Stabilization of finite-dimensional control systems: a survey

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Purpose of the talk

Given a control system

$$\dot{x} = f(x, u)$$

with an equilibrium point

$$f(O, 0) = 0$$

which is globally asymptotically controllable, study the existence and regularity of stabilizing feedbacks.

Globally asymptotically controllable systems

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- **Attractivity**: For each $x \in M$, there is a control $u(\cdot) : [0, \infty) \longrightarrow U$ such that the corresponding trajectory $x_{x,u}(\cdot) : [0, \infty) \longrightarrow M$ tends to O as $t \to \infty$.

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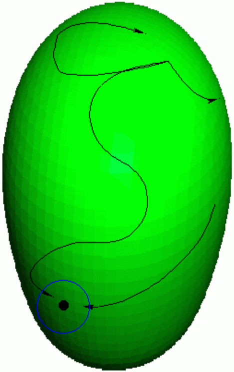
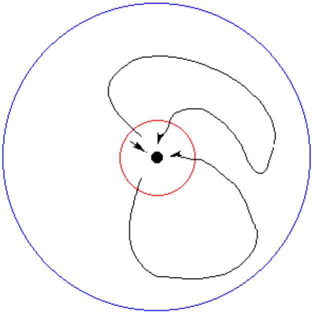
- **Lyapunov Stability**: For each neighborhood $\mathcal{V}$ of O, there exists some neighborhood $\mathcal{U}$ of $\mathcal{O}$ such that if $x \in \mathcal{U}$ then the above control can be chosen such that $x_{x,u}(t) \in \mathcal{V}, \forall t \geq 0$.

Attractivity



Lyapunov Stability



Examples

- If a linear control system

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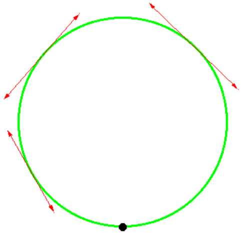

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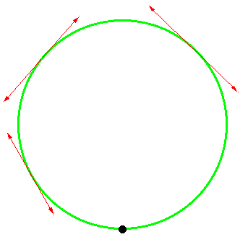
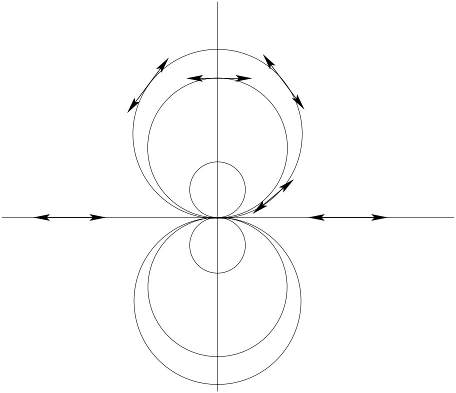


$$\dot{x} = f(x,u) \\ \Updownarrow \\ \dot{x} \in F(x)$$

Examples

- Artstein's circles



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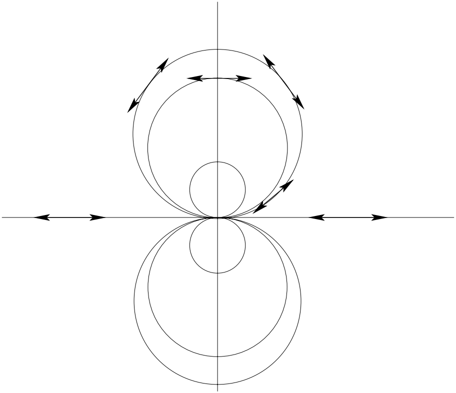

$$\left\{ \begin{array}{rcl} \dot{x} & = & u(x^2 - y^2) \\ \dot{y} & = & u(2xy) \end{array} \right. \qquad u \in [-1, 1]$$

Examples

- The nonholonomic integrator (shopping cart)

$$\begin{cases} \dot{x} &= u_1 \\ \dot{y} &= u_2 \\ \dot{z} &= u_2 x - u_1 y \end{cases} \qquad u = (u_1, u_2) \in \mathbb{R}^2$$

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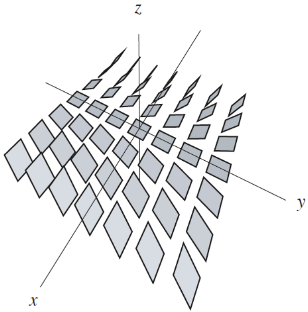


Examples

- If M is a connected manifold and the control system has the form

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x)$$

where $f_1, \ldots, f_m$ is a family of smooth vector fields satisfying the Hörmander bracket generating condition

$$\text{Lie}\left\{f_1, \ldots, f_m\right\}(x) = T_x M \qquad \forall x \in M,$$

then the Chow-Rashevski Theorem implies that the system is GAC at any $x \in M$.

The stabilization problem

Given a GAC control system

$$\dot{x} = f(x,u), \qquad x \in M, \quad u \in U,$$

can one find a feedback

$$k : M \longmapsto U$$

which makes the closed-loops system

$$\dot{x} = f(x,k(x))$$

Globally Asymptotically Stable ?

Back to examples

Proposition

If a linear control system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,$$

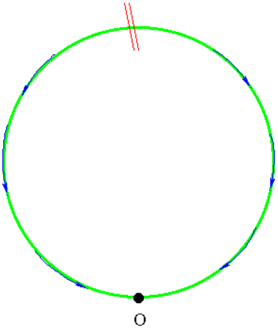
is GAC at the origin, then there is $K \in M_{m,n}(\mathbb{R})$ such that the closed-loop system

$$\dot{x} = (A + BK)x$$

is GAS at the origin.

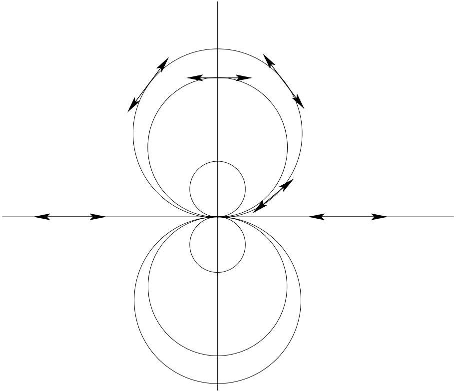
Back to examples

On the circle $\mathbb{S}^1$



Back to examples

Artstein's circles



A global obstruction

Proposition

If X is a continuous vector field on M which is GAS at some $O \in M$, then M is diffeomorphic to an Euclidean space $\mathbb{R}^n$.

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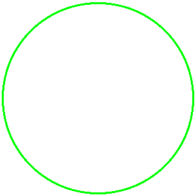

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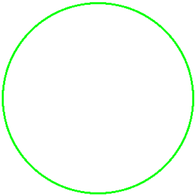

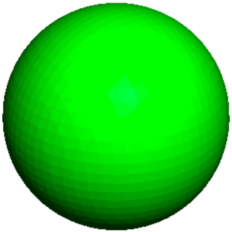

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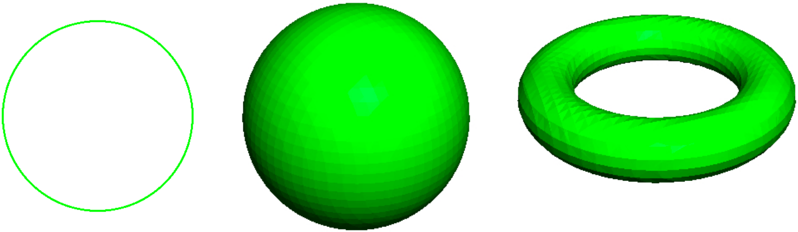

A local obstruction : The Brockett condition

Theorem

Let X be a continuous vector field in a neighborhood of the origin in $\mathbb{R}^n$. If X is GAS at 0, then for $\epsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\delta B \subset X(\epsilon B).$$

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Corollary

Let $\dot{x} = f(x,u)$ be a control system in a neighborhood of the origin with $f(0,0) = 0$. If it admits a feedback $k : \mathbb{R}^n \to U$ which is continuous and such that $X = f(x, k(x))$ is GAS at 0, then for $\epsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\delta B \subset f(\epsilon B, U).$$

Back to examples

The nonholonomic integrator

$$\begin{cases} \dot{x} &= u_1 \\ \dot{y} &= u_2 \\ \dot{z} &= u_2 x - u_1 y \end{cases} \qquad u = (u_1, u_2) \in \mathbb{R}^2$$

Vertical vectors of the form

$$\begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix} \quad \text{with} \quad \delta \neq 0$$

do not belong to $f(\mathbb{R}^3, \mathbb{R}^2)$!!

Smooth CLF

Let $\dot{x} = f(x, u)$ be a control system with $x \in \mathbb{R}^n$ and $u \in U$.

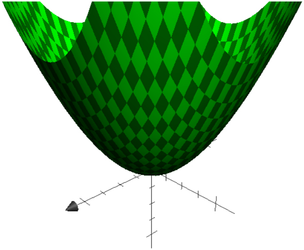
Definition

A function $V : \mathbb{R}^n \to \mathbb{R}$ is called a smooth control-Lyapunov function (CLF) for $\dot{x} = f(x, u)$ at the origin if it satisfies the following properties:

- V is smooth
- V is positive definite;
- V is proper;
- for every $x \in \mathbb{R}^n \setminus \{0\}$,

$$\inf_{u \in U} \Big\{ \langle \nabla V(x), f(x, u) \rangle \Big\} < 0.$$

Smooth CLF (Picture)



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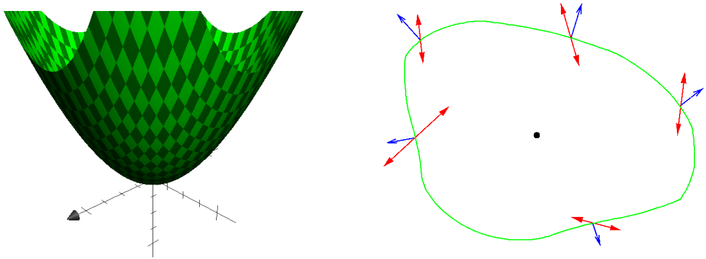


The Artstein theorem

Theorem

If the control system

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x) \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,$$

admits a smooth CLF at the origin, then it admits a smooth stabilizing feedback, that is a smooth mapping $k : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *such that the closed loop system*

$$\dot{x} = f(x, k(x))$$

is GAS at the origin.

Semiconcave CLF

Definition

A continuous function $V : M \longrightarrow \mathbb{R}$ is called a semiconcave control-Lyapunov function (CLF) for $\dot{x} = f(x, u)$ at the origin if it satisfies the following properties:

- V is locally semiconcave on $M \setminus \{O\}$;
- V is positive definite;
- V is proper;
- V is a viscosity supersolution of the Hamilton-Jacobi equation

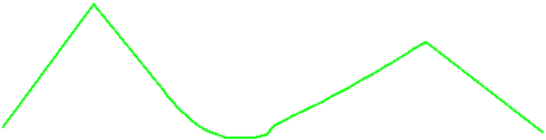
$$\sup_{u \in U} \Big\{ -\langle \nabla V(x), f(x, u) \rangle \Big\} - V(x) \geq 0.$$

Locally semiconcave functions

A function $f : \Omega \rightarrow \mathbb{R}$ is semiconcave in a neighborhood of $x \in \Omega$ if it can be written locally as

$$f = g + h,$$

with g concave and h smooth.



The weak decreasing condition

V is a viscosity supersolution of the Hamilton-Jacobi equation

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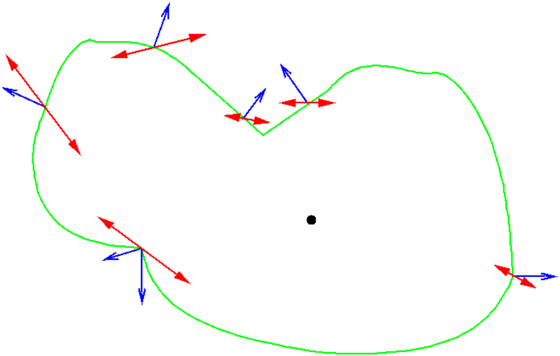

Picture of a semiconcave CLF

An existence theorem

Proposition

Let $\dot{x} = f(x,u)$ be a control system with $x \in M$ and $u \in U$. Assume that it admits a semiconcave CLF at $O \in M$. Then it is GAC at O.

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Theorem

If the control system $\dot{x} = f(x, u)$ is GAC at O, then it admits a semiconcave CLF at O.

Discontinuous stabilizing feedbacks

Let

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x) \qquad x \in M, \quad u \in \mathbb{R}^m,$$

be a control system and $O \in M$ be fixed.

Theorem

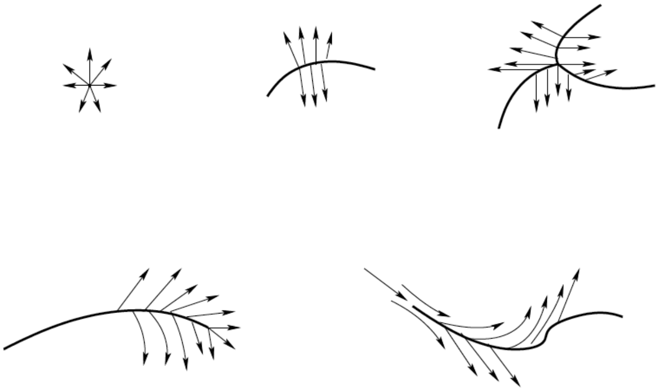
If $\dot{x} = f(x,u)$ is GAC at O, then there exists an open dense set of full measure in $M \setminus \{O\}$ and a feedback $k : M \longrightarrow \mathbb{R}^m$ such that

- *k is smooth on $\mathcal{D}$;*
- *the closed-loop system $\dot{x} = f(x, k(x))$ is GAS at O in the sense of Carathéodory, that is for the solutions of*

$$\dot{x}(t) = f(x(t), k(x(t))) \qquad a.e. \;\; t \geq 0.$$

Stabilizing the skieur

Singularities on surfaces



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- **Attractivity**: For almost every $x \in M$, the solution of $\dot{x} = X(x)$ starting at x converges to O;
- **Lyapunov Stability**: For each neighborhood $\mathcal{V}$ of O, there exists some neighborhood $\mathcal{U}$ of $\mathcal{O}$ such that if $x \in \mathcal{U}$ then the above trajectory remains in $\mathcal{V}$, $\forall t \geq 0$.

Theorem

If $\dot{x} = \sum_{i=1}^m u_i f_i(x)$ is GAC at O, then there is a smooth feedback $k : M \to \mathbb{R}^m$ such that $\dot{x} = f(x, k(x))$ is AGAS at O.

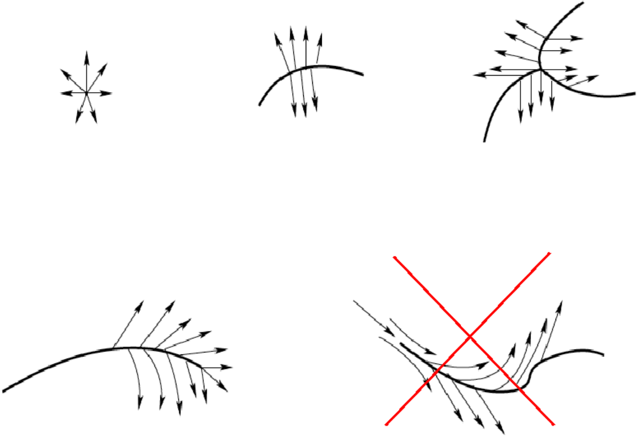
Almost stabilization of skieurs

SRS feedbacks

The feedback $k : M \longrightarrow \mathbb{R}^n$ is said to be a *smooth repulsive stabilizing* feedback at $O \in M$ (SRS) if the following properties are satisfied:

- there is a set $\mathcal{S} \subset M \setminus \{O\}$ which is closed in $M \setminus \{O\}$ and of full measure;
- k is smooth outside O;
- the closed-loop system is GAS at O in the sense of Carathéodory;
- for all $t > 0$, the trajectories of the closed-loop system do not belong to $\mathcal{S}$.

Repulsive singularities on surfaces



SRS feedbacks on surfaces

Let M be a smooth surface and

$$\dot{x} = u_1 X(x) + u_2 Y(x)$$

be a control system with X, Y two smooth vector fields on M and $O \in M$ be fixed.

Theorem

Assume that

$$Lie\{X, Y\}(x) = T_x M \qquad \forall x \in M.$$

Then it admits a SRS feedback on M at O. Moreover the feedback can be taken to be continuous around the origin.

Thank you for your attention !!