Geometric control and sub-Riemannian geodesics

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Outline

- Lecture 1: The Chow-Rashevsky Theorem
- Lecture 2: Sub-Riemannian geodesics
- Lecture 3: A closer look at singular curves
- Lecture 4: Open questions

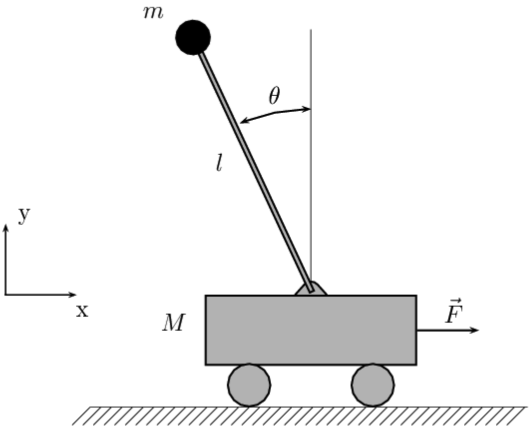
References

- A. Bellaïche. "The tangent space in sub-Riemannian geometry".
- R. Montgomery. "A tour of subriemannian geometries, their geodesics and applications".
- A. Agrachev, D. Barilari, U. Boscain. "Introduction to Riemannian and sub-Riemannian geometry".
- F. Jean. "Control of Nonholonomic Systems: From Sub-Riemannian Geometry to Motion Planning".
- L. Rifford. "Sub-Riemannian Geometry and Optimal Transport".

Lecture 1

The Chow-Rashevsky Theorem

Control of an inverted pendulum



Control systems

A general control system has the form

$$\dot{x} = f(x, u)$$

where

- x is the state in M
- u is the control in U

Proposition

Under classical assumptions on the datas, for every $x \in M$ and every measurable control $u : [0, T] \to U$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & a.e.\ t \in [0, T], \\ x(0) = x \end{cases}$$

admits a unique solution

$$x(\cdot) = x(\cdot; x, u) : [0, T] \longrightarrow M.$$

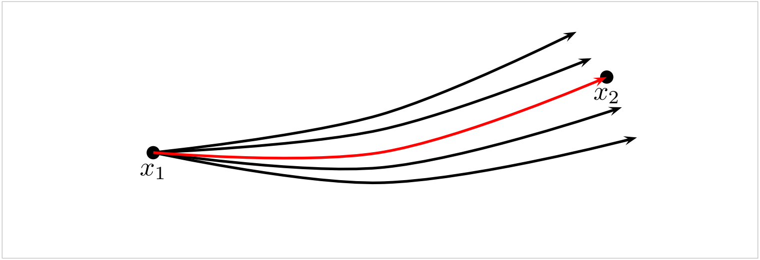
Controllability issues

Given two points x_1, x_2 in the state space M and $T > 0$, can we find a control u such that the solution of

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x_1 \end{cases}$$

satisfies

$$x(T) = x_2 \quad ?$$



The Chow-Rashevsky Theorem

Theorem (Chow 1939, Rashevsky 1938)

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

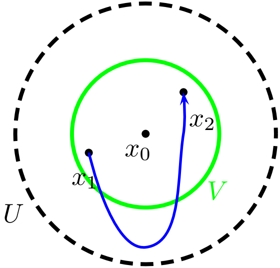
Then the control system

$$\dot{x} = \sum_{i=1}^{m} u_i \, X^i(x)$$

is locally controllable in any time at every point of M.

Comment I

The local controllability in any time at every point means that for every $x_0 \in M$, every $T > 0$ and every neighborhood U of x_0, there is a neighborhood $V \subset U$ of x_0 such that for any $x_1, x_2 \in V$, there is a control $u \in L^1([0,T];\mathbb{R}^m)$ such that the trajectory $x(\cdot; x_1, u) : [0,T] \to M$ remains in U and steers x_1 to x_2, *i.e.* $x(T; x_1, u) = x_2$.



Local controllability in time $T > 0$

$\Rightarrow$ Local controllability in time $T' > 0, \quad \forall T' > 0$

Comment II

If M is **connected** then

Local controllability $\Rightarrow$ Global controllability

Let $x \in M$ be fixed. Denote by $\mathcal{A}(x)$ the accessible set from x, that is

$$\begin{aligned} \mathcal{A}(x) &:= \left\{x\left(T;x,u\right) \mid T \geq 0, u \in L^1\right\} \\ &= \left\{x\left(1;x,u\right) \mid u \in L^1\right\}. \end{aligned}$$

- By local controllability, $\mathcal{A}(x)$ is open.
- Let y be in the closure of $\mathcal{A}(x)$. The set $\mathcal{A}(y)$ contains a small ball centered at y and there are points of $\mathcal{A}(x)$ in that ball. Then $\mathcal{A}(x)$ is closed.

By connectedness of M, we infer that $\mathcal{A}(x) = M$ for every $x \in M$, and in turn that the control system is globally controllable in any time.

The Chow-Rashevsky Theorem

Theorem (Chow 1939, Rashevsky 1938)

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then the control system $\dot{x} = \sum_{i=1}^m u_i X^i(x)$ is locally controllable in any time at every point of M.

The condition in red is called Hörmander's condition or bracket generating condition. Families of vector fields satisfying that condition are called nonholonomic, completely nonholonomic, or totally nonholonomic.

Comment III

Definition

Given two smooth vector fields X, Y on $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

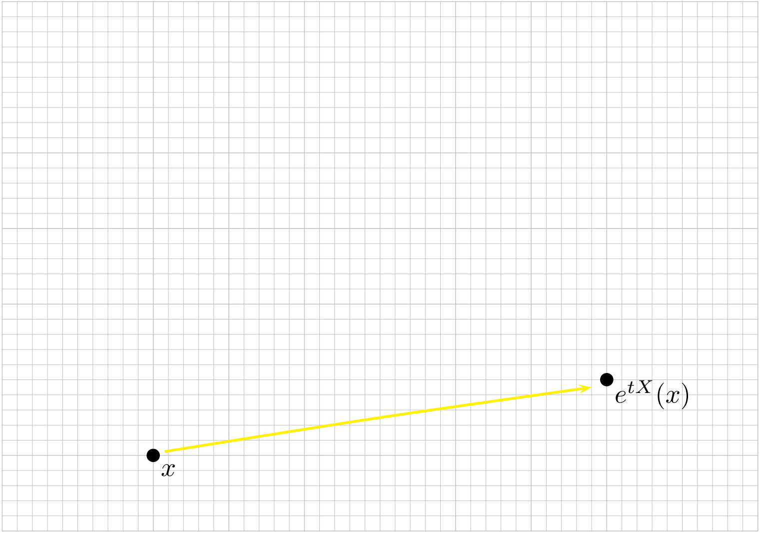
$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

The Lie brackets of two smooth vector fields on M can be defined in charts with the above formula.

Given a family $\mathcal{F}$ of smooth vector fields on M, we denote by $\mathrm{Lie}\{\mathcal{F}\}$ the Lie algebra generated by $\mathcal{F}$. It is the smallest vector subspace S of smooth vector fields containing $\mathcal{F}$ that also satisfies

$$[X, Y] \in S \qquad \forall X \in \mathcal{F}, \forall Y \in S.$$

Comment III



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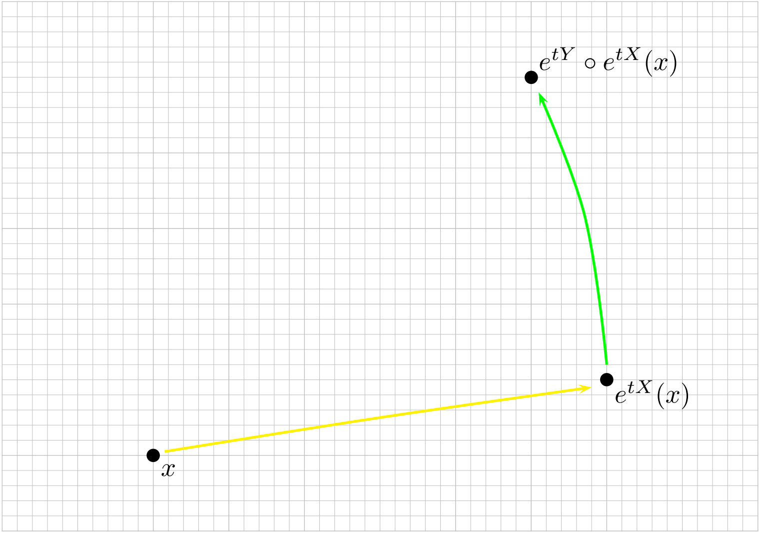

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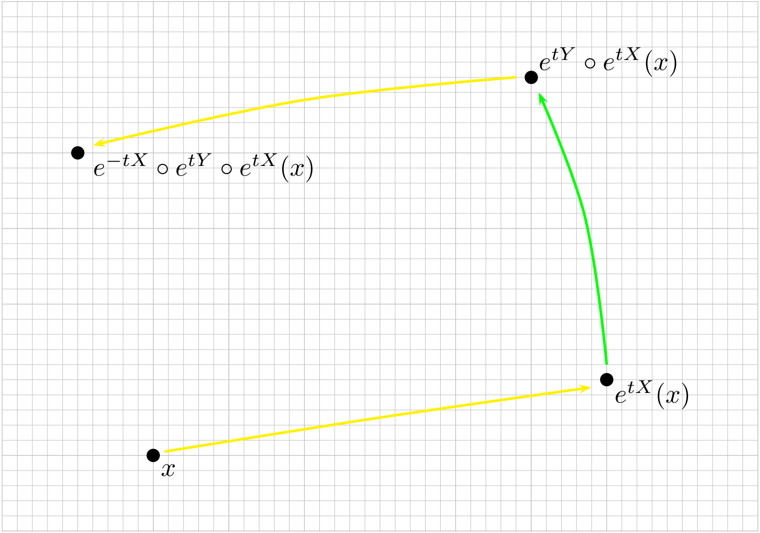

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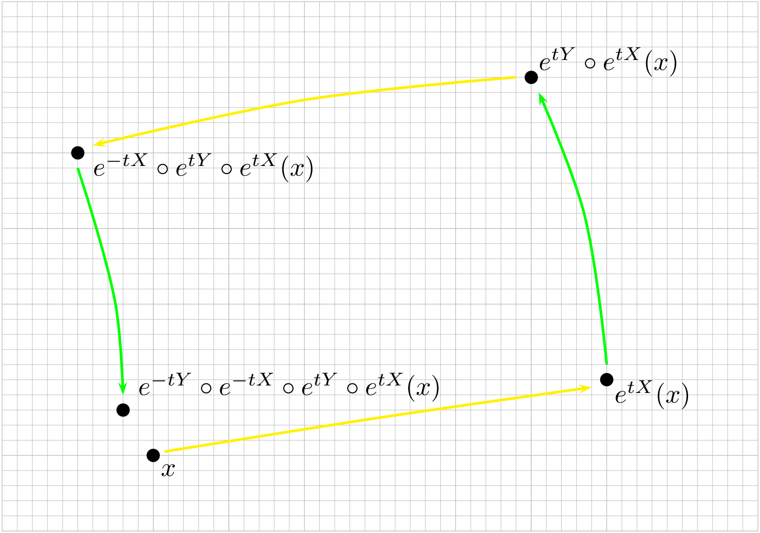

$e^{tY} \circ e^{tX}(x)$
$e^{-tX} \circ e^{tY} \circ e^{tX}(x)$
$e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(x)$
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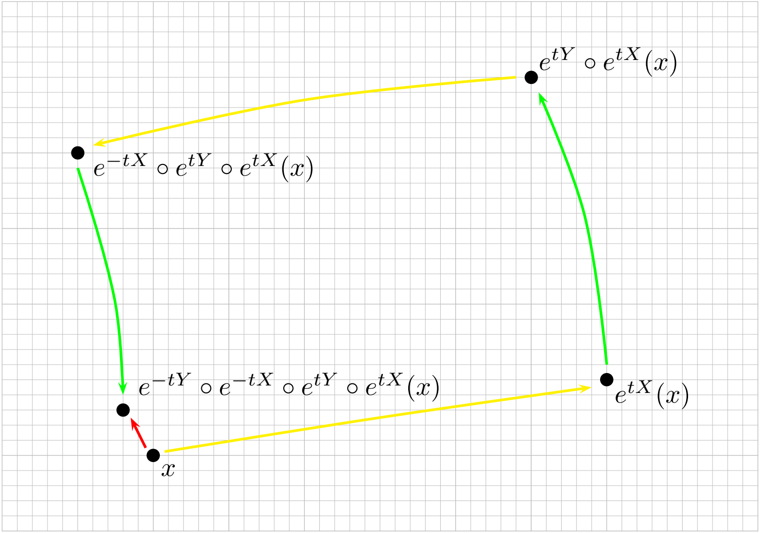
Comment III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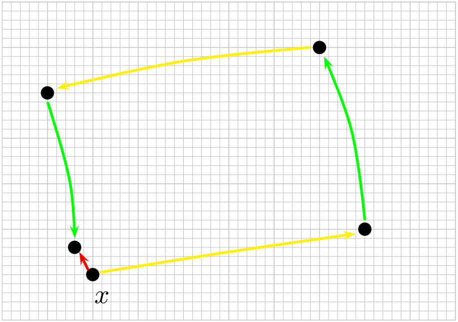
$e^{tX}(x)$
x

Comment III

Exercise

We have

$$[X, Y](x) = \lim_{t \downarrow 0} \frac{\left(e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}\right)(x) - x}{t^2}.$$



Comment III

Given a family $\mathcal{F}$ of smooth vector fields on M, we set $\mathrm{Lie}^1(\mathcal{F}) := \mathrm{Span}(\mathcal{F})$, and define recursively $\mathrm{Lie}^k(\mathcal{F})$ $(k = 2, 3, \ldots)$ by

$$\mathrm{Lie}^{k+1}(\mathcal{F}) := \mathrm{Span}\Big(\mathrm{Lie}^k(\mathcal{F}) \cup \Big\{[X, Y] \,|\, X \in \mathcal{F}, Y \in \mathrm{Lie}^k(\mathcal{F})\Big\}\Big).$$

We have

$$\mathrm{Lie}\{\mathcal{F}\} = \bigcup_{k \geq 1} \mathrm{Lie}^k(\mathcal{F}).$$

For example, the Lie algebra $\mathrm{Lie}\{X^1, \ldots, X^m\}$ is the vector subspace of smooth vector fields which is spanned by all the brackets (made from $X^1, \ldots, X^m$) of length $1, 2, 3, \ldots$. Since M has finite dimension, for every $x \in M$, there is $r = r(x) \geq 1$ (called degree of nonholonomy at x) such that

$$T_xM \supset \mathrm{Lie}\{X^1, \ldots, X^m\}(x) = \mathrm{Lie}^r\{X^1, \ldots, X^m\}(x).$$

Comment IV

We can prove the Chow-Rashevsky Theorem in the contact case in $\mathbb{R}^3$ as follows:

Exercise

Let X^1, X^2 be two smooth vector fields in $\mathbb{R}^3$ such that

$$Span\left\{X^1(0), X^2(0), [X^1, X^2](0)\right\} = \mathbb{R}^3.$$

Then the mapping $\varphi_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\varphi_\lambda(t_1, t_2, t_3) = e^{\lambda X^1} \circ e^{t_3 X^2} \circ e^{-\lambda X^1} \circ e^{t_2 X^2} \circ e^{t_1 X^1}(0)$$

is a local diffeomorphism at the origin for $\lambda > 0$ small.

$\rightsquigarrow$ F. Jean's monograph "Control of Nonholonomic Systems: from Sub-Riemannian Geometry to Motion Planning"

The End-Point mapping

Given a control system of the form

$$\dot{x} = \sum_{i=1}^{m} u_i X^i(x) \qquad (x \in M, u \in \mathbb{R}^m),$$

we define the **End-Point mapping** from x in time $T > 0$ as

$$\begin{array}{rccc} E^{x,T} : & L^2\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ & u & \longmapsto & x\big(T;x,u\big) \end{array}$$

Proposition

The mapping $E^{x,T}$ is of class C^1 (on its domain) and its differential is given by

$$D_u E^{x,T}(v) =$$

$$D_x\Phi^u(T,x) \cdot \int_0^T \left(D_x\Phi^u(t,x)\right)^{-1} \cdot \sum_{i=1}^{m} v_i(t) X^i\left(E^{x,t}(u)\right) dt.$$

Linearized control system

Writing the trajectory $x_{u+\epsilon v} : [0,T] \to \mathbb{R}^n$ as

$$x_{u+\epsilon v} = x_u + \epsilon \delta_x + o(\epsilon),$$

we have for every $t \in [0,T]$,

$$\begin{aligned}
x_{u+\epsilon v}(t) &= x + \int_0^t \sum_{i=1}^m \big(u_i(s) + \epsilon v_i(s)\big) X^i\big(x_{u+\epsilon v}(s)\big)\, ds \\
&= x + \int_0^t \sum_{i=1}^m u_i(s) X^i\big(x_u(s)\big)\, ds \\
&\quad + \epsilon \int_0^t \left(\sum_{i=1}^m u_i(s) D_{x_u(s)} X^i \right) \cdot \delta_x(s) + \sum_{i=1}^m v_i(s) X^i(x_u(s))\, ds \\
&\quad + o(\epsilon).
\end{aligned}$$

Linearized control system

So we have $D_u E^{x,T}(v) = \xi(T)$, where $\xi : [0,T] \to \mathbb{R}^n$ is a trajectory of the linearized system

$$\dot{\xi} = \left(\sum_{i=1}^m u_i D_{x_u} X^i \right) \cdot \xi + \sum_{i=1}^m v_i X^i(x_u), \quad \xi(0) = 0.$$

Consequently, setting for every $t \in [0,T]$,

$$A_u(t) := \sum_{i=1}^m u_i(t) D_{x_u(t)} X^i,$$

$$B_u(t) := \left(X^1(x_u(t)), \ldots, X^m(x_u(t)) \right).$$

we have

$$D_u E^{x,T}(v) = S_u(T) \int_0^T S_u(t)^{-1} B_u(t) v(t)\, dt$$

with S_u solution of $\dot{S}_u = A_u S_u$ a.e. $t \in [0,T]$, $S_u(0) = I_n$.

About the End-Point mapping

Proposition

For every $u \in L^2([0,T];\mathbb{R}^m)$ *and any* $i = 1, \ldots, m$, *we have*

$$X^i\left(E^{x,T}(u)\right) \in D_u E^{x,T}\left(L^2([0,T];\mathbb{R}^m)\right).$$

Proof.

Any linear form $\bar{p} \perp \mathrm{Im}\left(D_u E^{x,T}\right)$ satisfies

$$\bar{p} \cdot S_u(T) \int_0^T S_u(t)^{-1} B_u(t) v(t)\, dt = 0,$$

for any $v \in L^2\left([0,T];\mathbb{R}^m\right)$. Taking
$v(t) := \left(\bar{p} \cdot S_u(T) S_u(t)^{-1} B_u(t)\right)^*$ we get $\bar{p} \perp X^i(y)$ for all i. □

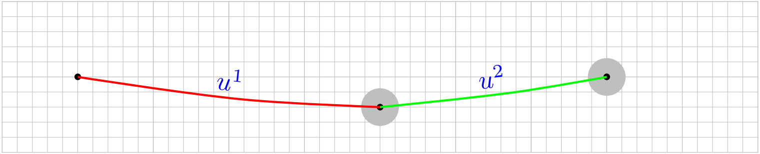
Regular controls vs. Singular controls

Definition

A control $u \in L^2([0,T];\mathbb{R}^m)$ is called **regular** with respect to $E^{x,T}$ if $E^{x,T}$ is a submersion at u. If not, u is called **singular**.

Exercise

The concatenations $u^1 * u^2$ and $u^2 * u^1$ of a regular control u^1 with another control u^2 are regular.



Rank of a control

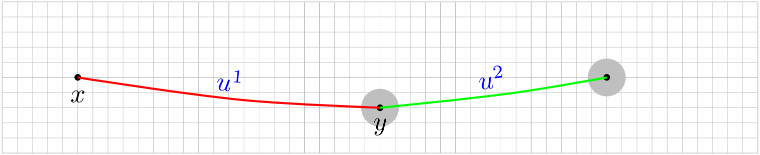
Definition

The rank of a control $u \in L^2([0,T];\mathbb{R}^m)$ (with respect to $E^{x,T}$) is defined as the dimension of the image of the linear mapping $D_u E^{x,T}$. We denote it by $\mathrm{rank}^{x,T}(u)$.

Exercise

The following properties hold:

- $rank^{x,T_1+T_2}(u^1 * u^2) \geq \max\{rank^{x,T_1}(u^1), rank^{y,T_2}(u^2)\}$.
- $rank^{y,T_1}(\check{u}^1) = rank^{x,T_1}(u^1)$.



Openness: Statement

The Chow-Rashevsky will follow from the following result:

Proposition

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then, for every $x \in M$ and every $T > 0$, the End-Point mapping

$$\begin{array}{rcl} E^{x,T} \ : \ L^2\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ u & \longmapsto & x\big(T; x, u\big) \end{array}$$

is open (on its domain).

Openness: Sketch of proof

Let $x \in M$ and $T > 0$ be fixed. Set for every $\epsilon > 0$,

$$d(\epsilon) = \max\left\{ \mathrm{rank}^{x,\epsilon}(u) \mid \|u\|_{L^2} < \epsilon \right\}.$$

Claim: $d(\epsilon) = n \qquad \forall \epsilon > 0.$

If not, we have $d(\epsilon) = d_0 \in \{1, \ldots, n-1\}$ for some $\epsilon > 0$. Given u^ϵ s.t. $\mathrm{rank}^{x,\epsilon}(u^\epsilon) = d_0$, there are d_0 controls $v^1, \ldots, v^{d_0}$ such that the mapping

$$\mathcal{E} \,:\, \lambda = (\lambda^1, \ldots, \lambda^{d_0}) \in \mathbb{R}^{d_0} \mapsto E^{x,\epsilon}\left(u^\epsilon + \sum_{j=1}^{d_0} \lambda^j v^j \right)$$

is an immersion near 0. Thus, its local image N is a d_0 dimensional submanifold of M of class C^1 such that

$$X^i\big(\mathcal{E}(\lambda)\big) \in \mathrm{Im}\big(D_\lambda \mathcal{E}\big) = T_y N. \qquad \textit{Contradiction}!!!$$

Openness: Sketch of proof (the return method)

To conclude, we pick (for any $\epsilon > 0$ small) a regular control u^ϵ in $L^2([0,\epsilon];\mathbb{R}^m)$ and define $\tilde{u} \in L^2([0, T + 2\epsilon];\mathbb{R}^m)$ by

$$\tilde{u} := u^\epsilon * \check{u}^\epsilon * u.$$

Up to reparametrizing u into a control v on $[0, T - 2\epsilon]$, the new control $\tilde{v} = u^\epsilon * \check{u}^\epsilon * v$ is regular, close to u in L^2 provided $\epsilon > 0$ is small, and steers x to $E^{x,T}(u)$.

The openness follows from the Inverse Function Theorem.

Remarks

Proposition

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then, for every $x \in M$ and every $T > 0$, the set of controls which are regular w.r.t. $E^{x,T}$ is open and dense in L^2.

The above result hods indeed in the smooth topology.

Proposition (Sontag)

Under the same assumptions, the set of controls which are regular w.r.t. $E^{x,T}$ is open and dense in C^∞.

Thank you for your attention !!

Lecture 2

Sub-Riemannian geodesics

Sub-Riemannian structures

Let M be a smooth connnected manifold of dimension $n \geq 2$.

Definition

A sub-Riemannian structure on M is a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \in [2, n]$, that is it is defined locally as

$$\Delta(x) = \mathrm{Span}\Big\{X^1(x), \ldots, X^m(x)\Big\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ are m linearly independent vector fields satisfying the Hörmander condition.
- g_x is a **scalar product** on $\Delta(x)$.

Sub-Riemannian structures

Remark

- *In general* Δ *does not admit a global frame. However we can always construct* $k = m \cdot (n+1)$ *smooth vector fields* $Y^1, \ldots, Y^k$ *such that*

$$\Delta(x) = Span\Big\{Y^1(x), \ldots, Y^k(x)\Big\} \qquad \forall x \in M.$$

- *If* (M, g) *is a Riemannian manifold, then any totally nonholonomic distribution* Δ *gives rise to a SR structure* (Δ, g) *on* M*.*

Example (Heisenberg)

Take in $\mathbb{R}^3$, $\Delta = \text{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x - \frac{y}{2}\partial_z, \quad X^2 = \partial_y + \frac{x}{2}\partial_z \text{ and } g = dx^2 + dy^2.$$

The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any path $\gamma \in W^{1,2}([0,1]; M)$ satisfying

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$

We observe that if $\Delta = \text{Span}\{Y^1, \ldots, Y^k\}$, for any $x \in M$ and any control $u \in L^2([0,1]; \mathbb{R}^k)$, the solution to

$$\dot{\gamma} = \sum_{i=1}^{k} u_i X^i(\gamma), \quad \gamma(0) = x$$

is an horizontal path joining x to $\gamma(1)$.

Theorem (Chow-Rashevsky)

Let Δ be a totally nonholonomic distribution on M then any pair of points can be joined by an horizontal path.

The sub-Riemannian distance

The **length** (w.r.t g) of an horizontal path γ is defined as

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)} \, dt$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is

$$d_{SR}(x,y) := \inf\Big\{\text{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y\Big\}.$$

Proposition

The manifold M equipped with the distance d_{SR} is a metric space whose topology coincides with the topology of M (as a manifold).

Minimizing horizontal paths and geodesics

Definition

Given $x, y \in M$, we call **minimizing horizontal path** between x and y any horizontal path $\gamma : [0, T] \to M$ connecting x to y such that

$$d_{SR}(x, y) = \text{length}^g(\gamma).$$

The **sub-Riemannian energy** between x and y is defined as

$$e_{SR}(x, y) := \inf \left\{ \text{energy}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|^g_{\gamma(t)} \right)^2 dt \,|\, \gamma \ldots \right\}.$$

Definition

We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0, 1] \to M$ connecting x to y such that

$$e_{SR}(x, y) = \text{energy}^g(\gamma).$$

A SR Hopf-Rinow Theorem

Theorem

Let (Δ, g) be a sub-Riemannian structure on M. Assume that (M, d_{SR}) is a complete metric space. Then the following properties hold:

- *The closed balls $\bar{B}_{SR}(x, r)$ are compact (for any $r \geq 0$).*
- *For every $x, y \in M$, there exists at least one minimizing geodesic joining x to y.*

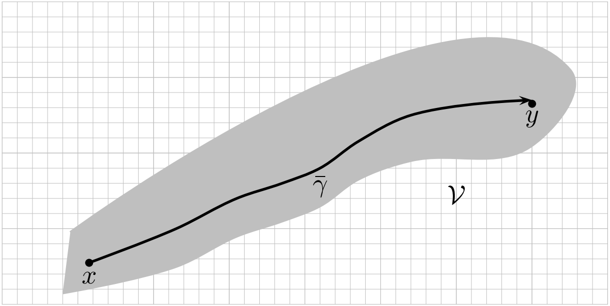
Remark

Given a complete Riemannian manifold (M, g), for any totally nonholonomic distribution Δ on M, the SR structure (Δ, g) is complete. As a matter of fact, since $d_g \leq d_{SR}$ any Cauchy sequence w.r.t. d_{SR} is Cauchy w.r.t. d_g.

The Hamiltonian geodesic equation

Let $x, y \in M$ and a **minimizing geodesic** $\bar{\gamma}$ joining x to y be fixed. The SR structure admits **an orthonormal frame** along $\bar{\gamma}$, that is there is an open neighborhood $\mathcal{V}$ of $\bar{\gamma}([0,1])$ and an orthonormal family of m vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \text{Span}\left\{X^1(z), \ldots, X^m(z)\right\} \qquad \forall z \in \mathcal{V}.$$



The Hamiltonian geodesic equation

There is a control $\bar{u} \in L^2\big([0,1];\mathbb{R}^m\big)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\big(\bar{\gamma}(t)\big) \qquad \text{a.e. } t \in [0,1].$$

Moreover, on the one hand any control $u \in \mathcal{U} \subset L^2\big([0,1];\mathbb{R}^m\big)$ (u sufficiently close to $\bar{u}$) gives rise to a trajectory γ_u solution of

$$\dot{\gamma}_u = \sum_{i=1}^{m} u^i\, X^i\big(\gamma_u\big) \quad \text{on } [0,T], \quad \gamma_u(0) = x.$$

On the other hand, for any horizontal path $\gamma : [0,1] \to \mathcal{V}$ there is a (unique) control $u \in L^2\big([0,1];\mathbb{R}^m\big)$ for which the equation in red is satisfied.

The Hamiltonian geodesic equation

So, considering as previously the **End-Point mapping**

$$E^{x,1} \,:\, L^2\big([0,1];\mathbb{R}^m\big) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and setting $C(u) = \|u\|_{L^2}^2$, we observe that $\bar{u}$ is solution to the following **optimization problem with constraints**:

$$\bar{u} \text{ minimizes } C(u) \text{ among all } u \in \mathcal{U} \text{ s.t. } E^{x,1}(u) = y.$$

(Since the family $X^1, \ldots, X^m$ is orthonormal, we have

$$\text{energy}^g(\gamma_u) = C(u) \qquad \forall u \in \mathcal{U}.)$$

The Hamiltonian geodesic equation

Proposition (Lagrange Multipliers)

There are $p \in T_y^*M \simeq (\mathbb{R}^n)^*$ *and* $\lambda_0 \in \{0,1\}$ *with* $(\lambda_0, p) \neq (0,0)$ *such that*

$$p \cdot D_{\bar{u}} E^{x,1} = \lambda_0 D_{\bar{u}} C.$$

Proof.

The mapping $\Phi : \mathcal{U} \to \mathbb{R} \times M$ defined by

$$\Phi(u) := \big(C(u), E^{x,1}(u)\big)$$

cannot be a submersion at $\bar{u}$. As a matter of fact, if $D_{\bar{u}}\Phi$ is surjective, then it is open at $\bar{u}$, so it must contain elements of the form $(C(\bar{u}) - \delta, y)$ for $\delta > 0$ small. □

$\rightsquigarrow$ two cases: $\lambda_0 = 0$ or $\lambda_0 = 1$.

The Hamiltonian geodesic equation

First case: $\lambda_0 = 0$

Then we have

$$p \cdot D_{\bar{u}} E^{x,1} = 0 \text{ with } p \neq 0.$$

So $\bar{u}$ is **singular** (w.r.t. x and $T = 1$).

Remark

- *If Δ has rank n, that is $\Delta = TM$ (Riemannian case), then there are no singular control. So this case cannot occur.*
- *If there are no nontrivial singular control, then this case cannot occur.*
- *If there are no nontrivial singular minimizing control, then this case cannot occur.*

$\rightsquigarrow$ Next lecture

The Hamiltonian geodesic equation

Second case: $\lambda_0 = 1$

Define the Hamiltonian $H : \mathcal{V} \times (\mathbb{R}^n)^* \to \mathbb{R}$ by

$$H(x,p) := \frac{1}{2} \sum_{i=1}^{m} \left(p \cdot X^i(x) \right)^2.$$

Proposition

There is a smooth arc $p : [0,1] \to (\mathbb{R}^n)^*$ with $p(1) = p/2$ such that

$$\begin{cases} \dot{\bar{\gamma}} & = & \frac{\partial H}{\partial p}(\bar{\gamma}, p) = \sum_{i=1}^m \left[p \cdot X^i(\bar{\gamma}) \right] X^i(\bar{\gamma}) \\ \dot{p} & = & -\frac{\partial H}{\partial x}(\bar{\gamma}, p) = -\sum_{i=1}^m \left[p \cdot X^i(\bar{\gamma}) \right] p \cdot D\bar{\gamma} X^i \end{cases}$$

for a.e. $t \in [0,1]$ and $\bar{u}_i(t) = p \cdot X^i(\bar{\gamma}(t))$ for a.e. $t \in [0,1]$ and any i. In particular, the path $\bar{\gamma}$ is smooth on $[0,1]$.

The Hamiltonian geodesic equation

Proof.

We have $D_{\bar{u}}C(v) = 2\langle \bar{u}, v\rangle_{L^2}$ and we remember that

$$D_{\bar{u}}E^{x,T}(v) = S(1)\int_0^1 S(t)^{-1}B(t)v(t)\,dt$$

with

$$\left\{ \begin{array}{lcl} A(t) & = & \sum_{i=1}^m u_i(t) D_{\bar{\gamma}(t)}X^i, \\ B(t) & = & \left(X^1(\bar{\gamma}(t)),\ldots,X^m(\bar{\gamma}(t))\right) \end{array} \right. \qquad \forall t \in [0,1],$$

and S solution of

$$\dot{S}(t) = A(t)S(t) \text{ for a.e. } t \in [0,1], \quad S(0) = I_n.$$

□

The Hamiltonian geodesic equation

Proof.

Then $p \cdot D_{\bar{u}} E^{x,1} = \lambda_0 D_{\bar{u}} C$ yields

$$\int_0^1 \left[p \cdot S(1) S(t)^{-1} B(t) - 2\bar{u}(t)^* \right] v(t)\, dt = 0 \quad \forall v \in L^2.$$

We infer that

$$\bar{u}(t) = \frac{1}{2} \left(p \cdot S(1) S(t)^{-1} B(t) \right)^* \quad \text{a.e. } t \in [0,1],$$

and that the absolutely continuous arc $p : [0,1] \to (\mathbb{R}^n)^*$ defined by

$$p(t) := \frac{1}{2} p \cdot S(1) S(t)^{-1}$$

satisfies the desired equations. □

The Hamiltonian geodesic equation

Define the **Hamiltonian** $H : T^*M \to \mathbb{R}$ by

$$H(x,p) = \frac{1}{2} \max \left\{ \frac{p(v)^2}{g_x(v,v)} \,|\, v \in \Delta_x \setminus \{0\} \right\}.$$

We call **normal extremal** any curve $\psi : [0,T] \to T^*M$ satisfying

$$\dot{\psi}(t) = \vec{H}\big(\psi(t)\big) \qquad \forall t \in [0,T].$$

Theorem

Let $\gamma : [0,1] \to M$ be a minimizing geodesic. One of the two following non-exclusive cases occur:

- *γ is singular.*
- *γ admits a normal extremal lift.*

It's time to take a small break !!

Examples

Example 1: The Riemannian case

Let $\Delta(x) = T_x M$ for any $x \in M$ so that ANY curve is horizontal. There are no singular curve, so any minimizing geodesic is the projection of a normal extremal.

Example 2: Heisenberg

Recall that in $\mathbb{R}^3$, $\Delta = \mathrm{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x - \frac{y}{2}\partial_z, \quad X^2 = \partial_y + \frac{x}{2}\partial_z \text{ and } g = dx^2 + dy^2.$$

Examples

Any horizontal path has the form $\gamma_u = (x, y, z) : [0, 1 \to \mathbb{R}^3$ with

$$\begin{cases} \dot{x}(t) & = & u_1(t) \\ \dot{y}(t) & = & u_2(t) \\ \dot{z}(t) & = & \frac{1}{2}\left(u_2(t)x(t) - u_1(t)y(t)\right), \end{cases}$$

for some $u \in L^2$. This means that

$$z(1) - z(0) = \int_\alpha \frac{1}{2}(xdy - ydx),$$

where α is the projection of γ to the plane $z = 0$. By Stokes' Theorem, we get

$$z(1) - z(0) = \int_{\mathcal{D}} dx \wedge dy + \int_c \frac{1}{2}(xdy - ydx)$$

where $\mathcal{D}$ is the domain enclosed by α and the segment $c = [\alpha(0), \alpha(1)]$. $\rightsquigarrow$ Projections of minimizing horizontal paths must be circles.

Examples

Let $\gamma_u = (x, y, z) : [0,1] \to \mathbb{R}^3$ be a minimizing geodesic from $P_1 := \gamma_u(0)$ to $P_2 := \gamma_u(1) \neq P_1$. Since u is necessarily regular ($\rightsquigarrow$ next lecture), there is a smooth arc $p : [0,1] \to (\mathbb{R}^3)^*$ s.t.

$$\begin{cases} \dot{x} &= p_x - \frac{y}{2} p_z \\ \dot{y} &= p_y + \frac{x}{2} p_z \\ \dot{z} &= \frac{1}{2}\left(\left(p_y + \frac{x}{2} p_z\right) x - \left(p_x - \frac{y}{2} p_z\right) y\right), \end{cases} \qquad \begin{cases} \dot{p}_x &= -\left(p_y + \frac{x}{2} p_z\right) \frac{p_z}{2} \\ \dot{p}_y &= \left(p_x - \frac{y}{2} p_z\right) \frac{p_z}{2} \\ \dot{p}_z &= 0. \end{cases}$$

Hence $p_z = \bar{p}_z$ for every t. Which implies that

$$\ddot{x} = -\bar{p}_z \dot{y} \quad \text{and} \quad \ddot{y} = \bar{p}_z \dot{x}.$$

If $\bar{p}_z = 0$, then the geodesic from P_1 to P_2 is a segment with constant speed. If $\bar{p}_z \neq 0$, we have or

$$\dddot{x} = -\bar{p}_z^2 \dot{x} \quad \text{and} \quad \dddot{y} = -\bar{p}_z^2 \dot{y}.$$

Which means that the curve $t \mapsto (x(t), y(t))$ is a circle.

Examples

Example 3: The Martinet distribution

In $\mathbb{R}^3$, let $\Delta = \text{Span}\{X^1, X^2\}$ with X^1, X^2 fo the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \left(1 + x_1\phi(x)\right)\partial_{x_2} + x_1^2\partial_{x_3},$$

where ϕ is a smooth function and g be a smooth metric on Δ.

Theorem

There is $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the (singular) horizontal path given by

$$\gamma(t) = (0, t, 0) \qquad \forall t \in [0, \epsilon],$$

minimizes the length (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$. Moreover if $\{X^1, X^2\}$ is orthonormal w.r.t. g and $\phi(0) \neq 0$, then γ can not be the projectionof a normal extremal.

The SR exponential mapping

Denote by $\psi_{x,p} : [0,1] \to T^*M$ the solution of

$$\dot{\psi}(t) = \vec{H}\big(\psi(t)\big) \qquad \forall t \in [0,1], \quad \psi(0) = (x,p)$$

and let

$$\mathcal{E}_x := \Big\{ p \in T_x^*M \,|\, \psi_{x,p} \text{ defined on } [0,1] \Big\}.$$

Definition

The **sub-Riemannian exponential map** from $x \in M$ is defined by

$$\begin{array}{rccc} \exp_x : \mathcal{E}_x \subset T_x^*M & \longrightarrow & M \\ p & \longmapsto & \pi\big(\psi_{x,p}(1)\big). \end{array}$$

Proposition

Assume that (M, d_{SR}) is complete. Then for every $x \in M$, $\mathcal{E}_x = T_x^*M$.

Thank you for your attention !!

Lecture 3

A closer look at singular curves

Setting

We are given a **complete sub-Riemannian structure** (Δ, g) of rank $m \in [2, n-1]$ in $M = \mathbb{R}^n$ which admits a global frame, i.e.

$$\Delta(x) = \mathrm{Span}\Big\{X^1(x), \ldots, X^m(x)\Big\} \qquad \forall x \in \mathbb{R}^n,$$

with $\{X^1, \ldots, X^m\}$ a family of m linearly independent smooth vector fields satisfying the Hörmander condition.
Given $x \in M$, there is a one-to-one correspondence between horizontal curves in $W^{1,2}([0,1];\mathbb{R}^m)$ starting at x and controls $u \in L^2([0,1];\mathbb{R}^m)$. An horizontal path $\gamma_u : [0,1] \to M$ is singular (w.r.t. $x = \gamma_u(0)$, $T = 1$) iff the control u is singular.

Exercise

Check that the definition "singular curve" does not depend on the family $\{X^1, \ldots, X^m\}$.

Characterization of singular curves

Let $h^1, \ldots, h^m : T^*M (= \mathbb{R}^n \times (\mathbb{R}^n)^*) \to \mathbb{R}$ be the **Hamiltonians** defined by

$$h^i(\psi) = p \cdot X^i(x) \qquad \forall \psi = (x,p) \in T^*M, \, \forall i = 1, \ldots, m$$

and denote by $\vec{h}^1, \ldots, \vec{h}^m$ the corresponding **Hamiltonian vector fields** which read $\vec{h}^i(x,p) = \left(\frac{\partial h^i}{\partial p}(x,p), -\frac{\partial h^i}{\partial x}(x,p) \right)$.

Proposition

The control u is singular iff there is an absolutely continuous arc $\psi = (x,p) : [0,1] \to T^*M$ with $p(t) \neq 0$ for all $t \in [0,1]$ such that

$$\dot{\psi}(t) = \sum_{i=1}^m u_i(t) \vec{h}^i(\psi(t)) \qquad a.e. \;\; t \in [0,1]$$

$$h^i(\psi(t)) = 0 \qquad \forall t \in [0,1], \, \forall i = 1, \ldots, m.$$

Characterization of singular curves (proof)

Proof.

If $D_u E^{x,1} : L^2([0,1];\mathbb{R}^m) \to \mathbb{R}^n$ is not surjective, there is $p \in (\mathbb{R}^n)^* \setminus \{0\}$ such that

$$p \cdot D_u E^{x,1}(v) = 0 \qquad \forall v \in L^2\big([0,1];\mathbb{R}^m\big).$$

Then we have

$$\int_0^1 p \cdot S(1)S(t)^{-1}B(t)v(t)dt = 0 \qquad \forall v \in L^2([0,T];\mathbb{R}^m).$$

which implies that $p \cdot S(1)S(t)^{-1}B(t) = 0$ for any $t \in [0,1]$. Let us now define, for each $t \in [0,1]$, $p(t) := p \cdot S(1)S(t)^{-1}$. By construction, $p : [0,1] \to (\mathbb{R}^n)^*$ is an absolutely continuous arc, it is solution of the desired equations, and it does not vanish on $[0,1]$. □

Reminder on minimizing geodesics

We call **normal extremal** any curve $\psi : [0, T] \to T^*M$ satisfying

$$\dot{\psi}(t) = \vec{H}(\psi(t)) \qquad \forall t \in [0, T].$$

where the **Hamiltonian** $H : T^*M \to \mathbb{R}$ is defined by

$$H(x,p) = \frac{1}{2} \max \left\{ \frac{p(v)^2}{g_x(v,v)} \,|\, v \in \Delta_x \setminus \{0\} \right\}.$$

Theorem

Let $\gamma : [0,1] \to M$ be a minimizing geodesic. One of the three following exclusive cases occur:

- *γ is singular and not the projection of a normal extremal.*
- *γ is singular and also the projection of a normal extremal.*
- *γ is regular and so is the projection of a normal extremal.*

$\rightsquigarrow$ Study of examples

Contact distribution in dimension three

Let Δ be a rank two distribution in $\mathbb{R}^3$ given by

$$\Delta(x) = \left\{ X^1(x), X^2(x) \right\}$$

with

$$\mathrm{Span}\left\{ X^1(x), X^2(x), [X^1, X^2](x) \right\} = \mathbb{R}^3.$$

Proposition

- *Any nontrivial control is regular.*
- *All (nontrivial) minimizing geodesics are regular and projection of a normal extremal.*
- *For every x, the SR exponential map $\exp_x : T_x^* M \to M$ is onto.*

Contact distribution in dimension three (proof)

Proof.

Argue by contradiction and let $\gamma_u : [0,1] \to M$ be a singular horizontal path associated with $u \neq 0 \in L^2$. Then there is $p : [0,1] \to (\mathbb{R}^3)^* \setminus \{0\}$ such that

$$\dot{p}(t) = -\sum_{i=1}^{2} u_i(t) p(t) \cdot D_{\gamma_u(t)} X^i \quad \text{a.e. } t \in [0,1]$$

and

$$p(t) \cdot X^i(\gamma_u(t)) = 0 \quad \forall t \in [0,1], \, \forall i = 1,2.$$

Taking a derivative in the last two equalities yields ($i = 1,2$)

$$u_1(t) p(t) \cdot [X^1, X^2](\gamma_u(t)) = u_2(t) p(t) \cdot [X^1, X^2](\gamma_u(t)) = 0,$$

so that for any t, $p(t) \perp X^1, X^2, u_1[X^1, X^2], u_2[X^1, X^2]$.
$\rightsquigarrow$ Contradiction !! □

Fat distributions

The distribution Δ in $M = \mathbb{R}^n$ is called **fat** if for every $x \in \mathbb{R}^n$ and every section X of Δ with $X(x) \neq 0$, there holds

$$T_x M = \Delta(x) + [X, \Delta](x)$$

where

$$[X, \Delta](x) := \Big\{ [X, Z](x) \,|\, Z \text{ section of } \Delta \Big\}.$$

Proposition

- *Any nontrivial control is regular.*
- *All minimizing geodesics are regular and projection of a normal extremal.*
- *For every x, the SR exponential map $\exp_x : T_x^* M \to M$ is onto.*

$\rightsquigarrow$ There are very few fat distributions, see Montgomery.

Fat distributions (proof)

Proof.

Argue by contradiction and let $\gamma_u : [0,1] \to M$ be a singular horizontal path associated with $u \neq 0 \in L^2$. Then there is $p : [0,1] \to (\mathbb{R}^n)^* \setminus \{0\}$ such that

$$\dot{p}(t) = -\sum_{i=1}^{m} u_i(t) p(t) \cdot D_{\gamma_u(t)} X^i \quad \text{a.e. } t \in [0,1],$$

$$p(t) \cdot X^i(\gamma_u(t)) = 0 \quad \forall t \in [0,1], \forall i = 1, m.$$

Taking a derivative in the last m equalities gives

$$p(t) \cdot \left[\sum_{i=1}^{m} u_j(t) X^j, X^i \right] (\gamma_u(t)) = 0 \qquad \forall i = 1, \ldots, m.$$

$\rightsquigarrow X = \sum_{j=1}^{m} u_j(t) X^j$ yields a contradiction !! □

The Martinet distribution

Let $\Delta = \mathrm{Span}\{X^1, X^2\}$ be a rank two distribution in $\mathbb{R}^3$ given by

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \partial_{x_2} + \frac{x_1^2}{2}\,\partial_{x_3}.$$

We check that

$$[X^1, X^2] = x_1\,\partial_{x_3} \quad \text{and} \quad \left[[X^1, X^2], X^1\right] = \partial_{x_3}.$$

Proposition

The horizontal paths which are singular are those which are tangent to the **Martinet surface**

$$\Sigma_\Delta := \left\{x \in \mathbb{R}^3 \,|\, x_1 = 0\right\}.$$

The Martinet distribution (proof)

Proof.

The distribution is contact outside Σ_Δ, so the singular paths are necessarily contained in Σ_Δ.
Conversely, an horizontal path $\gamma_u : [0, T] \to \Sigma_\Delta$ has the form

$$\gamma_u(t) = (0, x_2(t), z) \quad \text{with } z \in \mathbb{R}.$$

Any arc $p : [0, T] \to (\mathbb{R}^3)^*$ of the form $p(t) = (0, 0, p_3)$ with $p_3 \neq 0$ satisfies

$$\dot{p}(t) = -\sum_{i=1}^{2} u_i(t) p(t) \cdot D_{\gamma_u(t)} X^i \quad \text{a.e. } t \in [0, 1]$$

$$p(t) \cdot X^i(\gamma_u(t)) = 0 \quad \forall t \in [0, 1], \ \forall i = 1, 2.$$

This shows that γ_u is singular. □

Martinet like distributions

Given a rank two totally nonholonomic distribution Δ in $\mathbb{R}^3$, we define the **Martinet surface** of Δ as the set

$$\Sigma_\Delta := \left\{ x \in \mathbb{R}^3 \,|\, \Delta(x) + [\Delta, \Delta](x) \neq \mathbb{R}^3 \right\},$$

where

$$[\Delta, \Delta](x) = \left\{ [X, Y](x) \,|\, X, Y \text{ sections of } \Delta \right\}.$$

Proposition

The set Σ_Δ is closed and countably 2-rectifiable. Moreover, the horizontal paths which are singular are those which are tangent to Σ_Δ.

$\rightsquigarrow$ We can have singular or nonsingular minimizing controls.

Generic rank two distributions in $\mathbb{R}^4$

Let Δ be a rank two distribution in $\mathbb{R}^4$ such that for every $x \in \mathbb{R}^4$ the two following properties hold:

- $\Delta(x) + [\Delta, \Delta](x)$ has dimension three.
- $$\Delta(x) + [\Delta, \Delta](x) + \big[\Delta, [\Delta, \Delta]\big](x) = \mathbb{R}^4,$$

  where

  $$\big[\Delta, [\Delta, \Delta]\big](x) = \Big\{ \big[X, [Y, Z]\big](x) \,|\, X, Y, Z \text{ sections of } \Delta \Big\}.$$

Proposition

There is a rank one distribution (a smooth line field) $L \subset \Delta$ such that the horizontal paths which are singular are those which are tangent to L.

$\rightsquigarrow$ Such singular paths can be minimizing (cf. Liu-Sussmann).

It's time to take a small break !!

Medium Fat distributions

The distribution Δ in $M = \mathbb{R}^n$ is called **medium fat** if for every $x \in \mathbb{R}^n$ and every section X of Δ with $X(x) \neq 0$, there holds

$$\Delta(x) + [\Delta, \Delta](x) + [X, [\Delta, \Delta]](x) = \mathbb{R}^n$$

where

$$[X, [\Delta, \Delta]](x) := \Big\{ \big[X, [Y, Z]\big](x) \mid Y, Z \text{ sections of } \Delta \Big\}.$$

Proposition

Any minimizing geodesic of a medium fat distribution is the projection of a normal extremal. In particular, for every $x \in \mathbb{R}^4$, *the mapping* $\exp_x$ *is onto.*

$\rightsquigarrow$ For example, any two-generating distribution is medium fat.

Medium Fat distributions (sketch of proof)

The result is a consequence of the two following results:

Theorem

Let $\gamma : [0,1] \to M$ be a minimizing geodesic which is singular and not the projection of a normal extremal. Then there is an abnormal lift $\psi = (x,p) : [0,1] \to T^*M$ of γ such that

$$p(t) \cdot \left[X^i, X^j\right](\gamma(t)) = 0 \qquad \forall t \in [0,1],\ \forall i,j = 1,\dots,m.$$

A path satisfying the above property is called a **Goh path**.

Lemma

A medium fat distribution does not admit nontrivial Goh paths.

$\rightsquigarrow$ The proof of the lemma is left as an exercise.

Medium Fat distributions (sketch of proof)

The theorem follows from of a study of the End-Point mapping at second order. Let us give a flavor of the proof.

Theorem (Agrachev-Sarychev)

Let $F : \mathbb{R}^k \to \mathbb{R}^n$ be a mapping of class C^2 and $\bar{u}$ be a critical point of F of corank $r \geq 1$. If

$$ind_- \left(\lambda^* \left(D^2_{\bar{u}} F \right)_{|Ker(D_{\bar{u}}F)} \right) \geq r \quad \forall \lambda \in \left(Im\left(D_{\bar{u}} F \right) \right)^{\perp} \setminus \{0\},$$

then the mapping F is locally open at $\bar{u}$, that is the image of any neighborhood of $\bar{u}$ is a neighborhood of $F(\bar{u})$.

Maybe we should recall that the **negative index** of a quadratic form $Q : \mathbb{R}^k \to \mathbb{R}$ is defined by

$$\mathrm{ind}_-(Q) = \max \left\{ \dim(L) \mid Q_{|L \setminus \{0\}} < 0 \right\}.$$

Codimension one distributions

Given a rank $(n-1)$ totally nonholonomic distribution Δ in $\mathbb{R}^n$, we define the **singular set** associated to Δ as the set

$$\Sigma_\Delta := \left\{ x \in \mathbb{R}^n \,|\, \Delta(x) + [\Delta, \Delta](x) \neq \mathbb{R}^n \right\}.$$

Proposition

The set Σ_Δ is closed and countably $(n-1)$-rectifiable. Moreover, any Goh path is contained Σ_Δ.

$\rightsquigarrow$ As a consequence all minimizing geodesics joining x to y with either $x \notin \Sigma_\Delta$ or $y \notin \Sigma_\Delta$ is the projection of a normal extremal (it can be singular).

Generic sub-Riemannian structures

Theorem (Agrachev-Gauthier,Chitour-Jean-Trélat)

Generic SR structures of rank ≥ 3 do not admit (nontrivial) minimizing geodesics which are singular.

Proposition

For a generic SR structure of rank ≥ 3, the following holds:

- *All (nontrivial) minimizing geodesics are regular and projection of a normal extremal.*
- *For every x, the SR exponential map $\exp_x : T_x^*M \to M$ is onto.*

Mind the gap

Contact Distributions

$\Downarrow$

Fat Distributions

$\Downarrow$

Two-generating Distributions

$\Downarrow$

Medium fat

$\Downarrow$

Thank you for your attention !!

Lecture 4

Open questions

Open questions

- The Sard conjecture
- Regularity of minimizing geodesics
- Small sub-Riemannian balls

The Sard Conjecture

Let M be a smooth connected manifold of dimension n and $\mathcal{F} = \{X^1, \ldots, X^k\}$ be a family of smooth vector fields on M satisfying the Hörmander condition. Given $x \in M$ and $T > 0$, the **End-Point mapping** $E^{x,T}$ is defined as

$$\begin{array}{rcl} E^{x,T} \ : \ L^2\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ u & \longmapsto & x\big(T;x,u\big) \end{array}$$

where $x(\cdot) = x(\cdot;x,u) : [0,T] \longrightarrow M$ is solution to the Cauchy problem

$$\dot{x} = \sum_{i=1}^{m} u_i \, X^i(x), \quad x(0) = x.$$

Proposition

The map $E^{x,T}$ is smooth on its domain.

The Sard Conjecture

Theorem (Morse 1939, Sard 1942)

Let $f : \mathbb{R}^d \to \mathbb{R}^p$ be a function of class C^k, then

$$k \geq \max\{1, d-p+1\} \quad \Longrightarrow \quad \mathcal{L}^p\Big(f\big(Crit(f)\big)\Big) = 0,$$

where $Crit(f)$ is the set of critical points of f, i.e. the points where $D_x f$ is not onto.

Let

$$\mathrm{Sing}_{\mathcal{F}}^{x,T} := \Big\{ u \in L^2\big([0,T];\mathbb{R}^m\big) \mid u \text{ singular} \Big\}.$$

Conjecture

The set $E^{x,T}\left(Sing_{\mathcal{F}}^{x,T}\right) \subset M$ has Lebesgue measure zero.

The Sard Conjecture (very partial answers)

Assump: $\Delta = \mathrm{Span}\{X^1, \ldots, X^k\}$ is a tot. nonholon. distrib.

Proposition

If Δ is **fat**, *then any nontrivial horizontal path is nonsingular. As a consequence for every $x \in M$,*

$$E^{x,T}\left(Sing^{x,T}\right) = \{x\} \quad \textit{and} \quad \mathcal{L}^n\left(E^{x,T}\left(Sing^{x,T}\right)\right) = 0.$$

Proposition

If Δ has **rank two in dimension three**, *then for every $x \in M$, we have*

$$\begin{array}{ccc} x \notin \Sigma_\Delta & \Longrightarrow & E^{x,T}\left(Sing^{x,T}\right) = \{x\} \\ x \in \Sigma_\Delta & \Longrightarrow & E^{x,T}\left(Sing^{x,T}\right) \subset \Sigma_\Delta. \end{array}$$

In any case $E^{x,T}\left(Sing^{x,T}\right)$ has Lebesgue measure zero.

The Sard Conjecture (very partial answers)

Proposition

If Δ *is a* **rank two distributions in dimension four** *satisfying for every* $x \in M$,

$$\Delta(x) + [\Delta, \Delta](x) \text{ has dimension three}$$

and

$$\Delta(x) + [\Delta, \Delta](x) + \big[\Delta, [\Delta, \Delta]\big](x) = \mathbb{R}^4,$$

then there is a rank one distribution (a smooth line field) $L \subset \Delta$ *such that the horizontal paths which are singular are those which are tangent to* L*. As a consequence we have for every* $x \in M$,

$$E^{x,T}\left(Sing^{x,T}\right) = \mathcal{O}_x^L\{x\},$$

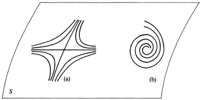
the orbit of x *w.r.t.* L*, which has measure zero.*

The Sard Conjecture (very partial answer)

The Sard conjecture for general totally nonholonomic distributions is unsolved. We even don't know if $E^{x,T}\left(\mathrm{Sing}^{x,T}\right)$ has in general nonempty interior in M !!

In fact, there are even stronger conjectures on the size of the set $E^{x,T}\left(\mathrm{Sing}^{x,T}\right)$.

For example in the case of a **rank two distributions in dimension three**, we can hope that the Hausdorff dimension of $E^{x,T}\left(\mathrm{Sing}^{x,T}\right)$ is ≤ 1.



The minimizing Sard Conjecture

Given a complete SR structure (Δ, g) and $x \in M$, we define the following sets:

- Denoting by $\mathrm{Sing}^x_{\mathrm{min}}$ the set of **singular minimizing geodesics starting at** x, we set

$$\mathcal{S}^x_{\mathrm{min}} := E^{x,1}\left(\mathrm{Sing}^x_{\mathrm{min}}\right).$$

- Denoting by $\mathrm{Sing}^x_{\mathrm{min,\,strict}}$ the set of **singular minimizing geodesics starting at** x, we set

$$\mathcal{S}^x_{\mathrm{min,strict}} := E^{x,1}\left(\mathrm{Sing}^x_{\mathrm{min,strict}}\right) \subset \mathcal{S}^x_{\mathrm{min}}.$$

Conjecture

The set $\mathcal{S}^x_{min}$ (resp. $\mathcal{S}^x_{min,strict}$) has Lebesgue measure zero.

The minimizing Sard Conjecture (partial answer)

Proposition (Agrachev-Trélat-LR)

The following properties hold:

- *The set $\mathcal{S}^x_{min,strict}$ has empty interior.*
- *The set $\mathcal{S}^x_{min}$ has empty interior.*

Lemma

Let $y \neq x$ in M be such that there is a function $\phi : M \to \mathbb{R}$ differentiable at y such that

$$\phi(y) = d^2_{SR}(x,y) \quad \text{and} \quad d^2_{SR}(x,z) \geq \phi(z) \quad \forall z \in M.$$

Then there is a unique minimizing geodesic $\gamma : [0,1] \to M$ between x and y. It is the projection of a normal extremal $\psi : [0,1] \to T^*M$ satisfying $\psi(1) = (y, \frac{1}{2} D_y\phi)$. In particular $x = \exp_y(-\frac{1}{2}D_y\phi)$.

The minimizing Sard Conjecture (partial answer)

Proof.

Let $y \neq x$ in M satisfying the assumption and $\bar{\gamma} = \gamma_{\bar{u}} : [0,1] \to M$ be a minimizing geodesic from x to y. We have for every $u \in \mathcal{U} \subset L^2([0,1];\mathbb{R}^m)$ (close to $\bar{u}$),

$$\|u\|_{L^2}^2 = C(u) \geq e_{SR}\left(x, E^{x,1}(u)\right) \geq \phi\left(E^{x,1}(u)\right),$$

with equality if $u = \bar{u}$. So $\bar{u}$ is solution to the following **optimization problem**:

$$\bar{u} \text{ minimizes } C(u) - \phi\left(E^{x,1}(u)\right) \text{ among all } u \in \mathcal{U}.$$

We infer that there is $p \neq 0$ such that

$$p \cdot D_u E^{x,1} = D_u C \quad \text{with } p = D_{E^{x,1}(u)}\phi$$

and in turn get the result. □

It's time to take a small break !!

Regularity of minimizing geodesics

Let (Δ, g) be complete SR structure on a smooth manifold M.

Open Question

Do the minimizing geodesics anjoy some regularity ? Are they at least of class C^1 ?

$\rightsquigarrow$ Results by Monti, Leonardi and later Monti

$\rightsquigarrow$ An interesting result in the real analytic case by Sussmann

Regularity of minimizing geodesics

Let (Δ, g) be complete SR structure on a smooth manifold M.

Theorem (Leonardi-Monti, 2008)

Assume that Δ satisfies the following assumptions:

- *there is $r \geq 2$ such that the dimensions of the spaces*

$$\Delta(x) = Lie^1(\Delta)(x) \subset Lie^2(\Delta)(x), \ldots, Lie^r(\Delta)(x) = T_x M$$

  does not depend upon $x \in M$.

- *We have the following technical condition*

$$\left[Lie^i(\Delta), Lie^j(\Delta)\right](x) \subset Lie^{i+j+1}(\Delta)(x) \qquad \forall x \in M.$$

Then a curve with **corner** *cannot be a minimizing geodesic.*

(A Lipschitz curve has a corner at some point if the left and right derivatives at that point exist and are lin. independent.)

Regularity of singular horizontal paths

Let (Δ, g) be complete real analytic SR structure on a **real analytic** manifold M.

Theorem (Sussmann, recent)

Every singular minimizing geodesic is real analytic on an open dense subset of its interval of definition.

The proof is based on stratification results in sub-analytic geometry.

Small SR spheres

Let (Δ, g) be complete SR structure on a smooth manifold M.

Open Question

Are small SR spheres homeomorphic to Euclidean spheres ?

$\rightsquigarrow$ Almost no result on this problem.

Baryshnikov claims the small SR-balls are momeomorphic to Euclidean balls in the contact case.

In a paper of mine, I show that if there are no singular minimizing geodesics, then almost all spheres are Lipschitz hypersurfaces.

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Thank you for your attention !!