# On Riemannian manifolds satisfying the Transport Continuity Property

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(Joint work with A. Figalli and C. Villani)

# Statement of the problem

# Optimal transport on Riemannian manifolds

Let $(M, g)$ be a smooth compact connected Riemannian manifold of dimension $n \geq 2$.

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Denote by $d_g$ the geodesic distance on $M$ and define the quadratic cost $c : M \times M \longrightarrow [0, \infty)$ by

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Given two Borelian probability measures $\mu_0, \mu_1$ on $M$, find a mesurable map $T : M \rightarrow M$ satisfying

$$T_\sharp \mu_0 = \mu_1 \quad \text{(i.e. } \mu_1(B) = \mu_0\left(T^{-1}(B)\right), \forall B \text{ borelian } \subset M),$$

and minimizing

$$\int_M c(x, T(x)) d\mu_0(x).$$

# The Brenier-McCann Theorem

## Theorem (McCann '01)

*Let* $\mu_0, \mu_1$ *be two probability measures on* $M$. *If* $\mu_0$ *is absolutely continuous w.r.t. the Lebesgue measure, then there is a unique optimal transport map* $T : M \to M$ *satisfying* $T_\sharp \mu_0 = \mu_1$ *and minimizing*

$$\int_M c(x, T(x)) d\mu_0(x).$$

*It is characterized by the existence of a semiconvex function* $\psi : M \to \mathbb{R}$ *such that*

$$T(x) = \exp_x (\nabla \psi(x)) \qquad \text{for } \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

# The Transport Continuity Property

We say that $(M, g)$ satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:
For any pair of probability measures $\mu_0, \mu_1$ associated locally with **continuous positive densities** $\rho_0, \rho_1$, that is

$$\mu_0 = \rho_0 \mathcal{L}^n, \quad \mu_1 = \rho_1 \mathcal{L}^n,$$

the optimal transport map between $\mu_0$ and $\mu_1$ is **continuous**.

# References

- Cordero-Erausquin (1999)
- Ma, Trudinger, Wang (2005)
- Loeper
- Kim, McCann
- Delanoe-Ge
- Villani
- Figalli, Rifford
- Figalli, Rifford, Villani
- Figalli, Kim, McCann

# Necessary conditions for **TCP**

# Necessary conditions

**Theorem (Loeper '08, Villani '09, Figalli-R-Villani)**

*Assume that $(M, g)$ satisfies* **(TCP)** *then the following properties hold:*

- *all the injectivity domains are convex,*
- *the cost $c$ is regular.*

# Injectivity domains

Let $x \in M$ be fixed. We call exponential mapping from $x$, the mapping defined as

$$\begin{array}{rcl} \exp_x : T_xM & \longrightarrow & M \\ v & \longmapsto & \exp_x(v) := \gamma_v(1), \end{array}$$

where $\gamma_v : [0,1] \rightarrow M$ is the unique geodesic starting at $x$ with speed $\dot{\gamma}_v(0) = v$.

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$$\mathcal{I}(x) := \left\{ v \in T_xM \,\middle|\, \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minimizing} \\ \text{geodesic between } x \text{ and } \exp_x(tv) \end{array} \right\}.$$

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It is a star-shaped (w.r.t. $0 \in T_xM$) domain with Lipschitz boundary.

# Regular costs

The cost $c = d^2/2 : M \times M \longrightarrow \mathbb{R}$ is called **regular**, if for every $x \in M$ and every $v_0, v_1 \in \mathcal{I}(x)$, there holds

$$v_t := (1-t)v_0 + tv_1 \in \mathcal{I}(x) \qquad \forall t \in [0,1],$$

and

$$c(x, y_t) - c(x', y_t) \leq \max\Big(c(x, y_0) - c(x', y_0),\, c(x, y_1) - c(x', y_1)\Big),$$

for any $x' \in M$, where $y_t := \exp_x v_t$.

# An obvious remark

**Remark**

*Assume that all the injectivity domains of $(M, g)$ are convex. Then the cost $c$ is regular if and only if for every $x, x' \in M$, the mapping*

$$F_{x,x'} \, : \, v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

*is quasiconvex (its sublevels sets are always convex).*

# An easy lemma

**Lemma**

*Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$, the following property holds*

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F\, w, w \rangle > 0.$$

*Then $F$ is quasiconvex.*

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Let $v_0, v_1 \in U$ be fixed.

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$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

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$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) > \max\{h(0), h(1)\}.$$

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{and} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

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Since $\tau$ is a local maximum, we get a contradiction. □

# Exercice 1

The following lemma is false !!

## FALSE Lemma

*Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n$, the following property holds*

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle \geq 0.$$

*Then $F$ is quasiconvex.*

# Exercice 2

However, the following result holds true.

**Lemma**

*Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that there is a constant $C > 0$ such that*

$$\langle \nabla_v^2 F\, w, w \rangle \geq -C \left| \langle \nabla_v F, w \rangle \right| |w| \qquad \forall v \in U, \forall w \in \mathbb{R}^n.$$

*Then $F$ is quasiconvex.*

# The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\bigg|_{s=0}\,\frac{d^2}{dt^2}\bigg|_{t=0} c\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

# The Ma-Trudinger-Wang tensor

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for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

**Proposition (Villani '09, Figalli-R-Villani)**

*Assume that all the injectivity domains are convex. Then, the two following properties are equivalent:*

- *the cost $c$ is regular,*
- *the* **MTW** *tensor $\mathfrak{S}$ is $\succeq 0$, that is, for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

# Necessary conditions

**Theorem (Villani '09, Figalli-R-Villani)**

*Assume that* $(M, g)$ *satisfies* **(TCP)** *then the following properties hold:*

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- *the cost* $c$ *is regular,*
- *the* **MTW** *tensor* $\mathfrak{S}$ *is* $\succeq 0$.

Loeper noticed that for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_xM$, there holds

$$\mathfrak{S}_{(x,0)}(\xi,\eta) = \sigma_x(P),$$

where $P$ is the plane generated by $\xi$ and $\eta$. Consequently, any $(M,g)$ satisfying **TCP** must have nonnegative sectional curvatures.

# Sufficient conditions for **TCP**

# Sufficient conditions

**Theorem (Figalli-R-Villani)**

*Assume that $(M, g)$ satisfies the following properties:*

- *all the injectivity domains are strictly convex,*
- *the* **MTW** *tensor $\mathfrak{S}$ is $\succ 0$, that is, for every $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM \setminus \{0\}$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) > 0.$$

*Then $(M, g)$ satisfies* **TCP***.*

# Examples

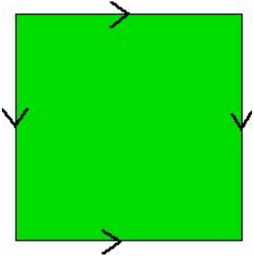
# The flat torus

The **MTW** tensor of the flat torus $\left(\mathbb{T}^n, g^0\right)$ satisfies

$$\mathfrak{S}_{(x,v)} \equiv 0 \qquad \forall x \in \mathbb{T}^n, \forall v \in \mathcal{I}(x)$$

**Theorem (Cordero-Erausquin '99)**

*The flat torus* $\left(\mathbb{T}^n, g^0\right)$ *satisfies* **TCP**.



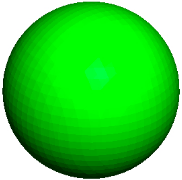
# Round spheres

Loeper checked that the **MTW** tensor of the round sphere $(\mathbb{S}^n, g^0)$ satisfies for any $x \in \mathbb{S}^n$, $v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x\mathbb{S}^n$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq \|\xi\|_x^2 \|\eta\|_x^2.$$

**Theorem (Loeper '06)**

*The round sphere* $(\mathbb{S}^n, g^0)$ *satisfies* **TCP**.



# Riemannian quotients

Let $G$ be a discrete group of isometries of $(M,g)$ acting freely and properly. Then there exists on the quotient manifold $N = M/G$ a unique Riemannian metric $h$ such that the canonical projection $p : M \longrightarrow N$ is a Riemannian covering map.

**Theorem (Delanoe-Ge '08)**

*If* $(M,g)$ *satisfies* **TCP**, *then* $\left(N = M/G, h\right)$ *satisfies* **TCP**.

Examples: $\left(\mathbb{RP}^n, g^0\right)$, the flat Klein bottle.

# Riemannian submersions

We call **Riemannian submersion** from $(M, g)$ to $(N, h)$ any smooth submersion $p : M \to N$ such that for every $x \in M$, the differential mapping $d_x p$ is an isometry between $H_x$ and $T_{p(x)}N$, where $H_x \subset T_x M$ is the **horizontal subspace** defined as

$$H_x := \left\{ \left(d_x p\right)^{-1}(0) \right\}^{\perp}.$$

**Theorem (Kim-McCann '08)**

*If $(M, g)$ satisfies $\mathfrak{S} \succ 0$ (resp. $\succeq 0$), then $(N, h)$ satisfies $\mathfrak{S} \succ 0$ (resp. $\succeq 0$).*

Examples: complex projective spaces $\left(\mathbb{CP}^k, g^0\right)$ (dim $= 2k$), quaternionic projective spaces $\left(\mathbb{HP}^k, g^0\right)$ (dim $= 4k$).

# Small deformations of $\left(\mathbb{S}^2, g^0\right)$

On $\left(\mathbb{S}^2, g^0\right)$, the **MTW** tensor is given by

$$\begin{aligned}\mathfrak{S}_{(x,v)}\left(\xi,\xi^\perp\right) &= 3\left[\frac{1}{r^2}-\frac{\cos(r)}{r\sin(r)}\right]\xi_1^4+3\left[\frac{1}{\sin^2(r)}-\frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4\\ &\qquad +\frac{3}{2}\left[-\frac{6}{r^2}+\frac{\cos(r)}{r\sin(r)}+\frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,\end{aligned}$$

with $x\in\mathbb{S}^2, v\in\mathcal{I}(x), r:=\|v\|_x, \xi=(\xi_1,\xi_2), \xi^\perp=(-\xi_2,\xi_1)$.

# Small deformations of $(\mathbb{S}^2, g^0)$

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&\qquad\qquad + \frac{3}{2}\left[-\frac{6}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,
\end{aligned}
$$

with $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := \|v\|_x, \xi = (\xi_1, \xi_2), \xi^\perp = (-\xi_2, \xi_1)$.

**Theorem (Figalli-R '09)**

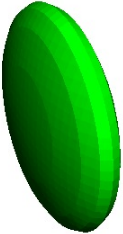
*Any small deformation of the round sphere $(\mathbb{S}^2, g^0)$ in $C^4$ topology satisfies* **TCP**.

# Ellipsoids

The oblate ellipsoid of revolution $(E_\epsilon)$ given by the equation

$$\frac{x^2}{\epsilon^2} + y^2 + z^2 = 1, \qquad \text{with } \epsilon = 0.29,$$

does not satisfies **MTW** $\geq 0$.



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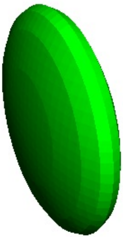
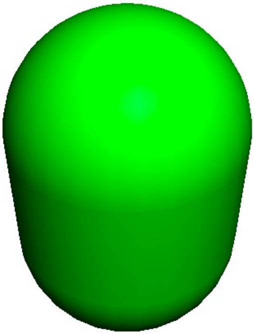


In consequence, $(E_\epsilon)$ does not satisfy **TCP**.

# Jump of curvature

The surface made with two half-balls joined by a cylinder has not a regular cost.



Then, it does not satisfy **TCP**.

# Small deformations of $(\mathbb{S}^n, g^0)$

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# Small deformations of $\left(\mathbb{S}^n, g^0\right)$

**Theorem (Figalli-R-Villani'09)**

*Any small deformation of the round sphere $\left(\mathbb{S}^n, g^0\right)$ in $C^4$ topology satisfies* **TCP**.

As a by-product, we obtain that the injectivity domains on small $C^4$ deformations of $\left(\mathbb{S}^n, g^0\right)$ are convex.

# Perspectives

# Mind the gap !

**Theorem (Necessary conditions)**

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There is a gap !

# More examples ?

List of $(M, g)$ satisfying $\mathfrak{S} \succeq 0$ or $\mathfrak{S} \succ 0$:

- Flat tori ( $\mathfrak{S} \succeq 0$).
- Round spheres ($\mathfrak{S} \succ 0$).
- Complex projective spaces ($\mathfrak{S} \succ 0$).
- Quaternionic projective spaces ($\mathfrak{S} \succ 0$).
- $C^4$ small deformations of round spheres ($\mathfrak{S} \succ 0$).
- Riemannian products some of the above spaces ($\mathfrak{S} \succeq 0$).
- (Riemannian quotients of the above spaces.)
- ?

# $\succeq$ vs. $\geq$

Let us say that $(M, g)$ satisfies $\mathfrak{S} \geq 0$ if

$$\mathfrak{S}_{(x,v)}(\xi, \eta) \geq 0 \qquad \forall x \in M, \forall v \in \mathcal{I}(x), \forall \xi, \eta \in T_x M.$$

# $\succeq$ vs. $\geq$

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All the following examples

- flat tori,
- round spheres,
- complex projective spaces,
- quaternionic projective spaces,
- $C^4$ small deformations of $\left(\mathbb{S}^2, g^0\right)$,
- Riemannian products some of the above spaces,

satisfy $\mathfrak{S} \geq 0$ !!!

Thank you for your attention !