

The Sard Conjecture on Martinet Surfaces

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New Trends in Control Theory and PDEs
In honor of Piermarco Cannarsa
July 3-7, 2017

Levico Terme, 1998



Sub-Riemannian structures

Let M be a smooth connected manifold of dimension n .

Definition

A sub-Riemannian structure of rank m in M is given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally by

$$\Delta(x) = \text{Span} \left\{ X^1(x), \dots, X^m(x) \right\} \subset T_x M,$$

where X^1, \dots, X^m is a family of m linearly independent smooth vector fields satisfying the **Hörmander condition**.

- g_x is a **scalar product** over $\Delta(x)$.

The Hörmander condition

We say that a family of smooth vector fields X^1, \dots, X^m , satisfies the **Hörmander condition** if

$$\text{Lie} \{X^1, \dots, X^m\} (x) = T_x M \quad \forall x,$$

where $\text{Lie}\{X^1, \dots, X^m\}$ denotes the Lie algebra generated by X^1, \dots, X^m , i.e. the smallest subspace of smooth vector fields that contains all the X^1, \dots, X^m and which is stable under Lie brackets.

Reminder

Given smooth vector fields X, Y in \mathbb{R}^n , the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

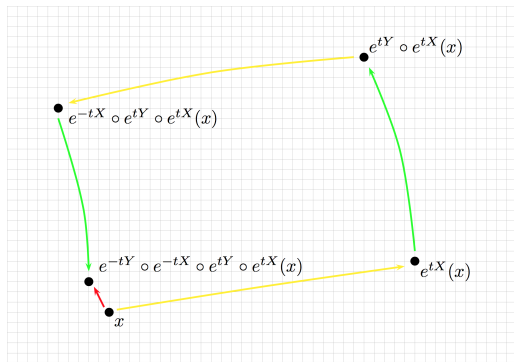
$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

Lie Bracket: Dynamic Viewpoint

Exercise

There holds

$$[X, Y](x) = \lim_{t \downarrow 0} \frac{(e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX})(x) - x}{t^2}.$$



The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any $\gamma \in W^{1,2}([0, 1]; M)$ such that

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \quad \text{a.e. } t \in [0, 1].$$

The following result is the cornerstone of the sub-Riemannian geometry. (Recall that M is assumed to be connected.)

Theorem (Chow-Rashevsky, 1938)

Let Δ be a totally nonholonomic distribution on M , then every pair of points can be joined by an horizontal path.

Since the distribution is equipped with a metric, we can measure the lengths of horizontal paths and consequently we can associate a metric with the sub-Riemannian structure.

Examples of sub-Riemannian structures

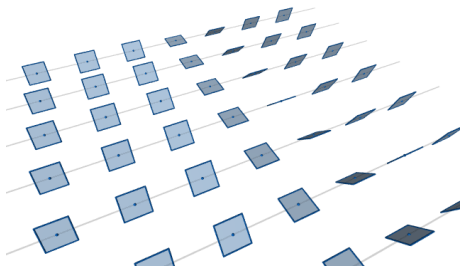
Example (Riemannian case)

Every Riemannian manifold (M, g) gives rise to a sub-Riemannian structure with $\Delta = TM$.

Example (Heisenberg)

In \mathbb{R}^3 , $\Delta = \text{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z \quad \text{et} \quad g = dx^2 + dy^2.$$



Examples of sub-Riemannian structures

Example (Martinet)

In \mathbb{R}^3 , $\Delta = \text{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x^2 \partial_z.$$

Since $[X^1, X^2] = 2x\partial_z$ and $[X^1, [X^1, X^2]] = 2\partial_z$, only one bracket is sufficient to generate \mathbb{R}^3 if $x \neq 0$, however we need two brackets if $x = 0$.

Example (Rank 2 distribution in dimension 4)

In \mathbb{R}^4 , $\Delta = \text{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z + z\partial_w$$

satisfies $\text{Vect}\{X^1, X^2, [X^1, X^2], [[X^1, X^2], X^2]\} = \mathbb{R}^4$.

Sub-Riemannian geodesics

The **length** and **energy** of an horizontal path γ are defined by

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|_{\gamma(t)}^g dt, \text{ener}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|_{\gamma(t)}^g \right)^2 dt.$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is defined by

$$d_{SR}(x, y) := \inf \left\{ \text{length}^g(\gamma) \mid \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y \right\}.$$

Definition

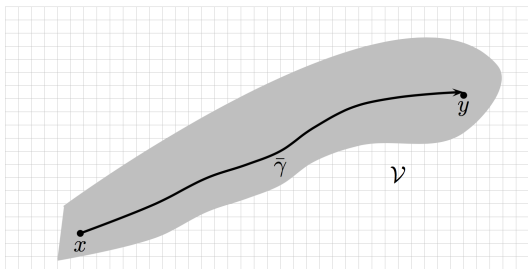
We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0, 1] \rightarrow M$ joining x to y such that

$$d_{SR}(x, y)^2 = \text{ener}^g(\gamma).$$

Study of minimizing geodesics

Let $x, y \in M$ and $\bar{\gamma}$ be a **minimizing geodesic** between x and y be fixed. The SR structure admits an orthonormal parametrization along $\bar{\gamma}$, which means that there exists a neighborhood \mathcal{V} of $\bar{\gamma}([0, 1])$ and an orthonormal family of m vector fields X^1, \dots, X^m such that

$$\Delta(z) = \text{Span}\{X^1(z), \dots, X^m(z)\} \quad \forall z \in \mathcal{V}.$$



Study of minimizing geodesics

There exists a control $\bar{u} \in L^2([0, 1]; \mathbb{R}^m)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^m \bar{u}_i(t) X^i(\bar{\gamma}(t)) \quad \text{a.e. } t \in [0, 1].$$

Moreover, any control $u \in \mathcal{U} \subset L^2([0, 1]; \mathbb{R}^m)$ (u sufficiently close to \bar{u}) gives rise to a trajectory γ_u solution of

$$\dot{\gamma}_u = \sum_{i=1}^m u^i X^i(\gamma_u) \quad \text{sur } [0, T], \quad \gamma_u(0) = x.$$

Furthermore, for every horizontal path $\gamma : [0, 1] \rightarrow \mathcal{V}$ there exists a unique control $u \in L^2([0, 1]; \mathbb{R}^m)$ for which the above equation is satisfied.

Study of minimizing geodesics

Consider the **End-Point mapping**

$$E^{x,1} : L^2([0, 1]; \mathbb{R}^m) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and set $C(u) = \|u\|_{L^2}^2$, then \bar{u} is a solution to the following **optimization problem with constraints**:

\bar{u} minimize $C(u)$ among all $u \in \mathcal{U}$ s.t. $E^{x,1}(u) = y$.

(Since the family X^1, \dots, X^m is orthonormal, we have

$$\text{ener}^g(\gamma_u) = C(u) \quad \forall u \in \mathcal{U}.)$$

Study of minimizing geodesics

Proposition (Lagrange Multipliers)

There exist $p \in T_y^*M \simeq (\mathbb{R}^n)^*$ and $\lambda_0 \in \{0, 1\}$ with $(\lambda_0, p) \neq (0, 0)$ such that

$$p \cdot d_{\bar{u}}E^{x,1} = \lambda_0 d_{\bar{u}}C.$$

First case : $\lambda_0 = 1$

This is the good case, the Riemannian-like case.

Second case : $\lambda_0 = 0$

In this case, we have

$$p \cdot D_{\bar{u}}E^{x,1} = 0 \text{ with } p \neq 0,$$

which means that \bar{u} is **singular** as a critical point of the mapping $E^{x,1}$.

Singular horizontal paths and Examples

Definition

An horizontal path is called **singular** if it is, through the correspondence $\gamma \leftrightarrow u$, a critical point of the End-Point mapping $E^{x,1} : L^2 \rightarrow M$.

Example 1: Riemannian case

Let $\Delta(x) = T_x M$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

Example 2: Heisenberg, fat distributions

In \mathbb{R}^3 , Δ given by $X^1 = \partial_x, X^2 = \partial_y + x\partial_z$ does not admit nontrivial singular horizontal paths.

Example 3: Martinet-like distributions

In \mathbb{R}^3 , let $\Delta = \text{Vect}\{X^1, X^2\}$ with X^1, X^2 of the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = (1 + x_1\phi(x)) \partial_{x_2} + x_1^2 \partial_{x_3},$$

where ϕ is a smooth function.

The Sard Conjectures

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed.

$$\mathcal{S}_{\Delta, \min^g}^x = \{\gamma(1) \mid \gamma : [0, 1] \rightarrow M, \gamma(0) = x, \gamma \text{ hor.}, \text{ sing.}, \text{ min.}\}.$$

Conjecture (SR or minimizing Sard Conjecture)

The set $\mathcal{S}_{\Delta, \min^g}^x$ has Lebesgue measure zero.

$$\mathcal{S}_{\Delta}^x = \{\gamma(1) \mid \gamma : [0, 1] \rightarrow M, \gamma(0) = x, \gamma \text{ hor.}, \text{ sing.}\}.$$

Conjecture (Sard Conjecture)

The set \mathcal{S}_{Δ}^x has Lebesgue measure zero.

The Brown-Morse-Sard Theorem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function of class C^k .

Definition

- We call **critical point** of f any $x \in \mathbb{R}^n$ such that $d_x f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is not surjective and we denote by C_f the set of critical points of f .
- We call **critical value** any element of $f(C_f)$. The elements of $\mathbb{R}^m \setminus f(C_f)$ are called **regular values**.



H.C. Marston Morse

(1892-1977)



Arthur B. Brown

(1905-1999)



Anthony P. Morse

(1911-1984)



Arthur Sard

(1909-1980)

The Brown-Morse-Sard Theorem

Theorem (Arthur B. Brown, 1935)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be of class C^k . If $k = \infty$ (or large enough) then $f(C_f)$ has empty interior.

Theorem (Anthony P. Morse, 1939)

Assume that $m = 1$ and $k \geq m$, then $f(C_f)$ has Lebesgue measure zero.

Theorem (Arthur Sard, 1942)

If $k \geq \max\{1, n - m + 1\}$, $\mathcal{L}^m(f(C_f)) = 0$.

Remark

Thanks to a construction by Hassler Whitney (1935), the assumption in Sard's theorem is sharp.

Infinite dimension (Bates-Moreira, 2001)

The Sard Theorem is false in infinite dimension. Let $f : \ell^2 \rightarrow \mathbb{R}$ be defined by

$$f \left(\sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{\infty} (3 \cdot 2^{-n/3} x_n^2 - 2x_n^3).$$

The function f is polynomial ($f^{(4)} \equiv 0$) with critical set

$$C(f) = \left\{ \sum_{n=1}^{\infty} x_n e_n \mid x_n \in \{0, 2^{-n/3}\} \right\},$$

and critical values

$$f(C(f)) = \left\{ \sum_{n=1}^{\infty} \delta_n 2^{-n} \mid \delta_n \in \{0, 1\} \right\} = [0, 1].$$

Back to the Sard Conjecture

Let (Δ, g) be a SR structure on M . Set

$$\Delta^\perp := \left\{ (x, p) \in T^*M \mid p \perp \Delta(x) \right\} \subset T^*M$$

and (we assume here that Δ is generated by m vector fields X^1, \dots, X^m) define

$$\vec{\Delta}(x, p) := \text{Span} \left\{ \vec{h}^1(x, p), \dots, \vec{h}^m(x, p) \right\} \quad \forall (x, p) \in T^*M,$$

where $h^i(x, p) = p \cdot X^i(x)$ and \vec{h}^i is the associated Hamiltonian vector field in T^*M .

Proposition

An horizontal path $\gamma : [0, 1] \rightarrow M$ is singular if and only if it is the projection of a path $\psi : [0, 1] \rightarrow \Delta^\perp \setminus \{0\}$ which is horizontal w.r.t. $\vec{\Delta}$.

The case of Martinet surfaces

Let M be a smooth manifold of dimension 3 and Δ be a totally nonholonomic distribution of rank 2 on M . We define the **Martinet surface** by

$$\Sigma_{\Delta} = \{x \in M \mid \Delta(x) + [\Delta, \Delta](x) \neq T_x M\}$$

If Δ is generic, Σ_{Δ} is a surface in M . If Δ is analytic then Σ_{Δ} is analytic of dimension ≤ 2 .

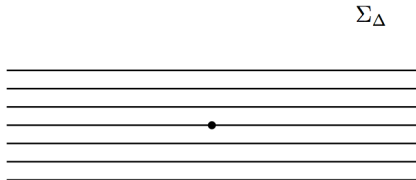
Proposition

The singular horizontal paths are the orbits of the trace of Δ on Σ_{Δ} .

\rightsquigarrow Let us fix x on Σ_{Δ} and see how its orbit look like.

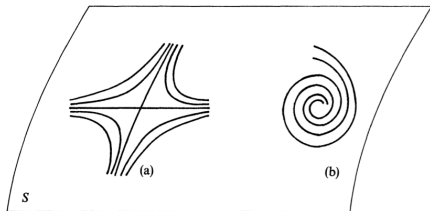
The Sard Conjecture on Martinet surfaces

Transverse case



The Sard Conjecture on Martinet surfaces

Generic tangent case (Zelenko-Zhitomirskii, 1995)



The Sard Conjecture on Martinet surfaces

Let M be of dimension 3 and Δ of rank 2.

$$\mathcal{S}_\Delta^x = \{\gamma(1) \mid \gamma : [0, 1] \rightarrow M, \gamma(0) = x, \gamma \text{ hor.}, \text{sing.}\}.$$

Conjecture (Sard Conjecture)

The set \mathcal{S}_Δ^x has vanishing \mathcal{H}^2 -measure.

Theorem (Belotto-R, 2016)

The above conjecture holds true under one of the following assumptions:

- *The Martinet surface is smooth.*
- *All datas are analytic and*

$$\Delta(x) \cap T_x \text{Sing}(\Sigma_\Delta) = T_x \text{Sing}(\Sigma_\Delta) \quad \forall x \in \text{Sing}(\Sigma_\Delta).$$

Ingredients of the proof

- Control of the divergence of vector fields which generates the trace of Δ over Σ_Σ of the form

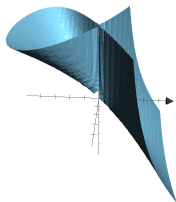
$$|\operatorname{div} \mathcal{Z}| \leq C |\mathcal{Z}|.$$

- Resolution of singularities.

An example

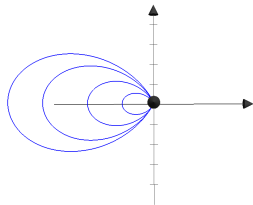
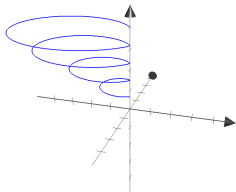
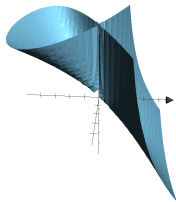
In \mathbb{R}^3 ,

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[\frac{y^3}{3} - x^2 y(x+z) \right] \partial_z.$$



$$\text{Martinet Surface: } \Sigma_{\Delta} = \left\{ y^2 - x^2(x+z) = 0 \right\}.$$

An example



Thank you for your attention !!