ is at least discontinuous. Therefore in general the product of distributions $a\ u$ is not *a priori* defined. However, except in Examples 2 and 3, this product is in fact perfectly well defined by the results of Section 3. In particular, we can easily see in Example 1 or 5 that the product $a\ u_s$ (where $u_s$ is the singular part of $u$) is perfectly well defined, and in fact is 0, since in either case the part of the solution which has already concentrated at time $t$ will keep traveling with a *zero* speed. This is exactly the physical meaning of Theorem 3.4.
(iii) In particular, if $a$ is discontinuous and compressive across a co-dimension 1 manifold, as in the above examples 1 and 4, we see that the"shock speed" is only governed by the vector-field $a$, and is independent of the solution. Moreover as in examples 5, 6 and 7, the co-dimension of the set of singular points can be strictly bigger than 1. Therefore the notion of Rankine-Hugoniot condition is totally irrelevant here.
(iv) In [9], DiPerna and Lions studied the more general non-conservative transport equation

$$\frac{\partial}{\partial t}u + a(t,x).\nabla u + b(t,x)u = 0. \tag{4.9}$$

Under suitable hypotheses of regularity, essentially assuming that

$$a \in L^1(0,T;(W^{1,1}_{loc}(I\!R^n))^n) \tag{4.10}$$

and

$$\text{div } a \in L^1(0,T;L^\infty(I\!R^n)), \tag{4.11}$$

they proved that (4.9) as well as the associated ordinary differential equation (1.3) has a unique *renormalized* solution. The corresponding trajectories are defined for all $t$, but only *almost everywhere* with respect to $x$. Therefore on one hand the characteristics cannot intersect as in the above examples. On the other hand, since the characteristics are not defined *everywhere*, it would be impossible to construct a solution associated to a singular initial datum $u_0$.
We now take a look at the inviscid Burgers equation.

# 5 Burgers Equation revisited.

We consider the one-dimensional case and rewrite this equation under the form

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}((u/2)\ u) = 0\ ,\ x \in I\!R. \tag{5.1}$$

It is interesting to remark that the natural speed of propagation in (5.1) is $u/2$ and not $u$ as usually.
We consider the Riemann problem

$$u(0,x) = u_\pm \text{ for } \ \pm x > 0. \tag{5.2}$$

Let us define

$$a(t,x) = u(t,x)/2 = u_\pm/2 \text{ for } \pm(x-st) > 0. \tag{5.3}$$

We are going to show that even in the case of a rarefaction wave, the Filippov characteristics, and therefore the measure solution $\tilde{u}$ is well defined.

**Theorem 5.1**
*(i) The Filippov characteristics associated to (5.3) are unique. Therefore there exists a unique measure solution $\tilde{u}$ to (5.1), (5.2).*
*(ii) This measure solution $\tilde{u}$ has no singular part and*

$$\tilde{u} = u. \tag{5.4}$$

*In other words the two natural notions of solutions coincide.*

**Proof.**
We study separately the two cases.

### 5.1 The case of a shock.

We first assume

$$u_+ < u_-.$$

The usual solution $u$ to (5.1), (5.2) consists of a single shock wave, with speed

$$s = (u_+ + u_-)/2.$$

(i) We first describe the Filippov characteristics. They are given by (1.3), (5.3). There are three cases.

**Case 1:** $u_+ < 0 < u_-$.
That implies

$$u_+/2 < s < u_-/2.$$

In this case the Filippov characteristics are compressive, and therefore unique.

**Case 2:** $0 < u_+ < u_-$ (or $u_+ < u_- < 0$).
In this case the $u_+/2$ characteristics enter the shock and the $u_-/2$ characteristics leave it. Therefore they are not compressive. However, by (2.7) they are uniquely defined, since

$$(a(t,x) - a(t,y)).(x-y) \leq (u_+ - u_-).|x-y|/2 \leq 0.$$

**Case 3:** $u_+ = 0 < u_-$ (or $u_+ < u_- = 0$).
Obviously the $u/2$ characteristics are still uniquely defined.

(ii) It is interesting to remark that a shock wave for the Burgers equation is not compressive in the sense defined here, except in the first case! Now let us describe the measure solution $\tilde{u}$ in each case.

**Case 1:** As in (4.2) the solution is given here by

$$\tilde{u}(t,x) = u_-.H(-x+st) + u_+.H(x-st) + \left(\int_{x-tu_-/2}^{x+tu_+/2} u_0(y)dy\right).\delta(x-st). \quad (5.5)$$

But we note that for $x = st$ the above integral is

$$-(s - u_-/2)tu_- + (s - u_+/2)tu_+ = 0.$$

Therefore (fortunately!) the singular term in (5.5) vanishes, which implies (5.4) in this first case.

**Remark 5.1**

As we just said, the above singular term vanishes, provided that we are able to compute *exactly* the characteristic speed $u/2$. Of course, if that were not the case, any error in this computation would immediately produce a small (but non zero) delta-function in the solution, in other words would produce a numerical *spike*.

**Case 2:** The characteristics are given by

$$X(t,x_0)=\begin{cases} x_0+tu_-/2 & \text{for } x_0\le 0\ ,\\ x_0+tu_+/2 & \text{for } 0<t<2x_0/u_-\ ,\\ (x_0(u_++u_-)/u_-)+(t-x_0/2u_-)u_-/2 & \text{for } t>2x_0/u_->0\ .\end{cases}$$

Therefore, for any $t>0$

$$(X(t,.))^{-1}(x)=\begin{cases} x-tu_-/2 & \text{for } x\le tu_-/2<st\ ,\\ x-tu_+/2 & \text{for } x\ge st\ ,\\ (x-tu_-/2)(u_-/u_+) & \text{for } tu_-/2<x<st.\end{cases}$$

In particular, the measure solution satisfies for any $t>0$

$$\begin{aligned}\tilde u(t)([a,b[) &= u_0(\Big[(X(t,.))^{-1}(a),(X(t,.))^{-1}(b)\Big[)\\ &= \begin{cases}(b-a)u_- & \text{for } a<b<tu_-/2\ ,\\ (b-a)u_+ & \text{for } st\le a<b\ ,\\ u_+(b-a)u_-/u_+ & \text{for } tu_-/2\le a<b\le st,\end{cases}\end{aligned}$$

and the obvious combinations for an arbitrary interval. We deduce that

$$\tilde u(t,x)=u_-.H(st-x)+u_+.H(x-st)=u(t,x).$$

**Case 3:** This case is obvious. We leave it to the reader.

The second case is more surprising.

## 5.2 The case of a rarefaction.

We consider again the Riemann problem, but this time we assume

$$u_-<u_+.$$

Of course the classical solution $u$ to (5.1), (5.2) is now a centered rarefaction wave. Clearly, the usual characteristics, given by

$$\frac{dX}{dt}=u(t,x)$$

are not uniquely defined. However, the surprising thing is that the Filippov characteristics associated to (5.3) are well defined. Indeed, there are three cases.

**Case 1.** $u_-<0<u_+$.

The Filippov characteristics are given by

$$X(t,x_0)=\begin{cases} x_0+tu_-/2 & \text{for } X\le tu_-\ ,\\ x_0+tu_+/2 & \text{for } tu_+\le X\ ,\\ \sqrt{2x_0\ u_-\ .t} & \text{for } tu_-<X<0\ ,\\ \sqrt{2x_0\ u_+\ .t} & \text{for } 0<X<tu_+\ .\end{cases} \tag{5.6}$$

We deduce from the above relations that for $t > 0$

$$(X(t,.))^{-1}(x) = \begin{cases} x - tu_-/2 & \text{for } x \le tu_- \ , \\ x - tu_+/2 & \text{for } tu_+ \le x \ , \\ \frac{x^2}{2tu_-} & \text{for } tu_- < x < 0 \ , \\ \frac{x^2}{2tu_+} & \text{for } 0 < x < tu_+ \ . \end{cases} \tag{5.7}$$

In particular, we deduce that for any $t > 0$ and $tu_- < a < 0 < b < tu_+$

$$\begin{aligned} \tilde{u}(t)([a,b[) &= \int_{a^2/2tu_-}^{b^2/2tu_+} u_0 dy \\ &= -u_- .\, a^2/2tu_- + u_+ .\, b^2/2tu_+ = -a^2/2t + b^2/2t = \textstyle\int_a^b (x/t)dx. \end{aligned}$$

Therefore in this case we obtain (5.4).

**Case 2.** $0 < u_- < u_+$ (or $u_- < u_+ < 0$).
For $x_0 > 0$ the Filippov characteristics are now given by

$$X(t, x_0) = \begin{cases} x_0 + tu_-/2 & \text{for } X \le tu_-/2 \ , \\ x_0 + tu_+/2 & \text{for } tu_+ \le X \ , \\ \sqrt{2x_0 \ u_+ \ .t} & \text{for } tu_- < X < tu_+ \ , \\ (t - t_2)u_-/2 + \ u_- .2x_0 u_+/(u_-^2) & \text{for } tu_-/2 < X < tu_- \ . \end{cases}$$

Conversely,

$$(X(t,.))^{-1}(x) = \begin{cases} x - tu_-/2 & \text{for } x \le tu_-/2 \ , \\ x - tu_+/2 & \text{for } tu_+ \le x \ , \\ \frac{x^2}{2tu_+} & \text{for } 0 < x < tu_+ \ , \\ (x - tu_-/2)u_-/u_+ & \text{for } tu_-/2 < X < tu_- \ . \end{cases} \tag{5.8}$$

We easily check that the measure solution $\tilde{u}$ still satisfies (5.4). In particular if $tu_-/2 \le a < b \le tu_-$ then

$$\tilde{u}(t)([a,b[) = \int_{(a-tu_-/2)u_-/u_+}^{(b-tu_-/2)u_-/u_+} u_0 dy = (b-a)u_- .$$

**Case 3.** $u_- = 0 < u_+$ or $u_- < u_+ = 0$.
We leave it to the reader.

Therefore the proof of Theorem 5.1 is complete. In some sense the above description says that the Burgers equation is *really* a conservation law!

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