# Closing Aubry sets II

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**Abstract**

Given a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class $C^k$, with $k \geq 4$, we prove the following results: (1) Assume there is a critical viscosity subsolution which is of class $C^{k+1}$ in an open neighborhood of a positive orbit of a recurrent point of the projected Aubry set. Then, there exists a potential $V : M \to \mathbb{R}$ of class $C^{k-1}$, small in $C^2$ topology, for which the Aubry set of the new Hamiltonian $H + V$ is either an equilibrium point or a periodic orbit. (2) For every $\epsilon > 0$ there exists a potential $V : M \to \mathbb{R}$ of class $C^{k-2}$, with $\|V\|_{C^1} < \epsilon$, for which the Aubry set of the new Hamiltonian $H + V$ is either an equilibrium point or a periodic orbit. The latter result solves in the affirmative the Mañé density conjecture in $C^1$ topology.

## Contents



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# 1 Introduction

In this paper, the sequel of [8], we continue our investigation on how to close trajectories in the Aubry set by adding a small potential, as suggested by Mañé (see [11, 8]). More precisely, in [8] we proved the following: Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ $(k \geq 2)$ on a $n$-dimensional smooth compact Riemannian manifold without boundary $M$. Then we can "close" the Aubry set in the following cases:
(1) Assume there exist a recurrent point of the projected Aubry set $\bar{x}$, and a critical viscosity subsolution $u$, such that $u$ is a $C^1$ critical solution in an open neighborhood of the positive orbit of $\bar{x}$. Suppose further that $u$ is "$C^2$ at $\bar{x}$". Then, for any $\epsilon > 0$ there exists a potential $V : M \to \mathbb{R}$ of class $C^k$, with $\|V\|_{C^2} < \epsilon$, for which the Aubry set of the new Hamiltonian $H + V$ is either an equilibrium point or a periodic orbit.
(2) If $M$ is two dimensional, the above result holds replacing "$C^1$ critical solution + $C^2$ at $\bar{x}$" by "$C^3$ critical subsolution".

The aim of this paper is twofold: first of all, we want to extend (2) above to arbitrary dimension (Theorem 1.1 below), and to prove such a result, new techniques and ideas (with respect to the ones introduced in [8]) are needed. Then, as a by-product of these techniques, we will show the validity of the Mañé density Conjecture in $C^1$ topology (Theorem 1.2 below).

For convenience of the reader, we will recall through the paper the main notation and assumptions, referring to [8] for more details.

In the present paper, the space $M$ will be a smooth compact Riemannian manifold without boundary of dimension $n \geq 2$, and $H : T^*M \to \mathbb{R}$ a $C^k$ *Tonelli Hamiltonian* (with $k \geq 2$), that is, a Hamiltonian of class $C^k$ satisfying the two following properties:

(H1) *Superlinear growth:* For every $K \geq 0$, there is a finite constant $C^*(K)$ such that

$$H(x,p) \geq K\|p\|_x + C^*(K) \qquad \forall\,(x,p) \in T^*M.$$

(H2) *Strict convexity:* For every $(x,p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

We say that a continuous function $u : M \to \mathbb{R}$ is a *critical viscosity solution* (resp. *subsolution*) if $u$ is a viscosity solution (resp. subsolution) of the critical Hamilton-Jacobi equation

$$H\big(x, du(x)\big) = c[H] \qquad \forall\, x \in M, \tag{1.1}$$

where $c[H]$ denotes the critical value of $H$. Denoting by $\mathcal{SS}^1$ the set of critical subsolutions $u : M \to \mathbb{R}$ of class $C^1$, we recall that, thanks to the Fathi-Siconolfi Theorem [7] (see also [8, Subsection 1.2]), the Aubry set can be seen as the nonempty compact subset of $T^*M$ defined by

$$\tilde{\mathcal{A}}(H) := \bigcap_{u \in \mathcal{SS}^1} \Big\{ (x, du(x)) \,|\, x \in M \text{ s.t. } H(x, du(x)) = c[H] \Big\}.$$

Then the *projected Aubry set* $\mathcal{A}(H)$ can be defined for instance as $\pi^*\big(\tilde{\mathcal{A}}(H)\big)$, where $\pi^* : T^*M \to M$ denotes the canonical projection map. We refer the reader to our first paper [8] or to the monograph [5] for more details on Aubry-Mather theory.

As we said above, the aim of the present paper is to show that we can always close an Aubry set in $C^2$ topology if there is a critical viscosity subsolution which is sufficiently regular in a neighborhood of a positive orbit of a recurrent point of the projected Aubry set: Let $x \in \mathcal{A}(H)$, fix $u : M \to \mathbb{R}$ a critical viscosity subsolution, and denote by $\mathcal{O}^+(x)$ its positive orbit in the projected Aubry set, that is,

$$\mathcal{O}^+(x) := \Big\{ \pi^*\big(\phi_t^H(x, du(x))\big) \mid t \geq 0 \Big\}. \tag{1.2}$$

A point $x \in \mathcal{A}(H)$ is called *recurrent* if there is a sequence of times $\{t_k\}$ tending to $+\infty$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \pi^*\big(\phi_{t_k}^H(x, du(x))\big) = x.$$

As explained in [8, Section 2], since $x \in \mathcal{A}(M)$, both definitions of $\mathcal{O}^+(x)$ and of recurrent point do not depend on the choice of the subsolution $u$. From now on, given a potential $V : M \to \mathbb{R}$, we denote by $H_V$ the Hamiltonian $H_V(x,p) := H(x,p) + V(x)$. The following result extends [8, Theorem 2.4] to any dimension:

**Theorem 1.1.** *Assume that* $\dim M \geq 3$. *Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian of class* $C^k$ *with* $k \geq 4$, *and fix* $\epsilon > 0$. *Assume that there are a recurrent point* $\bar{x} \in \mathcal{A}(H)$, *a critical viscosity subsolution* $u : M \to \mathbb{R}$, *and an open neighborhood* $\mathcal{V}$ *of* $\mathcal{O}^+(\bar{x})$ *such that* $u$ *is at least* $C^{k+1}$ *on* $\mathcal{V}$. *Then there exists a potential* $V : M \to \mathbb{R}$ *of class* $C^{k-1}$, *with* $\|V\|_{C^2} < \epsilon$, *such that* $c[H_V] = c[H]$ *and the Aubry set of* $H_V$ *is either an equilibrium point or a periodic orbit.*

As a by-product of our method, we show that we can always close Aubry sets in $C^1$ topology:

**Theorem 1.2.** *Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian of class* $C^k$ *with* $k \geq 4$, *and fix* $\epsilon > 0$. *Then there exists a potential* $V : M \to \mathbb{R}$ *of class* $C^{k-2}$, *with* $\|V\|_{C^1} < \epsilon$, *such that* $c[H_V] = c[H]$ *and the Aubry set of* $H_V$ *is either an equilibrium point or a periodic orbit.*

Let us point out that in both results above we need more regularity on $H$ with respect to the assumptions in [8]. This is due to the fact that here, to connect Hamiltonian trajectories, we do a construction "by hand" where we explicitly define our connecting trajectory by taking a convex combination of the original trajectories and a suitable time rescaling (see Proposition 2.1). With respect to the "control theory approach" used in [8], this construction has the advantage of forcing the connecting trajectory to be "almost tangent" to the Aubry set, though we still need the results of [8] to control the action, see Subsection 4.4.

By Theorem 1.1 above and the same argument as in [8, Section 7], we see that the Mañé Conjecture in $C^2$ topology for smooth Hamiltonians (of class $C^\infty$) is equivalent to the[1]:

**Mañé regularity Conjecture for viscosity subsolutions.** For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class $C^\infty$ there is a set $\mathcal{D} \subset C^\infty(M)$ which is dense in $C^2(M)$ (with respect to the $C^2$ topology) such that the following holds: For every $V \in \mathcal{D}$, there are a recurrent point

[1] Although the "Mañé regularity Conjecture for viscosity subsolutions" could be stated as in [8, Section 7] using $C^k$ topologies, we prefer to state it with $C^\infty$ because the statement becomes simpler and nicer.

$\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$ such that $u$ is of class $C^\infty$ on $\mathcal{V}$.

The paper is organized as follows: In Section 2, we refine [8, Propositions 3.1 and 4.1] by proving that we can connect two Hamiltonian trajectories with small potential with a state constraint on the connecting trajectory. In Section 3, we prove a refined version of the Mai Lemma with constraints which is essential for the proof of Theorem 1.2. Then the proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5, respectively.

# 2 A connection result with constraints

## 2.1 Statement of the result

Let $n \geq 2$ be fixed. We denote a point $x \in \mathbb{R}^n$ either as $x = (x_1, \ldots, x_n)$ or in the form $x = (x_1, \hat{x})$, where $\hat{x} = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Let $\bar{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Hamiltonian[2] of class $C^k$, with $k \geq 2$, satisfying (H1), (H2), and the additional hypothesis

(H3) *Uniform boundedness in the fibers:* For every $R \geq 0$ we have

$$A^*(R) := \sup\Big\{\bar{H}(x,p) \,|\, |p| \leq R\Big\} < +\infty.$$

Note that, under these assumptions, the Hamiltonian $\bar{H}$ generates a flow $\phi_t^{\bar{H}}$ which is of class $C^{k-1}$ and complete (see [6, corollary 2.2]). Let $\bar{\tau} \in (0,1)$ be fixed. We suppose that there exists a solution

$$\big(\bar{x}(\cdot), \bar{p}(\cdot)\big) \,:\, \big[0, \bar{\tau}\big] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

of the Hamiltonian system

$$\left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & \nabla_p \bar{H}\big(\bar{x}(t), \bar{p}(t)\big) \\ \dot{\bar{p}}(t) & = & -\nabla_x \bar{H}\big(\bar{x}(t), \bar{p}(t)\big) \end{array} \right. \tag{2.1}$$

on $\big[0, \bar{\tau}\big]$ satisfying the following conditions:

(A1) $\bar{x}^0 = \big(0, \hat{\bar{x}}^0\big) := \bar{x}(0) = 0_n$ and $\dot{\bar{x}}(0) = e_1$;

(A2) $\bar{x}^{\bar{\tau}} = \big(\bar{\tau}, \hat{\bar{x}}^{\bar{\tau}}\big) := \bar{x}(\bar{\tau}) = \big(\bar{\tau}, 0_{n-1}\big)$ and $\dot{\bar{x}}(\bar{\tau}) = e_1$;

(A3) $\big|\dot{\bar{x}}(t) - e_1\big| < 1/2$ for any $t \in \big[0, \bar{\tau}\big]$;

(A4) $\det\Big(\frac{\partial^2 \bar{H}}{\partial \hat{p}^2}\big(\bar{x}^{\bar{\tau}}, \bar{p}^{\bar{\tau}}\big)\Big) + \bar{p}_1^{\bar{\tau}} \det\Big(\frac{\partial^2 \bar{H}}{\partial p^2}\big(\bar{x}^{\bar{\tau}}, \bar{p}^{\bar{\tau}}\big)\Big) \neq 0$ (where $\bar{p}^{\bar{\tau}} := \bar{p}\big(\bar{\tau}\big)$).

For every $(x^0, p^0) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $\bar{H}(x^0, p^0) = 0$, we denote by

$$\Big(X\big(\cdot\,; (x^0, p^0)\big), P\big(\cdot\,; (x^0, p^0)\big)\Big) \,:\, [0, +\infty) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

the solution of the Hamiltonian system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \nabla_p \bar{H}\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x \bar{H}\big(x(t), p(t)\big) \end{array} \right. \tag{2.2}$$

satisfying

$$x(0) = x^0 \quad \text{and} \quad p(0) = p^0.$$

[2] Note that we identify $T^*(\mathbb{R}^n)$ with $\mathbb{R}^n \times \mathbb{R}^n$. For that reason, throughout Section 2 the adjoint variable $p$ will always be seen as a vector in $\mathbb{R}^n$.

Since the curve $\bar{x}(\cdot)$ is transverse to the hyperplane $\Pi^{\bar\tau} := \big\{x = \big(\bar\tau, \hat{x}\big) \in \mathbb{R}^n\big\}$ at time $\bar\tau$, there is a neighborhood $\mathcal{V}^0$ of $\big(\bar{x}^0, \bar{p}^0 := \bar{p}(0)\big)$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that the Poincaré mapping $\tau : \mathcal{V}^0 \to \mathbb{R}$ with respect to the section $\Pi^{\bar\tau}$ is well-defined, that is, it is of class $C^{k-1}$ and satisfies

$$\tau\big(\bar{x}^0, \bar{p}^0\big) = \bar\tau \quad \text{and} \quad X_1\big(\tau(x^0, p^0); (x^0, p^0)\big) = \bar\tau \qquad \forall\, (x^0, p^0) \in \mathcal{V}^0. \tag{2.3}$$

Our aim is to show that, given $\big(x^1 = (0, \hat{x}^1), p^1\big)$ and $\big(x^2 = (0, \hat{x}^2), p^2\big)$ such that $\bar{H}(x^1, p^1) = \bar{H}(x^2, p^2) = 0$ which are both sufficiently close to $(\bar{x}^0, \bar{p}^0)$, there exists a time $T^f$ close to $\tau(x^1, p^1)$, together with a potential $V : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$ whose support and $C^2$-norm are controlled, such that the solution $\big(x(\cdot), p(\cdot)\big) : [0, T^f] \to \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian system

$$\begin{cases} \dot{x}(t) &= \nabla_p \bar{H}_V(x(t), p(t)) = \nabla_p \bar{H}(x(t), p(t)) \\ \dot{p}(t) &= -\nabla_x \bar{H}_V(x(t), p(t)) = -\nabla_x \bar{H}(x(t), p(t)) - \nabla V(x(t)) \end{cases} \tag{2.4}$$

starting at $\big(x(0), p(0)\big) = (x^1, p^1)$ satisfies

$$\big(x(T^f), p(T^f)\big) = \Big(X\big(\tau(x^2, p^2); (x^2, p^2)\big), P\big(\tau(x^2, p^2); (x^2, p^2)\big)\Big),$$

and $x(\cdot)$ is constrained inside a given "flat" set containing both curves

$$X\big(\,\cdot\,; (x^1, p^1)\big) \,:\, \big[0, \tau(x^1, p^1)\big] \longrightarrow \mathbb{R}^n \quad \text{and} \quad X\big(\,\cdot\,; (x^2, p^2)\big) \,:\, \big[0, \tau(x^2, p^2)\big] \longrightarrow \mathbb{R}^n.$$

(Roughly speaking, $x(\cdot)$ will be a convex combination of $X\big(\,\cdot\,; (x^1, p^1)\big)$ and $X\big(\,\cdot\,; (x^2, p^2)\big)$.)

We denote by $\bar{L} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Lagrangian associated to $\bar{H}$ by Legendre-Fenchel duality, and for every $(x^0, p^0) \in \mathbb{R}^n \times \mathbb{R}^n$, $T > 0$, and every $C^2$ potential $V : \mathbb{R}^n \to \mathbb{R}$, we denote by $\mathbb{A}_V\big((x^0, p^0); T\big)$ the action of the curve $\gamma : [0, T] \to \mathbb{R}^n$ defined as the projection (onto the $x$ variable) of the Hamiltonian trajectory $t \mapsto \phi_t^{\bar{H}_V}(x^0, p^0) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$, that is

$$\begin{aligned} \mathbb{A}_V\big((x^0, p^0); T\big) &:= \int_0^T \bar{L}_V \left(\pi^*\left(\phi_t^{\bar{H}_V}(x^0, p^0)\right), \frac{d}{dt}\left(\pi^*\left(\phi_t^{\bar{H}_V}(x^0, p^0)\right)\right)\right) dt \\ &= \int_0^T \bar{L} \left(\pi^*\left(\phi_t^{\bar{H}_V}(x^0, p^0)\right), \frac{d}{dt}\left(\pi^*\left(\phi_t^{\bar{H}_V}(x^0, p^0)\right)\right)\right) \\ &\qquad\qquad - V\left(\pi^*\left(\phi_t^{\bar{H}_V}(x^0, p^0)\right)\right) dt, \end{aligned}$$

where $\bar{L}_V = \bar{L} - V$ is the Lagrangian associated to $\bar{H}_V := \bar{H} + V$. Moreover, we denote by

$$\big(X_V\big(\,\cdot\,; (x^0, p^0)\big), P_V\big(\,\cdot\,; (x^0, p^0)\big)\big) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$$

the solution to the Hamiltonian system (2.4) starting at $(x^0, p^0)$. Finally, for every $r > 0$ we set

$$\mathcal{C}\Big((x^0, p^0); \tau(x^0, p^0); r\Big) := \Big\{X\big(t; (x^0, p^0)\big) + (0, \hat{y}) \,|\, t \in \big[0, \tau(x^0, p^0)\big], \, |\hat{y}| < r\Big\}, \tag{2.5}$$

and for every $x^f = \big(\bar\tau, \hat{x}^f\big)$,

$$\Delta\big((x^0, p^0); \tau(x^0, p^0); x^f\big) := \big\langle P\big(\tau(x^0, p^0); (x^0, p^0)\big), x^f - X\big(\tau(x^0, p^0); (x^0, p^0)\big)\big\rangle.$$

We also introduce the following sets, which measure how much our connecting trajectory leave the "surface" spanned by the trajectories $X\big(\,\cdot\,; (x^1, p^1)\big)$ and $X\big(\,\cdot\,; (x^2, p^2)\big)$: given $K_1, \eta > 0$ we define

$$\mathcal{R}^1\Big((x^1, p^1); (x^2, p^2); K_1\Big) := \mathcal{R}\Big((x^1, p^1); (x^2, p^2); K_1\Big) \cap \mathcal{E}^1, \tag{2.6}$$

$$\mathcal{B}^2\Big(\big(x^2,p^2\big);\eta\Big) := \mathcal{B}\Big(\big(x^2,p^2\big);\eta\Big)\cap\mathcal{E}^2, \tag{2.7}$$

where

$$\mathcal{R}\Big(\big(x^1,p^1\big);\big(x^2,p^2\big);K_1\Big) := \bigcup_{(t^1,t^2)\in\mathcal{K}}\Big[X\big(t^1;(x^1,p^1)\big),X\big(t^2;(x^2,p^2)\big)\Big] \tag{2.8}$$

(here and in the sequel, $[z^1,z^2]$ denotes the segment joining two points $z^1,z^2\in\mathbb{R}^n$),

$$\mathcal{K} := \Big\{\big(t^1,t^2\big)\,|\,\big|t^2-t^1\big| < K_1\big(|x^2-x^1|+|p^2-p^1|\big),\, t^j\in\big[0,\tau(x^j,p^j)\big],\, j=1,2\Big\}, \tag{2.9}$$

$$\mathcal{B}\Big(\big(x^2,p^2\big);\eta\Big) := \bigcup_{t\in[0,\tau(x^2,p^2)]}\Big\{z\,|\,\big|z-X\big(t;(x^1,p^1)\big)\big|\le\eta\Big\}, \tag{2.10}$$

$$\mathcal{E}^1 := \Big\{\big(t,\hat{z}\big)\,|\,t\in\big[0,\bar{\tau}/2\big],\,\hat{z}\in\mathbb{R}^{n-1}\Big\}, \qquad \mathcal{E}^2 := \Big\{\big(t,\hat{z}\big)\,|\,t\in\big[\bar{\tau}/2,\bar{\tau}\big],\,\hat{z}\in\mathbb{R}^{n-1}\Big\}. \tag{2.11}$$

We are now ready to state our result.

**Proposition 2.1.** *Let $\bar{H}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be a Hamiltonian of class $C^k$, with $k\ge 4$, satisfying (H1)-(H3), and let $\big(\bar{x}(\cdot),\bar{p}(\cdot)\big):\big[0,\bar{\tau}\big]\to\mathbb{R}^n\times\mathbb{R}^n$ be a solution of (2.2) satisfying (A1)-(A4) on both subintervals $\big[0,\bar{\tau}/2\big]$ and $\big[\bar{\tau}/2,\bar{\tau}\big]$, i.e., (A1)-(A4) hold both when we replace $\bar{\tau}$ by $\bar{\tau}/2$, and when replacing $0$ by $\bar{\tau}/2$ (with obvious notation). Moreover, assume that $\bar{H}(\bar{x}^0,\bar{p}^0)=0$.*

*Then there are $\bar{\delta},\bar{r},\bar{\epsilon}\in(0,1)$ with $B^{2n}\big((\bar{x}^0,\bar{p}^0),\bar{\delta}\big)\subset\mathcal{V}^0$, and $K>0$, such that the following property holds: For every $r\in\big(0,\bar{r}\big),\epsilon\in\big(0,\bar{\epsilon}\big),\sigma>0$, and every $x^1=\big(0,\hat{x}^1\big),x^2=\big(0,\hat{x}^2\big),p^1,p^2\in\mathbb{R}^n$ satisfying*

$$\big|\hat{x}^1\big|,\big|\hat{x}^2\big|,\big|p^1-\bar{p}^0\big|,\big|p^2-\bar{p}^0\big| < \bar{\delta}, \tag{2.12}$$

$$\big|x^1-x^2\big|,\big|p^1-p^2\big| < r\epsilon, \tag{2.13}$$

$$\bar{H}\big(x^1,p^1\big) = \bar{H}\big(x^2,p^2\big) = 0, \tag{2.14}$$

$$|\sigma| < r^2\epsilon, \tag{2.15}$$

*there exist a time $T^f>0$ and a potential $V:\mathbb{R}^n\to\mathbb{R}$ of class $C^{k-1}$ such that:*

*(i)* $\mathrm{Supp}(V)\subset\mathcal{C}\Big(\big(x^0,p^0\big);\tau(x^0,p^0);r\Big)$;

*(ii)* $\|V\|_{C^2} < K\epsilon$;

*(iii)* $\big|T^f-\tau(x^1,p^1)\big| < Kr\epsilon$;

*(iv)* $\phi^{\bar{H}_V}_{T^f}\big(x^1,p^1\big) = \phi^{\bar{H}}_{\tau(x^2,p^2)}\big(x^2,p^2\big)$;

*(v)* $\mathbb{A}_V\big((x^1,p^1);T^f\big) = \mathbb{A}\big((x^1,p^1);\tau(x^1,p^1)\big) + \Delta\left((x^1,p^1);\tau(x^1,p^1);X\big(\tau(x^2,p^2);(x^2,p^2)\big)\right) + \sigma$;

*(vi) for every $t\in[0,T^f]$,*

$$\begin{aligned} &X_V\left(t;\big(x^1,p^1\big)\right) \\ &\quad\in\mathcal{R}^1\Big(\big(x^1,p^1\big);\big(x^2,p^2\big);K\Big)\cup\mathcal{B}^2\Big(\big(x^2,p^2\big);K\left(\big|(x^2,p^2)-(x^1,p^1)\big|^2+|\sigma|\right)\Big). \end{aligned}$$

As we will see in the next subsection, the proof of Proposition 2.1 offers an alternative proof for [8, Proposition 3.1] in the case of Hamiltonians of class at least $C^4$. Before giving the proof, we recall that the Lagrangian $\bar{L}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ associated with $\bar{H}$ by Legendre-Fenchel duality has the same regularity as $\bar{H}$ and satisfies:

$$p=\nabla_v\bar{L}(x,v)\quad\Longleftrightarrow\quad v=\nabla_p\bar{H}(x,p)\tag{2.16}$$

for all $x,v,p\in\mathbb{R}^n$.

## 2.2 Proof of Proposition 2.1

First, let us forget about assertion (v). That is, we will first show how to connect two Hamiltonian trajectories by a potential of class $C^{k-1}$ satisfying assertions (i)-(iv) and "to some extent" (vi), and then we will take care of (v).

For every $x\in\mathbb{R}^n$, denote by $S(x)\subset\mathbb{R}^n$ the set of vectors $p\in\mathbb{R}^n$ such that $\bar{H}(x,p)=0$, and define

$$\Lambda(x):=\Big\{\nabla_p\bar{H}(x,p)\,|\,p\in S(x)\Big\}.$$

Then we define the function $\lambda_x:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ by

$$\lambda_x(v):=\inf\Big\{s>0\,|\,sv\in\Lambda(x)\Big\}\qquad\forall\,v\in\mathbb{R}^n\setminus\{0\},$$

so that by (2.16) we have

$$\bar{H}\left(x,\nabla_v\bar{L}\big(x,\lambda_x(v)v\big)\right)=0\qquad\forall\,x\in\mathbb{R}^n,\ v\in\mathbb{R}^n\setminus\{0\}.\tag{2.17}$$

Consider now the map

$$\mathcal{H}\,:\,(x,v,\lambda)\longmapsto\bar{H}\left(x,\nabla_v\bar{L}(x,\lambda v)\right).$$

We observe that it is of class $C^{k-1}$, and since by assumption $\bar{H}(\bar{x}^0,\bar{p}^0)=0$ we have

$$\mathcal{H}\big(\bar{x}(t),\dot{\bar{x}}(t),1\big)=\bar{H}\left(\bar{x}(t),\nabla_v\bar{L}\big(\bar{x}(t),\dot{\bar{x}}(t)\big)\right)=\bar{H}\left(\bar{x}(t),\bar{p}(t)\right)=0\qquad\forall\,t\in\big[0,\bar{\tau}\big].$$

Moreover, by uniform convexity of $\bar{L}$ in the $v$ variable and (A3),

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\lambda}\big(\bar{x}(t),\dot{\bar{x}}(t),1\big)&=\left\langle\nabla_p\bar{H}\big(\bar{x}(t),\bar{p}(t)\big),\frac{\partial^2\bar{L}}{\partial v^2}\big(\bar{x}(t),\dot{\bar{x}}(t)\big)\dot{\bar{x}}(t)\right\rangle\\&=\left\langle\dot{\bar{x}}(t),\frac{\partial^2\bar{L}}{\partial v^2}\big(\bar{x}(t),\dot{\bar{x}}(t)\big)\dot{\bar{x}}(t)\right\rangle>0.\end{aligned}$$

Therefore, there exist $\mathcal{V}$ an open neighborhood of the set

$$\Big\{\big(\bar{x}(t),\dot{\bar{x}}(t)\big)\,|\,t\in\big[0,\bar{\tau}\big]\Big\}\subset\mathbb{R}^n\times\mathbb{R}^n$$

and a function $\lambda:\mathcal{V}\to(1/2,3/2)$ of class $C^{k-1}$ such that

$$\mathcal{H}\big(x,v,\lambda(x,v)\big)=0\qquad\forall\,(x,v)\in\mathcal{V}.$$

By uniform convexity of the sets $\Lambda(x)$ and by (2.17), we deduce

$$\lambda_x(v)=\lambda(x,v)\qquad\forall\,(x,v)\in\mathcal{V}.$$

Now, let us fix a smooth function $\phi:[0,1]\to[0,1]$ satisfying

$$\phi(s)=0\quad\text{for }s\in[0,1/3],\qquad\phi(s)=1\quad\text{for }s\in[2/3,1],$$

and fix $\mathcal{W}^0 \subset \mathcal{V}^0$ an open neighborhood of $\big(\bar{x}^0, \bar{p}^0\big)$ such that

$$\phi_t^{\bar{L}}\big(x, \nabla_v \bar{L}(x,p)\big) \in \mathcal{V} \qquad \forall\, t \in \big[0, \tau(x,p)\big],\, \forall\, (x,p) \in \mathcal{W}^0.$$

Given $x^1 = \big(0, \hat{x}^1\big), x^2 = \big(0, \hat{x}^2\big), p^1, p^2 \in \mathbb{R}^n$ such that $\big(x^1, p^1\big), \big(x^1, p^2\big) \in \mathcal{W}^0$ and $\bar{H}\big(x^1, p^1\big) = \bar{H}\big(x^2, p^2\big) = 0$, we set

$$\tau^1 := \tau\big(x^1, p^1\big), \quad \tau^2 := \tau\big(x^2, p^2\big),$$

$$\left\{ \begin{array}{lcl} x^1(t) & := & X\big(t; (x^1, p^1)\big) \\ p^1(t) & := & P\big(t; (x^1, p^1)\big) \end{array} \right. \quad \forall\, t \in [0, \tau^1], \qquad \left\{ \begin{array}{lcl} x^2(t) & := & X\big(t; (x^2, p^2)\big) \\ p^2(t) & := & P\big(t; (x^2, p^2)\big) \end{array} \right. \quad \forall\, t \in [0, \tau^2].$$

Then we define a trajectory $y(\cdot) : \big[0, \tau^1\big] \to \mathbb{R}^n$ of class $C^k$ which connects $x^1(0)$ to $x^2(\tau^2)$:

$$y(t) := \left(1 - \phi\left(\frac{t}{\tau^1}\right)\right) x^1(t) + \phi\left(\frac{t}{\tau^1}\right) x^2\left(\frac{\tau^2}{\tau^1} t\right) \qquad \forall\, t \in \big[0, \tau^1\big]. \tag{2.18}$$

We observe that, a priori, the above curve will not be the projection of a Hamiltonian trajectory of (2.4) for some potential $V$. However, we can slightly modify it so that it becomes a Hamiltonian trajectory of (2.4) for a suitable $V$ which will be constructed below.

To achieve this, let $\alpha : \big[0, \tau^1\big] \to [0, +\infty)$ be defined as

$$\alpha(t) := \int_0^t \frac{1}{\lambda_{y(s)}\big(\dot{y}(s)\big)}\, ds \qquad \forall\, t \in \big[0, \tau^1\big]. \tag{2.19}$$

We observe that $\alpha$ is strictly increasing and of class $C^k$. Let $\theta : \big[0, T^f := \alpha(\tau^1)\big] \to \big[0, \tau^1\big]$ denote its inverse, which is of class $C^k$ as well, and satisfies

$$\dot{\theta}(t) = \lambda_{y(\theta(t))}\big(\dot{y}(\theta(t))\big) \qquad \forall\, t \in [0, T^f].$$

Then, we define a new trajectory $x(\cdot) : [0, T^f] \to \mathbb{R}^n$ of class $C^k$ connecting $x^1(0)$ to $x^2(\tau^2)$:

$$x(t) := y\big(\theta(t)\big) \qquad \forall\, t \in [0, T^f]. \tag{2.20}$$

We claim that $x(t)$ is the projection of a Hamiltonian trajectory of (2.4) for some potential $V$ satisfying (i)-(ii). Indeed, first of all we have

$$\dot{x}(t) = \dot{\theta}(t)\dot{y}\big(\theta(t)\big) = \lambda_{y(\theta(t))}\big(\dot{y}(\theta(t))\big)\dot{y}\big(\theta(t)\big) \in \Lambda\left(y\big(\theta(t)\big)\right) = \Lambda\left(x(t)\right) \qquad \forall\, t \in [0, T^f],$$

which means that the adjoint trajectory $p(\cdot) : [0, T] \to \mathbb{R}^n$ (of class $C^{k-1}$) given by

$$p(t) := \nabla_v \bar{L}\big(x(t), \dot{x}(t)\big) \qquad \forall\, t \in [0, T^f],$$

satisfies

$$\dot{x}(t) = \nabla_p \bar{H}\big(x(t), p(t)\big), \quad \bar{H}\big(x(t), p(t)\big) = 0 \qquad \forall\, t \in [0, T^f]. \tag{2.21}$$

We now define the function $u : [0, T^f] \to \mathbb{R}^n$ (of class $C^{k-2}$) by

$$\begin{array}{rcl} u(t) & := & -\dot{p}(t) - \nabla_x \bar{H}\big(x(t), p(t)\big) \\ & = & -\dfrac{\partial^2 \bar{L}}{\partial x \partial v}\big(x(t), \dot{x}(t)\big) \cdot \dot{x}(t) - \dfrac{\partial^2 \bar{L}}{\partial v^2}\big(x(t), \dot{x}(t)\big) \cdot \ddot{x}(t) \\ & & \qquad -\nabla_x \bar{H}\Big(x(t), \nabla_v \bar{L}\big(x(t), \dot{x}(t)\big)\Big). \end{array} \tag{2.22}$$

By construction we have

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & \nabla_p \bar{H}(x(t), p(t)) \\ \dot{p}(t) & = & -\nabla_x \bar{H}(x(t), p(t)) - u(t), \end{array} \right. \tag{2.23}$$

$$\big(x(0),p(0)\big)=\big(x^1,p^1)\big), \qquad \big(x(T^f),p(T^f)\big)=\big(x^2(\tau^2),p^2(\tau^2)\big). \tag{2.24}$$

As in the proof of [8, Proposition 3.1], we now want to show that assertion (iii) is satisfied, and that we can construct a potential $V$ such that $\nabla V(x(t))=u(t)$, and which satisfies both assertions (i) and (ii). To this aim, we first compute the first derivative of $u$ on $[0,T^f]$:

$$\begin{aligned}
\dot u(t) \;=\; & -\frac{\partial^3\bar L}{\partial x^2\partial v}\big(x(t),\dot x(t)\big)\cdot\dot x(t)\cdot\dot x(t)-2\frac{\partial^3\bar L}{\partial x\partial v^2}\big(x(t),\dot x(t)\big)\cdot\ddot x(t)\cdot\dot x(t)\\
& -\frac{\partial^2\bar L}{\partial x\partial v}\big(x(t),\dot x(t)\big)\cdot\ddot x(t)-\frac{\partial^3\bar L}{\partial v^3}\big(x(t),\dot x(t)\big)\cdot\ddot x(t)\cdot\ddot x(t)\\
& -\frac{\partial^2\bar L}{\partial v^2}\big(x(t),\dot x(t)\big)\cdot x^{(3)}(t)-\frac{\partial^2\bar H}{\partial x^2}\Big(x(t),\nabla_v\bar L\big(x(t),\dot x(t)\big)\Big)\cdot\dot x(t)\\
& -\frac{\partial^2\bar H}{\partial p\partial x}\Big(x(t),\nabla_v\bar L\big(x(t),\dot x(t)\big)\Big)\left[\frac{\partial^2\bar L}{\partial x\partial v}\big(x(t),\dot x(t)\big)\cdot\dot x(t)+\frac{\partial^2\bar L}{\partial v^2}\big(x(t),\dot x(t)\big)\cdot\ddot x(t)\right].
\end{aligned}$$

Now, let $\mathcal S^0$ be the subset of $\mathcal W^0$ defined by

$$\mathcal S^0 := \Big\{\big(x^0,p^0\big)\in\mathcal W^0 \,|\, x^0=\big(0,\hat x^0\big),\ \bar H\big(x^0,p^0\big)=0\Big\},$$

which we can assume to be an open submanifold of $\mathbb R^{2n}$ of dimension $2n-2$ and of class $C^k$. Since $\bar H$ (and so also $\bar L$) is of class $C^k$ with $k\ge 4$, it is easily checked that the mapping

$$\begin{array}{cccc}
\mathcal Q: & \mathcal S^0\times\mathcal S^0\times[0,1] & \longrightarrow & \mathbb R\times\mathbb R\times\mathbb R^n\times\mathbb R^n\\
& \big(\big(x^1,p^1\big),\big(x^2,p^2\big),s\big) & \longmapsto & \big(T^f,\theta(sT^f)-s\tau^1,u\big(sT^f\big),\dot u(sT^f)\big)
\end{array}$$

is of class $C^1$ (recall that $T^f=\alpha(\tau^1)$, where $\tau^1=\tau(x^1,p^1)$ and $\alpha$ was defined in (2.19)). Therefore, since

$$\mathcal Q\left(\big(x^0,p^0\big),\big(x^0,p^0\big),s\right)=\big(\tau\big(x^0,p^0\big),0,0,0\big) \qquad \forall\, s\in[0,1],\ \forall\,\big(x^0,p^0\big)\in\mathcal S^0,$$

(as in this case $\lambda_{y(t)}(\dot y(t))\equiv 1$), there exists a constant $K>0$ such that, for every pair $\big(x^1,p^1\big),\big(x^2,p^2\big)\in\mathcal S^0$, it holds

$$\begin{aligned}
\big|T^f-\tau^1\big| &\le \left|\mathcal Q\left(\big(x^1,p^1\big),\big(x^2,p^2\big),0\right)-\mathcal Q\left(\big(x^1,p^1\big),\big(x^1,p^1\big),0\right)\right|\\
&\le K\left(\big|x^2-x^1\big|+\big|p^2-p^1\big|\right),
\end{aligned} \tag{2.25}$$

and analogously

$$\left|\theta(t)-\frac{\tau^1}{T^f}t\right|\le K\left(\big|x^2-x^1\big|+\big|p^2-p^1\big|\right) \qquad \forall\, t\in\big[0,T^f\big], \tag{2.26}$$

$$\big\|u\big\|_{C^1}\le K\left(\big|x^2-x^1\big|+\big|p^2-p^1\big|\right). \tag{2.27}$$

Furthermore, we notice that differentiating the second equality in (2.21) yields

$$\big\langle\nabla_x\bar H\big(x(t),p(t)\big),\dot x(t)\big\rangle+\big\langle\nabla_p\bar H\big(x(t),p(t)\big),\dot p(t)\big\rangle=0 \qquad \forall\, t\in[0,T^f],$$

which together with the first equality in (2.21) and with (2.22) gives

$$\big\langle u(t),\dot x(t)\big\rangle=0 \qquad \forall\, t\in[0,T^f]. \tag{2.28}$$

We observe that inequality (2.25) proves assertion (iii), while (2.26) yields

$$x(t)\in\mathcal R\Big(\big(x^1,p^1\big);\big(x^2,p^2\big);K\Big) \qquad \forall\, t\in[0,T^f], \tag{2.29}$$

that is the first part of (vi). Furthermore, inequality (2.27) is reminiscent of [8, Equation (3.36)], while (2.28) corresponds [8, Equation (3.37)]. Hence, as in the proof of [8, Proposition 3.1] we can apply [8, Lemma 3.3] together with (2.23) and (2.24) to deduce the existence of $\bar{\delta}, \bar{\rho}, \bar{\epsilon} \in (0,1)$ small, and a constant $K > 0$, such that for every pair $\left(x^1, p^1\right), \left(x^2, p^2\right) \in \mathcal{S}^0$ satisfying (2.12)-(2.14) there exist a time $T^f > 0$ and a potential $V : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$ such that assertions (i)-(iv) of Proposition 2.1 hold, and morever (2.29) is satisfied.

Now, it remains to control the action, and to achieve this we proceed as in the proof of [8, Proposition 5.2]: first we divide the interval $\left[0, \bar{\tau}\right]$ into two subintervals $\left[0, \bar{\tau}/2\right]$ and $\left[\bar{\tau}/2\bar{\tau}\right]$. Then we use the construction above on $\left[0, \bar{\tau}/2\right]$ to connect

$$\left(x^1, p^1\right) \quad \text{to} \quad \phi^{\bar{H}}_{\tau_{1/2}(x^2,p^2)}\left(x^2, p^2\right)$$

on some time interval $[0, T_1^f]$ with $T_1^f \sim \bar{\tau}/2$, where $\tau_{1/2}$ denotes the Poincaré mapping with respect to the hyperplane $\Pi^{\bar{\tau}/2} := \left\{x = \left(\bar{\tau}/2, \hat{x}\right) \in \mathbb{R}^n\right\}$. As in [8, Proposition 3.1(v)] (see in particular [8, Remark 3.4]), one can show that the action default is quadratic, that is,

$$\begin{aligned} \Big|\mathbb{A}_V\left((x^1,p^1);T_1^f\right) \quad &- \quad \mathbb{A}\left((x^1,p^1);\tau_{1/2}(x^1,p^1)\right) \\ &- \quad \Delta\left((x^1,p^1);\tau_{1/2}(x^1,p^1);X\left(\tau_{1/2}(x^2,p^2);(x^2,p^2)\right)\right)\Big| \\ &\leq \quad K\left|\phi^{\bar{H}}_{\tau_{1/2}(x^2,p^2)}(x^2,p^2) - \phi^{\bar{H}}_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\right|^2 \leq \tilde{K}\left|(x^2,p^2)-(x^1,p^1)\right|^2 \end{aligned} \tag{2.30}$$

for some uniform constant $\tilde{K} > 0$. Hence, up to choosing $\bar{\epsilon}$ sufficiently small so that $\tilde{K}\bar{\epsilon} \leq 1$, we can apply [8, Proposition 4.1] to connect

$$\phi^{\bar{H}}_{\tau_{1/2}(x^2,p^2)}\left(x^2, p^2\right) \quad \text{to} \quad \phi^{\bar{H}}_{\tau(x^2,p^2)}\left(x^2, p^2\right),$$

and, at the same time, fit the action by an amount $\sigma + O\left(\left|(x^2,p^2)-(x^1,p^1)\right|^2\right)$ so that (v) holds. We observe that [8, Equation (4.19)] shows that the potential $\tilde{V}$ needed to achieve this second step (which is constructed again using [8, Lemma 3.3]) satisfies the bound $\|\nabla\tilde{V}\|_\infty \leq \bar{K}\left(\left|(x^2,p^2)-(x^1,p^1)\right|^2 + |\sigma|\right)$. Thus, a simple Gronwall argument shows that this construction produces a connecting trajectory $X_V\left(\,\cdot\,;\left(x^1,p^1\right)\right) : [0,T] \to \mathbb{R}^n$ which satisfies (2.29) on the first interval $\left[0, T_1^f\right]$, and

$$X_V\left(t;\left(x^1,p^1\right)\right) \in \mathcal{B}^2\left((x^2,p^2);K'\left(\left|(x^2,p^2)-(x^1,p^1)\right|^2+|\sigma|\right)\right) \qquad \forall\, t \in \left[T_1^f, T^f\right],$$

for some uniform constant $K' > 0$.

This concludes the proof of Proposition 2.1.

## 2.3 A refined connecting result with constraints

Our aim is now to obtain a refined version of Proposition 2.1, where:

1) $\bar{\epsilon} \in (0,1)$ is not necessarily small;

2) the support of $V$ is still contained in a "cylinder" around the initial trajectory (see Proposition 2.1(i)), but now the section of the cylinder is a given convex set which is not a ball.

Indeed, this refined version is a key step in the proof of Theorem 1.2.

Given two points $y_1, y_2 \in \mathbb{R}^{n-1}$ and $\lambda > 0$, we denote by $\mathrm{Cyl}_0^\lambda\big(y_1; y_2\big) \subset \mathbb{R}^{n-1}$ the convex set defined by

$$
\begin{aligned}
\mathrm{Cyl}_0^\lambda\big(y_1; y_2\big) \quad &:= \quad \bigcup_{s\in[0,1]} B^{n-1}\Big((1-s)y_1 + sy_2, \lambda\big|y_1 - y_2\big|\Big) \\
&= \quad \Big\{y \in \mathbb{R}^{n-1} \,|\, \mathrm{dist}\big(y, [y_1, y_2]\big) < \lambda\big|y_1 - y_2\big|\Big\},
\end{aligned}
\tag{2.31}
$$

where $\mathrm{dist}\big(\,\cdot\,, [y_1, y_2]\big)$ denotes the distance function to the segment $[y_1, y_2]$. Let $\Pi^0$ denote the hyperplane $\Pi^0 := \big\{x = \big(0, \hat{x}\big) \in \mathbb{R}^n\big\}$. If $\bar{u} : \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^{1,1}$, then for every $x^1, x^2 \in \Pi^0$ and $\lambda > 0$ small enough, we define the set $\mathrm{Cyl}_{[0,\bar{\tau}]}^\lambda\big(x^1; x^2\big) \subset \mathbb{R}^n$ as

$$
\begin{aligned}
&\mathrm{Cyl}_{[0,\bar{\tau}]}^\lambda\big(x^1; x^2\big) \\
&\qquad := \Big\{X\big(t; (x, \nabla\bar{u}(x))\big) \,|\, x = (0, \hat{x}) \in \Pi^0,\ \hat{x} \in \mathrm{Cyl}_0^\lambda\big(\hat{x}^1, \hat{x}^2\big),\ t \in \big[0,\ \tau\big(x, \nabla\bar{u}(x)\big)\big]\Big\}.
\end{aligned}
$$

(Recall that $\tau(\cdot,\cdot)$ denotes the Poincaré mapping with respect to $\Pi^{\bar{\tau}}$, see (2.3).) Observe that this definition of "cylinder" is slightly different from the one in (2.5). Indeed, in (2.5) we were considering, for every time $t \geq 0$, a $(n-1)$-dimensional ball around the trajectory $X\big(t; (x^0, p^0)\big)$. Here, we take a $(n-1)$-dimensional convex set around the segment $[\hat{x}^1, \hat{x}^2]$ at time $t = 0$ and we let it flow. The reason for this choice is the following: since $\bar{\epsilon}$ will not be assumed to be small (or equivalently, $\lambda$ will not be assumed to be large), the trajectories starting from the two points $x^1$ and $x^2$ which we want to connect could exit from a cylinder like the one in (2.5). Hence, the definition of $\mathrm{Cyl}_{[0,\bar{\tau}]}^\lambda\big(x^1; x^2\big)$ ensures that both trajectories (and also the connecting one) will remain inside it.

Finally, given $x^1, x^2 \in \Pi^0$ and $\lambda > 0$ small enough, we also define an analogous version of $\mathcal{C}$ as in (2.5):

$$
\begin{aligned}
\mathcal{C}_{[0,\bar{\tau}]}^\lambda\big(x^1; x^2\big) := \Bigg\{X\left(t; \left(\frac{x^1 + x^2}{2}, \nabla\bar{u}\left(\frac{x^1 + x^2}{2}\right)\right)\right) + (0, \hat{y}) \,\Bigg|& \\
t \in \left[0, \tau\left(\frac{x^1 + x^2}{2}, \nabla\bar{u}\left(\frac{x^1 + x^2}{2}\right)\right)\right],\ \hat{y} \in \mathrm{Cyl}_0^\lambda\big(\hat{x}^1, \hat{x}^2\big)&\Bigg\}.
\end{aligned}
$$

We are now ready to state our refinement of Proposition 2.1.

**Proposition 2.2.** *Let $\bar{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 4$, satisfying (H1)-(H3), and let $\big(\bar{x}(\cdot), \bar{p}(\cdot)\big) : \big[0, \bar{\tau}\big] \to \mathbb{R}^n \times \mathbb{R}^n$ be a solution of (2.2) satisfying (A1)-(A4) on both subintervals $\big[0, \bar{\tau}/2\big]$ and $[\bar{\tau}/2, \bar{\tau}]$. Let $\mathcal{U}$ be an open neighborhood of the curve $\bar{\Gamma} := \bar{x}\big(\big[0, \bar{\tau}\big]\big)$ and $\bar{u} : \mathcal{U} \to \mathbb{R}$ be a function of class $C^{1,1}$ such that*

$$
\bar{H}\big(x, \nabla\bar{u}(x)\big) \leq 0 \qquad \forall\, x \in \mathcal{U}. \tag{2.32}
$$

*Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in (0, 1)$ be such that*

$$
\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5, \tag{2.33}
$$

*and assume that for any $x^1 = \big(0, \hat{x}^0\big), x^2 = \big(0, \hat{x}^2\big) \in \Pi^0$ with $\Big(\{0\} \times \mathrm{Cyl}_0^{\lambda_5}\big(\hat{x}^1; \hat{x}^2\big)\Big) \subset \mathcal{U}$, the following inclusions hold:*

$$
\mathrm{Cyl}_{[0,\bar{\tau}]}^{\lambda_1}\big(x^1; x^2\big) \subset \mathcal{C}_{[0,\bar{\tau}]}^{\lambda_2}\big(x^1; x^2\big), \tag{2.34}
$$

$$
\mathcal{C}_{[0,\bar{\tau}]}^{\lambda_3}\big(x^1; x^2\big) \subset \mathrm{Cyl}_{[0,\bar{\tau}]}^{\lambda_4}\big(x^1; x^2\big). \tag{2.35}
$$

*Then there are* $\bar{\delta}, \bar{r} \in (0,1)$ *and* $K > 0$ *such that the following property holds: For any* $r \in \big(0, \bar{r}\big)$ *and any* $x^1 = \big(0, \hat{x}^1\big), x^2 = \big(0, \hat{x}^2\big) \in \Pi^0$ *satisfying*

$$\big|\hat{x}^1\big|, \big|\hat{x}^2\big| < \bar{\delta}, \tag{2.36}$$

$$\big|x^1 - x^2\big| < r, \tag{2.37}$$

$$\bar{H}\Big(x^j(t), \nabla\bar{u}\big(x^j(t)\big)\Big) = 0 \qquad \forall\, t \in \big[0, \tau\big(x^j, p^j\big)\big],\ j = 1, 2, \tag{2.38}$$

*with*

$$p^j := \nabla\bar{u}(x^j), \quad x^j(t) := X\Big(t; \big(x^j, p^j\big)\Big) \qquad \forall\, t \in \big[0, \tau\big(x^j, \nabla\bar{u}(x^j)\big)\big],\ j = 1, 2,$$

*there exist a time* $T^f > 0$ *and a potential* $V : \mathbb{R}^n \to \mathbb{R}$ *of class* $C^{k-1}$ *such that:*

*(i)* $\mathrm{Supp}(V) \subset \mathrm{Cyl}^{\lambda_4}_{[0,\bar{\tau}]}\big(x^1; x^2\big)$*;*

*(ii)* $\|V\|_{C^2} < K$*;*

*(iii)* $\|V\|_{C^1} < Kr$*;*

*(iv)* $\big|T^f - \tau(x^1, p^1)\big| < Kr$*;*

*(v)* $\phi^{\bar{H}_V}_{T^f}\left(x^1, p^1\right) = \phi^{\bar{H}}_{\tau(x^2,p^2)}\big(x^2, p^2\big)$*;*

*(vi) for any* $\tau \in [0, \bar{\tau}]$*,* $t \in [0, \tau(x^1, p^1)]$ *and* $t_V \in [0, T^f]$ *such that*

$$X_V\big(t_V; (x^1, p^1)\big),\ X\big(t; (x^1, p^1)\big) \in \Pi^\tau,$$

*it holds:* $|t_V - t| \le K\left|x^1 - x^2\right|$ *and*

$$\Big|\mathbb{A}_V\left(\left(x^1, p^1\right); t_V\right) - \mathbb{A}\left(\left(x^1, p^1\right); t\right) \\ - \big\langle \nabla\bar{u}\big(X\big(t; (x^1, p^1)\big)\big), X_V\big(t_V; (x^1, p^1)\big) - X\big(t; (x^1, p^1)\big)\big\rangle\Big| \le K\left|x^1 - x^2\right|^2;$$

*(vii)* $\mathbb{A}_V\left(\left(x^1, p^1\right); T^f\right) = \bar{u}\Big(\pi^*\big(\phi^{\bar{H}}_{\tau(x^2,p^2)}(x^2, p^2)\big)\Big) - \bar{u}\big(x^1\big)$.

*Proof of Proposition 2.2.* We proceed as in the proof of Proposition 2.1. First of all, we forget about assertions (vi) and (vii). By the construction that we performed in the first part of the proof of Proposition 2.1 (when we connected the two trajectories, without taking care of the action), there are $K_1, \bar{\delta} > 0$ such that, for any $x^1, x^2 \in \Pi^0$ and any $p^1, p^2 \in \mathbb{R}^n$ with

$$\big|\hat{x}^1\big|, \big|\hat{x}^2\big|, \big|p^1 - \bar{p}^0\big|, \big|p^2 - \bar{p}^0\big| < \bar{\delta} \tag{2.39}$$

and

$$\bar{H}\big(x^1, p^1\big) = \bar{H}\big(x^2, p^2\big) = 0, \tag{2.40}$$

there exist a time $T^f > 0$, a curve $x(\cdot) : [0, T^f] \to \mathbb{R}^n$ of class $C^k$, and a function $u : [0, T^f] \to \mathbb{R}^n$ of class $C^{k-2}$, such that the following properties are satisfied (see the proof of Proposition 2.1, up to Equation (2.29)):

(a) $x(t) = X\left(t; (x^1, p^1)\right)$, for every $t \in \big[0, \bar{\delta}\big]$;

(b) $x(t) = X\left(t; (x^2, p^2)\right)$, for every $t \in \big[T^f - \bar{\delta}, T^f\big]$;

(c) $u = 0$ on $\big[0, \bar{\delta}\big] \cup \big[T^f - \bar{\delta}, T^f\big]$;

(d) $\big|T^f - \tau^1\big| < K_1 \left(\big|x^2 - x^1\big| + \big|p^2 - p^1\big|\right)$;

(e) $\big\|u\big\|_{C^1} < K_1 \left(\big|x^2 - x^1\big| + \big|p^2 - p^1\big|\right)$;

(f) $\big\langle u(t), \dot{x}(t)\big\rangle = 0$, for every $t \in [0, T^f]$;

(g) $x(t) \in \mathcal{R}\left(\big(x^1, p^1\big); \big(x^2, p^2\big); K_1\right)$ for every $t \in [0, T^f]$.

Fix $x^1 \neq x^2 \in \Pi^0$ satisfying (2.36)-(2.37) for some $r \in (0, \bar{r})$ where $\bar{r}$ will be choosen later. Set

$$x^0 := \frac{x^1 + x^2}{2}, \qquad p^0 := \nabla \bar{u}(x^0), \qquad v := \frac{x^2 - x^1}{\big|x^2 - x^1\big|}. \tag{2.41}$$

Define the trajectories $X^0(\cdot), X^1(\cdot), X^2(\cdot) : [0, +\infty) \to \mathbb{R}^n$ by

$$X^i(t) := X\big(t; (x^i, p^i)\big) \qquad \forall\, t \geq 0,\ i = 0, 1, 2.$$

By the construction performed in the proof of Proposition 2.1, for $|x^2 - x^1|$ small enough there exist a constant $K_2 > 0$ (depending on the Lipschitz constant of $\nabla \bar{u}$) and three functions $\nu, t^1, t^2 : [0, T^f] \to [0, 1]$ such that

$$x(t) = \nu(t) X^1\big(t^1(t)\big) + \big(1 - \nu(t)\big) X^2\big(t^2(t)\big) \qquad \forall\, t \in [0, T^f]$$

and

$$\big|t^2(t) - t^1(t)\big| < K_2 \big|x^2 - x^1\big| \qquad \forall\, t \in [0, T^f]. \tag{2.42}$$

Now, for every $i = 1, \cdots, 4$, denote by $\mathcal{N}_i^v$ the norm on $\mathbb{R}^{n-1}$ whose unit ball is given by

$$B_1^{\mathcal{N}_i^v} := \Big\{ y \in \mathbb{R}^{n-1} \,|\, \mathcal{N}_i^v(y) < 1 \Big\} = \mathrm{Cyl}_0^{\lambda_i}\left(-\frac{v}{2}; \frac{v}{2}\right),$$

with $v$ defined in (2.41). Then

$$\mathcal{N}_i^v(v) = \frac{1}{\frac{1}{2} + \lambda_i} = \frac{2}{1 + 2\lambda_i} \leq 2,$$

and by (2.33)

$$\mathcal{N}_4^v < \mathcal{N}_3^v < \mathcal{N}_2^v < \mathcal{N}_1^v.$$

Let us observe that the map $t \mapsto X_1^0(t) = X^0(t) \cdot e_1$ is strictly increasing, so we can define the $C^k$ function $\theta$ by the relation

$$X_1^0\big(\theta(s)\big) = s \qquad \forall\, s \geq 0.$$

By construction, there holds

$$x(t),\ X^0\big(\theta(x_1(t))\big) \in \Pi^{x_1(t)} := \Pi^0 + x_1(t) e_1 \qquad \forall\, t \in [0, T^f].$$

Let $t \in [0, T^f]$ be fixed. We have

$$\begin{aligned}
& \mathcal{N}_2^v\Big(\hat{x}(t) - \hat{X}^0\big(\theta(x_1(t))\big)\Big) \\
= \;& \mathcal{N}_2^v\Big(\nu(t)\hat{X}^1\big(t^1(t)\big) + \big(1-\nu(t)\big)\hat{X}^2\big(t^2(t)\big) - \hat{X}^0\big(\theta(x_1(t))\big)\Big) \\
= \;& \mathcal{N}_2^v\Big(\nu(t)\Big[\hat{X}^1\big(t^1(t)\big) - \hat{X}^0\big(\theta(X_1^1(t^1(t)))\big)\Big] \\
& \quad +\big(1-\nu(t)\big)\Big[\hat{X}^2\big(t^2(t)\big) - \hat{X}^0\big(\theta(X_1^2(t^2(t)))\big)\Big] \\
& \qquad +\nu(t)\hat{X}^0\big(\theta(X_1^1(t^1(t)))\big) + \big(1-\nu(t)\big)\hat{X}^0\big(\theta(X_1^2(t^2(t)))\big) - \hat{X}^0\big(\theta(x_1(t))\big)\Big) \\
\leq \;& \nu(t)\,\mathcal{N}_2^v\Big(\hat{X}^1\big(t^1(t)\big) - \hat{X}^0\big(\theta(X_1^1(t^1(t)))\big)\Big) \\
& \quad +\big(1-\nu(t)\big)\,\mathcal{N}_2^v\Big(\hat{X}^2\big(t^2(t)\big) - \hat{X}^0\big(\theta(X_1^2(t^2(t)))\big)\Big) \\
& \qquad +\mathcal{N}_2^v\Big(\nu(t)\hat{X}^0\big(\theta(X_1^1(t^1(t)))\big) + \big(1-\nu(t)\big)\hat{X}^0\big(\theta(X_1^2(t^2(t)))\big) - \hat{X}^0\big(\theta(x_1(t))\big)\Big).
\end{aligned}$$

Thanks to (2.34), both points $X^1\big(t^1(t)\big)$ and $X^2\big(t^2(t)\big)$ belong to $\mathcal{C}^{\lambda_2}_{[0,\bar{\tau}]}\big(x^1, x^2\big)$, which implies

$$\begin{aligned}
\nu(t)\,\mathcal{N}_2^v\Big(\hat{X}^1\big(t^1(t)\big) - \hat{X}^0\big(\theta(X_1^1(t^1(t)))\big)\Big) & \\
+ \big(1-\nu(t)\big)\,\mathcal{N}_2^v\Big(\hat{X}^2\big(t^2(t)\big) - \hat{X}^0\big(\theta(X_1^2(t^2(t)))\big)\Big) &\leq \big|x^1 - x^2\big|.
\end{aligned}$$

Furthermore, we notice that

$$\begin{aligned}
& \hat{X}^0\big(\theta(x_1(t))\big) \\
= \;& \hat{X}^0\Big(\theta\Big(X_1^2(t^2(t)) + \nu(t)\big(X_1^1(t^1(t)) - X_1^2(t^2(t))\big)\Big)\Big) \\
= \;& \big(\hat{X}^0\circ\theta\big)\Big(X_1^2(t^2(t))\Big) + \nu(t)\,\Big\langle\nabla\big(\hat{X}^0\circ\theta\big)\Big(X_1^2(t^2(t))\Big), X_1^1(t^1(t)) - X_1^2(t^2(t))\Big\rangle \\
& \qquad +O\Big(\Big|X_1^1(t^1(t)) - X_1^2(t^2(t))\Big|^2\Big),
\end{aligned}$$

which gives

$$\begin{aligned}
& \nu(t)\hat{X}^0\big(\theta(X_1^1(t^1(t)))\big) + \big(1-\nu(t)\big)\hat{X}^0\big(\theta(X_1^2(t^2(t)))\big) - \hat{X}^0\big(\theta(x_1(t))\big) \\
= \;& \nu(t)\Big[\big(\hat{X}^0\circ\theta\big)\big(X_1^1(t^1(t))\big) - \big(\hat{X}^0\circ\theta\big)\big(X_1^2(t^2(t))\big) \\
& \qquad - \Big\langle\nabla\big(\hat{X}^0\circ\theta\big)\Big(X_1^2(t^2(t))\Big), X_1^1(t^1(t)) - X_1^2(t^2(t))\Big\rangle\Big] \\
& \qquad +O\Big(\Big|X_1^1(t^1(t)) - X_1^2(t^2(t))\Big|^2\Big) \\
= \;& O\Big(\Big|X_1^1(t^1(t)) - X_1^2(t^2(t))\Big|^2\Big).
\end{aligned}$$

Combining all such estimates together, thanks to (e), (2.42), and Gronwall's Lemma, we obtain the existence of a constant $K_3$ such that

$$\mathcal{N}_2^v\Big(\hat{x}(t) - \hat{X}^0\big(\theta(x_1(t))\big)\Big) \leq \big|x^1 - x^2\big| + K_3\big|x^1 - x^2\big|^2. \tag{2.43}$$

This means that, if $r > 0$ is sufficiently small, then

$$\mathcal{N}_3^v\Big(\hat{x}(t) - \hat{X}^0\big(\theta(x_1(t))\big)\Big) < \big|x^1 - x^2\big|,$$

that is,

$$x(t) \in \mathcal{C}^{\lambda_3}_{[0,\bar\tau]}\big(x^1; x^2\big) \qquad \forall\, t \in [0,T].$$

By (2.35), this gives

$$x(t) \in \mathrm{Cyl}^{\lambda_4}_{[0,\bar\tau]}\big(x^1; x^2\big) \qquad \forall\, t \in [0,T].$$

Define the function $\Gamma : \big[0,\bar\tau\big] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ by

$$\Gamma(t,\hat z) := x\left(\frac{tT^f}{\bar\tau}\right) + (0,\hat z) \qquad \forall\,(t,\hat z) \in \big[0,\bar\tau\big] \times \mathbb{R}^{n-1}, \tag{2.44}$$

where $x(\cdot)$ is the trajectory associated to the control $u$ (see (a)-(g) above). Since $x_1(0) = 0$ and $x_1(T^f) = \bar\tau$, we can easily check that $\Gamma$ is a $C^k$ diffeomorphism from $\big[0,\bar\tau\big] \times \mathbb{R}^{n-1}$ onto $\big[0,\bar\tau\big] \times \mathbb{R}^{n-1}$. Let $\bar\mu > 0$ be small enough so that

$$(1+3\bar\mu)\mathcal{N}_3^v < \mathcal{N}_2^v,$$

and let $\mathcal{N}$ be a norm in $\mathbb{R}^{n-1}$, which is smooth on $\mathbb{R}^{n-1} \setminus \{0\}$, and such that

$$(1+3\bar\mu)\mathcal{N}_3^v < (1+2\bar\mu)\mathcal{N} < \mathcal{N}_2^v \qquad \text{on } \mathbb{R}^{n-1}\setminus\{0\}.$$

By (2.43), if $\bar r > 0$ is small enough, then

$$\Gamma\Big(\big[0,\bar\tau\big] \times B^{\mathcal{N}}_{\mu|x^1-x^2|}\Big) \subset \mathcal{C}^{\lambda_3}_{[0,\bar\tau]}\big(x^1;x^2\big) \subset \mathrm{Cyl}^{\lambda_4}_{[0,\bar\tau]}\big(x^1,x^2\big). \tag{2.45}$$

The following lemma is a simplified version of [8, Lemma 3.3] for general norms. For sake of completeness, its proof is given in Appendix B.

**Lemma 2.3.** *Let $\mathcal{N} : \mathbb{R}^{n-1} \to \mathbb{R}$ be a norm which is smooth on $\mathbb{R}^{n-1}\setminus\{0\}$, fix $\bar\tau, \delta, r \in (0,1)$ with $3r \le \delta < \bar\tau$, and let $\tilde v = \big(\tilde v_1,\ldots,\tilde v_n\big) : \big[0,\bar\tau\big] \to \mathbb{R}^n$ be a function of class $C^{k-2}$ with $k \ge 2$ satisfying*

$$\tilde v(t) = 0_n \qquad \forall\, t \in [0,\delta] \cup [\bar\tau-\delta,\bar\tau] \tag{2.46}$$

*and*

$$\tilde v_1(t) = 0 \qquad \forall\, t \in \big[0,\bar\tau\big]. \tag{2.47}$$

*Then there exist a constant $C > 0$, independent of $r$ and $v$, and a function $W : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$, such that the following properties hold:*

*(i)* $\mathrm{Supp}(W) \subset [\delta/2, \bar\tau-\delta/2] \times B^{\mathcal{N}}_{2r/3} \subset \mathbb{R}\times\mathbb{R}^{n-1}$*;*

*(ii)* $\|W\|_{C^1} \le C\left(\|\tilde v\|_\infty + \|\dot{\tilde v}\|_\infty\right)$*;*

*(iii)* $\|W\|_{C^2} \le C\left(\frac{1}{r}\|\tilde v\|_\infty + \|\dot{\tilde v}\|_\infty\right)$*;*

*(iv)* $\nabla W\big(t,0_{n-1}\big) = \tilde v(t)$ *for every* $t \in \big[0,\bar\tau\big]$*.*

Define the function $\tilde v = \big(\tilde v_1,\ldots,\tilde v_n\big) : \big[0,\bar\tau\big] \to \mathbb{R}^n$ by

$$\tilde v(t) := \big(d\Gamma\big(t,0_{n-1}\big)\big)^*\left(u\left(\frac{tT^f}{\bar\tau}\right)\right) \qquad \forall\, t \in \big[0,\bar\tau\big]. \tag{2.48}$$

The function $\tilde v$ is $C^{k-2}$; in addition, thanks to (f) and (2.44), for every $t \in \big[0,\bar\tau\big]$ we have

$$\tilde v_1(t) = 0 \quad \text{and} \quad \tilde v_i(t) = u_i\left(\frac{tT^f}{\bar\tau}\right) \quad \forall\, i = 2,\ldots,n.$$

Hence, thanks to (c), $\tilde{v}$ satisfies both (2.46) and (2.47), so we can apply Lemma 2.3 and obtain a function $W : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$ satisfying assertions (i)-(iv) of Lemma 2.3 with $r := \bar{\mu}|x^1 - x^2| \in (0,1)$. Define the $C^{k-1}$ potential $V : \mathbb{R}^n \to \mathbb{R}$ by

$$V(x) = \begin{cases} W\big(\Gamma^{-1}(x)\big) & \text{if } x \in \Gamma\Big(\big[0,\bar{\tau}\big] \times B^{\mathcal{N}}_{\bar{\mu}|x^1-x^2|}\Big) \\ 0 & \text{otherwise.} \end{cases}$$

We leave the reader to check that, if $\bar{r}$ is small enough, then assertions (i)-(v) of Proposition 2.2 are satisfied.

Now it remains to show how control the action (assertion (vii)) and to show the bound in (vi). We proceed as in the proof of Proposition 2.1: first, we divide the interval $\big[0,\bar{\tau}\big]$ into two subintervals $\big[0,\bar{\tau}/2\big]$ and $\big[\bar{\tau}/2\bar{\tau}\big]$. Then, we use the construction above on $\big[0,\bar{\tau}/2\big]$ to connect

$$\big(x^1,p^1\big) = \big(x^1, \nabla\bar{u}(x^1)\big) \quad \text{to} \quad \big(x^{1/2},p^{1/2}\big) = \big(x^{1/2}, \nabla\bar{u}(x^{1/2})\big) := \phi^{\bar{H}}_{\tau_{1/2}(x^2,p^2)}\big(x^2,p^2\big).$$

on some time interval $[0,T_1^f]$ with $T_1^f \sim \bar{\tau}/2$, where $\tau_{1/2}$ denotes the Poincaré mapping with respect to the hyperplane $\Pi^{\bar{\tau}/2} := \big\{x = \big(\bar{\tau}/2, \hat{x}\big) \in \mathbb{R}^n\big\}$. As in [8, Proposition 3.1(v)] (see also [8, Remark 3.4]), one can show that the action default is quadratic, see (2.30):

$$\begin{aligned} \Big|\mathbb{A}_V\big((x^1,p^1);T_1^f\big) &- \mathbb{A}\big((x^1,p^1);\tau_{1/2}(x^1,p^1)\big) \\ &- \Delta\left((x^1,p^1);\tau_{1/2}(x^1,p^1);X\big(\tau_{1/2}(x^2,p^2);(x^2,p^2)\big)\right)\Big| \\ &\leq K\left|\phi^H_{\tau_{1/2}(x^2,p^2)}(x^2,p^2) - \phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\right|^2 \leq \tilde{K}\left|(x^2,p^2)-(x^1,p^1)\right|^2 \end{aligned}$$

Now, thanks to assumptions (2.32) and (2.38), it is not difficult to check that

$$\begin{aligned} \Delta\left((x^1,p^1);\tau_{1/2}(x^1,p^1);X\big(\tau_{1/2}(x^2,p^2);(x^2,p^2)\big)\right) \\ = \Big\langle \nabla\bar{u}\big(\pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\big), x^{1/2} - \pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\Big\rangle, \end{aligned}$$

$$\mathbb{A}\big((x^1,p^1);\tau_{1/2}(x^1,p^1)\big) = \bar{u}\big(\pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\big) - \bar{u}(x^1).$$

Moreover, since $\bar{u}$ is $C^{1,1}$ on $\mathcal{U}$, if $K_{\bar{u}}$ denotes a bound for the Lipschitz constant of $\nabla\bar{u}$, we also have

$$\begin{aligned} \Big|\bar{u}(x^{1/2}) &- \bar{u}\big(\pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\big) \\ &- \Big\langle \nabla\bar{u}\big(\pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\big), x^{1/2} - \pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\Big\rangle\Big| \\ &\leq K_{\bar{u}}\big|x^{1/2} - \pi^*\big(\phi^H_{\tau_{1/2}(x^1,p^1)}(x^1,p^1)\big)\rangle\big|^2. \end{aligned}$$

Hence, combining the above estimates, we get

$$\mathbb{A}_V\big((x^1,p^1);T_1^f\big) = \bar{u}(x^{1/2}) - \bar{u}(x_1) + O\big(\big|(x^2,p^2)-(x^1,p^1)\big|^2\big).$$

Furthermore, we observe that (2.38) implies

$$\int_{\tau_{1/2}(x^2,p^2)}^{\tau(x^2,p^2)} \bar{L}\left(\phi_t^{\bar{H}_V}(x^2,p^2), \frac{d}{dt}\left(\pi^*\left(\phi_t^{\bar{H}_V}(x^2,p^2)\right)\right)\right) = \bar{u}\big(\pi^*\big(\phi^H_{\tau(x^2,p^2)}(x^2,p^2)\big)\big) - \bar{u}(x^{1/2}).$$

Hence, for $\bar{r}$ sufficiently small, we can apply [8, Proposition 4.1] on $\big[\bar{\tau}/2,\bar{\tau}\big]$ to compensate any default of action of the order $O\big(|(x^2,p^2)-(x^1,p^1)|^2\big)$, so that (vii) holds.

Finally, for any $\tau \in [0,\bar{\tau}]$, $t \in [0,\tau(x^1,p^1)]$ and $t_V \in [0,T^f]$ such that

$$X_V\big(t_V;(x^1,p^1)\big), X\big(t;(x^1,p^1)\big) \in \Pi^\tau,$$

thanks to [8, Remark 3.4, Equations (3.47)-(3.48)] and the $C^{1,1}$-regularity of $\bar{u}$, the above argument shows the validity of (vi), which concludes the proof. □

*Remark* 2.4. We supposed that assumptions (A1)-(A4) hold on both subintervals $\big[0,\bar{\tau}/2\big]$ and $[\bar{\tau}/2,\bar{\tau}]$. If instead we fix $0<\bar{\nu}_1<\bar{\nu}_2<\bar{\tau}$ and assume that (A1)-(A4) hold on both subintervals $[\bar{\nu}_1,\bar{\nu}_2],[\bar{\nu}_2,\bar{\tau}]$, then there exist $\bar{\delta},\bar{r}\in(0,1)$ and $K>0$ such that the property stated in Proposition 2.2 is satisfied with

$$\mathrm{Supp}(V) \subset \mathrm{Cyl}^{\lambda_4}_{[0,\bar{\tau}]}\big(x^1;x^2\big) \cap \mathcal{H}_{[\bar{\nu}_1,\bar{\tau}]},$$

where $\mathcal{H}_{[\bar{\nu}_1,\bar{\tau}]} := \big\{z=\big(z_1,\hat{z}\big)\in\mathbb{R}^n \mid z_1\in[\bar{\nu}_1,\bar{\tau}]\big\}$. Indeed, arguing as above, we first construct a potential supported on $\mathcal{H}_{[\bar{\nu}_1,\bar{\nu}_2]}$ to connect the trajectories, and then a potential supported on $\mathcal{H}_{[\bar{\nu}_2,\bar{\tau}]}$ to compensate the action (of course, $\bar{\delta},\bar{r}$, and $K$ depend on both $\bar{\nu}_2-\bar{\nu}_1$ and $\bar{\tau}-\bar{\nu}_2$).

# 3 A Mai Lemma with constraints

The aim of this section is to prove some refined versions of the Mai Lemma. Let us recall that the classical Mai Lemma was introduced in [10] to give a new and simpler proof of the closing lemma in $C^1$ topology (this consists in showing that, given a vector field $X$ with a recurrent point $x$, one can find a vector field $Y$ close to $X$ in $C^1$ topology which has a periodic orbit containing $x$). In our case, we already used the classical Mai Lemma in [8] to "close" the Aubry set, assuming the existence of a critical subsolution which is a $C^2$ critical solution in an open neighborhood of a positive orbit of a recurrent point of the projected Aubry set. Here, since in the statement of Theorem 1.1 we only assume to have a smooth subsolution, we have relevant information on $u$ only on the Aubry set. Hence, by using a Taylor development, we can still get some information in directions "tangent" to the Aubry set, but we have no controls in the orthogonal directions. For this reason, we need to prove a refined Mai Lemma where we connect two points by remaining "almost tangent" to a given subspace, see Lemma 3.4 below.

For proving our refined Mai Lemma, it will be useful to first recall the classical result.

## 3.1 The classical Mai Lemma

Let $\{E_i\}$ be a countable family of ellipsoids in $\mathbb{R}^k$, that is, a countable family of compact sets in $\mathbb{R}^k$ associated with a countable family of invertible linear mappings $P_i:\mathbb{R}^k\to\mathbb{R}^k$ such that

$$E_i = \Big\{v\in\mathbb{R}^k \mid |P_i(v)| \le \|P_i\|\Big\},$$

where $\big\|P_i\big\|$ denotes the operator norm of $P_i$. For every $x\in\mathbb{R}^k$, $r>0$ and $i\in\mathbb{N}$, we call $E_i$-ellipsoid centered at $x$ with radius $r$ the set defined by

$$E_i(x,r) := \Big\{x+rv \mid v\in E_i\Big\} = \Big\{x' \mid |P_i(x'-x)| < r\|P_i\|\Big\}.$$

We note that such an ellipsoid contains the open ball $B(x,r)$. Given an integer $N\ge 2$, we call $1/N$-kernel of $E_i(x,r)$ the ellipsoid $E_i(x,r/N)$. The Mai lemma can be stated as follows (see also [8, Subsection 5.3, Figure 4])[3]:

[3]Note that in [8, Lemma D.1] we stated it in a slightly weaker form. However, in order to be able to prove Lemmas 3.2 and 3.4, we need the full statement of [10, Theorem 2.1].

**Lemma 3.1** (Mai Lemma). *Let $N \geq 2$ be an integer. There exist a real number $\rho \geq 3$ and an integer $\eta \geq 2$, which depend on the family $\{E_i\}$ and on $N$ only, such that the following property holds: For every finite ordered set $X = \{x_1, \ldots, x_J\} \subset \mathbb{R}^k$, every $x \in \mathbb{R}^k$ and every $\delta > 0$ such that $B(x, \delta/4) \cap X$ contains at least two points, there are two points $x_j, x_l \in X \cap B(x, \rho\delta)$ $(j > l)$ and $\eta$ points $z_1, \ldots, z_\eta$ in $B(x, \rho\delta)$ satisfying:*

*(i)* $z_1 = x_j, z_\eta = x_l$;

*(ii)* *for any $i \in \{1, \ldots, \eta - 1\}$, the point $z_{i+1}$ belongs to the $1/N$-kernel of $E_i(z_i, r_i)$, where $r_i$ is the supremum of the radii $r > 0$ such that*

$$E_i(z_i, r) \cap \Big(\partial B(x, \rho\delta) \cup (X \setminus \{x_j, x_l\})\Big) = \emptyset.$$

The purpose of the next two subsections is to refine the construction of the points $z_1, \ldots, z_\eta$, and to show that, under additional assumption on $X$, these points can be chosen to belong to a Lipschitz submanifold of $\mathbb{R}^k$.

## 3.2 A first refined Mai Lemma

Our first goal is to provide a lower bound on the radii of the ellipsoids $E_i(z_i, r_i)$'s. This will be very important for the proof of Lemma 4.1, which is one of the key steps for proving Theorem 1.1.

Given an ellipsoid $E_i$ and a set $X \subset \mathbb{R}^k$, we denote by $\mathrm{dist}_i(\cdot, X)$ the distance function to the set $X$ with respect to $E_i$, that is

$$\mathrm{dist}_i(z, X) := \inf\Big\{r \geq 0 \,|\, E_i(z, r) \cap X \neq \emptyset\Big\} \qquad \forall\, z \in \mathbb{R}^k. \tag{3.1}$$

The following result is a slight improvement of Lemma 3.1:

**Lemma 3.2.** *Let $N \geq 2$ be an integer. There exist a real number $\bar{\rho} \geq 3$ and an integer $\eta \geq 3$, which depend on the family $\{E_i\}$ and on $N$ only, such that the following property holds: For every finite ordered set $X = \{x_1, \ldots, x_J\} \subset \mathbb{R}^k$, every $x \in \mathbb{R}^k$, and every $r > 0$ such that $X \cap B(x, r)$ contains at least two points, there are $\eta$ points $z_1, \ldots, z_\eta$ in $\mathbb{R}^k$ and $(\eta - 1)$ positive real numbers $\bar{r}_1, \ldots, \bar{r}_{\eta-1}$ satisfying:*

*(i)* *there exist $j, l \in \{1, \ldots, J\}$, with $j > l$, such that $z_1 = x_j$ and $z_\eta = x_l$;*

*(ii)* $\forall\, i \in \{1, \ldots, \eta - 1\}$, $E_i\big(z_i, \bar{r}_i\big) \subset B\big(x, \bar{\rho} r\big)$;

*(iii)* $\forall\, i \in \{1, \ldots, \eta - 1\}$, $E_i\big(z_i, \bar{r}_i\big) \cap \big(X \setminus \{x_j, x_l\}\big) = \emptyset$;

*(iv)* $\forall\, i \in \{1, \ldots, \eta - 1\}$, $z_{i+1} \in E_i\big(z_i, \bar{r}_i/N\big)$;

*(v)* $\forall\, i \in \{1, \ldots, \eta - 1\}$, $\bar{r}_i \geq dist_i\big(z_i, X\big)$.

Observe that, while in the classical Mai Lemma 3.1 one has $\eta \geq 2$, in the statement above $\eta \geq 3$. Indeed, as we will show below, with a simple argument one can always count one of the points twice so that $\eta \geq 3$. This is done because, for the application we have in mind, we would otherwise need to distinguish between the case $\eta = 2$ and $\eta \geq 3$

*Proof.* Let us apply Lemma 3.1 to the family $\{E_i\}$ and $N$: there exist $\rho \geq 3$ and an integer $\eta \geq 2$ such that assertions (i)-(ii) of Lemma 3.1 are satisfied. Set

$$\bar{\rho} := 13\,\rho\, \max\Big\{\|P_i\|\|P_i^{-1}\| \,|\, i = 1, \ldots, \eta - 1\Big\},$$

and let us show that we can choose positive numbers $\bar{r}_i$ $(i = 1, \dots, \eta - 1)$ so that assertions (i)-(v) are satisfied. Let $X = \{x_1, \dots, x_J\}$ be a finite ordered set in $\mathbb{R}^k$, fix a point $x \in \mathbb{R}^k$, and let $r > 0$ be such that $X \cap B(x, r)$ contains at least two points. By construction of $\rho$ and $\eta$, there exist $\eta$ points $z_1, \dots, z_\eta$ in $B(x, 4\rho r)$ such that assertions (i)-(ii) of Lemma 3.1 are satisfied. Now, for every $i = \{1, \dots, \eta - 1\}$ denote by $r_i'$ the supremum of the radii $r > 0$ such that $E_i(z_i, r) \cap \big(\partial B\big(x, \bar{\rho} r\big) \cup X\big) = \emptyset$, that is

$$r_i' := \mathrm{dist}_i\big(z_i, \partial B(x, \bar{\rho} r) \cup X\big).$$

Note that

$$\begin{aligned} z_1 \in \overline{E_i\left(z_i, |z_1 - z_i|\right)} &\subset \overline{E_i\left(z_i, |z_1 - x| + |z_i - x|\right)} \\ &\subset E_i\left(z_i, 8\rho r\right) \\ &\subset B\left(z_i, 8\rho r \|P_i\| \|P_i^{-1}\|\right) \\ &\subset B\left(x, 8\rho r \|P_i\| \|P_i^{-1}\| + 4\rho r\right) \\ &\subset B\left(x, 12\rho r \|P_i\| \|P_i^{-1}\|\right). \end{aligned}$$

Therefore, by definition of $\bar{\rho}$ and the fact that $z_1 \in X$, we deduce that

$$\overline{E_i\big(z_i, r_i'\big)} \cap \partial B\big(x, \bar{\rho} r\big) = \emptyset.$$

Two cases appear, depending whether $r_i'$ is larger or smaller than $r_i$, where

$$r_i := \mathrm{dist}_i\Big(z_i, \partial B(x, \bar{\rho} r) \cup \big(X \setminus \{z_1, z_\eta\}\big)\Big)$$

is as in Lemma 3.1(ii).
*Case I:* $r_i' < r_i$. Set $\bar{r}_i := r_i$. Then, since $\bar{\rho} > \rho$ we necessarily have either $z_1 \in E_i\big(z_i, r_i\big)$ or $z_\eta \in E_i\big(z_i, r_i\big)$, so that $\bar{r}_i \geq \mathrm{dist}_i(z_i, X)$.
*Case II:* $r_i' \geq r_i$. Set $\bar{r}_i := r_i'$. Then, by construction, the set $\overline{E_i\big(z_i, \bar{r}_i\big)} \cap X$ is nonempty, and we deduce as above that $\bar{r}_i \geq \mathrm{dist}_i(z_i, X)$.

Finally, we notice that if the number $\eta$ given by the Mai Lemma 3.1 is equal to 2, then we can set $\eta = 3, z_3 := z_2$, and choose any radius $\bar{r}_2 > 0$ sufficiently small so that $E_2(z_2, \bar{r}_2) \cap X = \{z_2\}$ and $E_2(z_2, \bar{r}_2) \subset B(x, \bar{\rho} r)$. □

### 3.3 A constrained Mai Lemma

As we explained above, we will need a version of the Mai Lemma where the sequence of points $z_1, \dots, z_\eta$ "almost" lies inside a given vector subspace, which, roughly speaking, represents the "tangent space" to a set $A$ at a given point. More precisely, let $A \subset \mathbb{R}^k$ be a compact set and assume that the origin is a cluster point. We recall that the *paratingent space* of $A$ at 0 is the vector space defined as

$$\Pi_0(A) := \mathrm{Span}\,\left\{\lim_{i \to \infty} \frac{x_i - y_i}{|x_i - y_i|} \mid \lim_{i \to \infty} x_i = \lim_{i \to \infty} y_i = 0,\, x_i \in A,\, y_i \in A,\, x_i \neq y_i \,\forall\, i\right\}.$$

The aim of this subsection is twofold: first, in Lemma 3.3 we show that inside a small ball $B_r$ around 0 the set $A$ is contained inside a Lipschitz graph $\Gamma_A$ with respect to $\Pi_0(A)$, with a Lipschitz constant going to 0 as $r \to 0$. Then, in Lemma 3.4 we show that if the ordered set of points $X$ is contained inside $A$, then the sequence $z_1, \dots, z_\eta$ provided by Mai Lemma can be chosen to belong to $\Gamma_A$.

In the statement below, for simplicity of notation we set $\Pi := \Pi_0(A)$. Let $d$ be the dimension of $\Pi$, and denote by $\Pi^\perp$ the orthogonal space to $\Pi$ in $\mathbb{R}^k$. We denote by $\mathrm{Proj}_\Pi$ the orthogonal projection onto the space $\Pi$ in $\mathbb{R}^k$, and set $H_A := \mathrm{Proj}_\Pi(A)$. Finally, for any $r, \nu > 0$ we define the cylinder

$$C(r,\nu) := \Big\{(h,v) \in \Pi \times \Pi^\perp \mid |h| < r, |v| < \nu\Big\}.$$

**Lemma 3.3.** *There exist a radius $r_A > 0$ and a Lipschitz function $\Psi_A : \Pi \cap \bar{B}_{r_A} \to \Pi^\perp$ such that the following properties hold:*

*(i) $A \cap C(r_A, r_A) \subset \mathrm{graph}(\Psi_A)|_{B_{r_A}} := \big\{h + \Psi_A(h) \mid h \in \Pi \cap B_{r_A}\big\}$;*

*(ii) $h + \Psi_A(h)$ belongs to $A \cap C(r_A, r_A)$ for every $h \in H_A \cap B_{r_A}$;*

*(iii) For any $r \in (0, r_A)$, let $L_A(r) > 0$ denote the Lipschitz constant of $\Psi_A$ on $\Pi \cap B_r$. Then $\lim_{r\downarrow 0} L_A(r) = 0$.*

*In particular, $\Psi_A(0) = 0$, $\Psi_A$ is differentiable at $0$, and $\nabla\Psi_A(0) = 0$.*

*Proof.* We claim that, if $r > 0$ is sufficiently small, then there exists a function $\psi : H_A \cap B_r \mapsto \Pi^\perp$ such that $A \cap C(r,r) \subset \mathrm{graph}(\psi)|_{B_r}$. Moreover, $\psi$ is Lipschitz on $H_A \cap B_r$, and its Lipschitz constant converges to 0 as $r \to 0$.

To prove the claim, let $\{h_l^1\}, \{h_l^2\} \subset H_A$ be two sequences converging to 0, and for any $l \in \mathbb{N}$ take vectors $v_l^1, v_l^2 \in \Pi^\perp$ such that $x_l^1 := h_l^1 + v_l^1, x_l^2 := h_l^2 + v_l^2 \in A$. We observe that

$$\frac{x_l^1 - x_l^2}{|x_l^1 - x_l^2|} = \frac{h_l^1 - h_l^2}{\sqrt{\big|h_l^1 - h_l^2\big|^2 + \big|v_l^1 - v_l^2\big|^2}} + \frac{v_l^1 - v_l^2}{\sqrt{\big|h_l^1 - h_l^2\big|^2 + \big|v_l^1 - v_l^2\big|^2}} =: g_l + w_l,$$

where $g_l \in \Pi$ and $w_l \in \Pi^\perp$. Hence, since by definition of $\Pi$ any cluster point of $\frac{x_l^1 - x_l^2}{|x_l^1 - x_l^2|}$ belongs to $\Pi$, we necessarily have that $w_l \to 0$ as $l \to \infty$, or equivalently

$$\lim_{l\to\infty} \frac{a_l}{\sqrt{1 + a_l^2}} = 0, \qquad \text{with} \quad a_l := \frac{\big|v_l^1 - v_l^2\big|}{\big|h_l^1 - h_l^2\big|}.$$

Since the function $s \mapsto \frac{s}{\sqrt{1+s^2}}$ is strictly increasing, we deduce that $a_l \to 0$ as $l \to \infty$.

Observe that by choosing $h_l^1 = h_l^2$ for all $l \in \mathbb{N}$, the above argument shows that, if $r > 0$ is sufficiently small, then for every $h \in \Pi$ with $|h| < r$ there is at most one $v = v(h) \in \Pi^\perp$ such that $h + v \in A$. So, we can define a function $\psi : H_A \cap B_r \mapsto \Pi^\perp$ by $\psi(h) := v(h)$ for every $h \in H_A \cap B_r$, and the fact that

$$\frac{\big|\psi(h_l^1) - \psi(h_l^2)\big|}{\big|h_l^1 - h_l^2\big|} = \frac{\big|v_l^1 - v_l^2\big|}{\big|h_l^1 - h_l^2\big|} \to 0 \qquad \text{as } l \to \infty$$

for any sequences $\{h_l^1\}, \{h_l^2\} \subset H_A$ converging to 0 proves that $\psi$ is Lipschitz on $H_A \cap B_r$, with Lipschitz constant converging to 0 as $r \to 0$. Consequently, there is $\bar{r} > 0$ such that $\psi : H_A \cap B_{\bar{r}} \to \Pi^\perp$ is Lipschitz and valued in $B_{\bar{r}}$, which proves assertions (i) and (ii). To conclude, it remains to extend the function $\psi : H_A \cap B_{\bar{r}} \to \Pi^\perp$ to a global Lipschitz function $\Psi_A : \Pi \cap B_{\bar{r}} \to \Pi^\perp$ which satisfies (iii).

For every $r \in (0, \bar{r})$, let $\lambda(r)$ denote the Lipschitz constant of $\psi$ on $H_A \cap B_r$, and recall that $\lambda(r) \to 0$ as $r \to 0$. Let $\psi_1, \ldots, \psi_{k-d}$ denote the coordinates of $\psi$. For every $r \in (0, \bar{r})$, each coordinate $\psi_j$ is a $\lambda(r)$-Lipschitz function from $H_A \cap B_r$ onto $\mathbb{R}$. For every $j = 1, \ldots, k-d$ and any integer $l \geq 1$, we define $\psi_j^l : \Pi \to \Pi^\perp$ by

$$\psi_j^l(x) := \min\Big\{\psi_j(y) + \lambda\big(2^{-l}\bar{r}\big)|y - x| \mid y \in H_A \cap B_{2^{-l}\bar{r}}\Big\} \qquad \forall\, x \in P.$$

It is easily checked that the function $\psi_j^l$ is $\lambda\big(2^{-l}\bar{r}\big)$-Lipschitz on $\Pi$ for any $j, l$, and moreover

$$\psi_j^l = \psi_j \qquad \text{on } H_A \cap B_{2^{-l}\bar{r}}. \tag{3.2}$$

Let $\{I_l\}_{l\geq 1}$ be the sequence of intervals in $\mathbb{R}$ defined by

$$I_l := \big(2^{-l-1}\bar r, 2^{1-l}\bar r\big).$$

The family $\{I_l\}$ forms a locally finite covering of the open interval $(0,\bar r)$. Let $\{\rho_l\}$ be a smooth approximation of unity in $(0,\bar r)$ associated with the covering $\{I_l\}$ such that

$$\big|\rho_l'(r)\big| \leq C2^l, \tag{3.3}$$

for some constant $C>0$ independent of $l$. Finally, define the function $\Psi = \big(\Psi_1,\ldots,\Psi_{k-d}\big) : \Pi\cap\bar B_{\bar r} \to \Pi^\perp$ by

$$\Psi_j(x) := \sum_{l=1}^{\infty} \rho_{l+1}\big(|x|\big)\psi_j^l(x) \qquad \forall\, x\in\Pi\cap\bar B_{\bar r},\ j=1,\ldots,k-d.$$

We claim that $\Psi_A := \Psi$ satisfies assumption (iii). Indeed, consider first a point $x\in\bar B_{\bar r}$ which satisfies $|x|\in\big(2^{-\bar l-1}\bar r, 2^{-\bar l}\bar r\big)$ for some integer $\bar l\geq 2$. Then

$$\rho_{l+1}\big(|x|\big) = 0 \qquad \forall\, l\notin\{\bar l-1,\bar l\}.$$

so that by (3.2) we get

$$\Psi_j(x) = \rho_{\bar l}\big(|x|\big)\psi_j^{\bar l-1}(x) + \rho_{\bar l+1}\big(|x|\big)\psi_j^{\bar l}(x) = \big(\rho_{\bar l}\big(|x|\big)+\rho_{\bar l+1}\big(|x|\big)\big)\,\psi_j(x) = \psi_j(x).$$

By the arbitrariness of $x$, this gives

$$\Psi = \psi \qquad \text{on } H_A\cap B_{\bar r/4}. \tag{3.4}$$

In addition, if $x\in B_{\bar r/4}\cap\{2^{-\bar l-1}\bar r\leq|x|\leq 2^{-\bar l}\bar r)$ is a point at which all functions $\psi_j^l$ are differentiable (since all functions $\psi_j^l$ are Lipschitz, by Rademacher's Theorem almost every point satisfies this assumption), then for any vector $h\in\mathbb{R}^d$ we have

$$\begin{aligned}\langle\nabla\Psi_j(x),h\rangle = \rho_{\bar l}\big(|x|\big)\langle\nabla\psi_j^{\bar l-1}(x),h\rangle + \rho_{\bar l+1}\big(|x|\big)\langle\nabla\psi_j^{\bar l}(x),h\rangle \\ + \frac{\rho_{\bar l}'\big(|x|\big)\psi_j^{\bar l-1}(x)}{|x|}\langle x,h\rangle + \frac{\rho_{\bar l+1}'\big(|x|\big)\psi_j^{\bar l}(x)}{|x|}\langle x,h\rangle.\end{aligned}$$

Using (3.3) together with the fact that $\psi_j^l$ is $\lambda\big(2^{-l}\bar r\big)$-Lipschitz and satisifies $\psi_j^l(0)=\psi(0)=0$, we obtain

$$\begin{aligned}\big|\nabla\Psi_j(x)\big| &\leq \lambda(2^{1-\bar l}\bar r\big) + \lambda\big(2^{-\bar l}\bar r\big) + C2^{\bar l}\lambda\big(2^{1-\bar l}\bar r\big)|x| + C2^{\bar l+1}\lambda\big(2^{-\bar l}\bar r\big)|x| \\ &\leq \lambda(2^{1-\bar l}\bar r\big) + \lambda\big(2^{-\bar l}\bar r\big) + C\bar r\lambda\big(2^{1-\bar l}\bar r\big) + 2C\bar r\lambda\big(2^{-\bar l}\bar r\big) \\ &\leq \lambda(2^{1-\bar l}\bar r\big)\,(2+3C\bar r)\,.\end{aligned}$$

Hence, recalling that $\lambda(2^{1-\bar l}\bar r\big)\to 0$ as $l\to\infty$, we conclude that $\Psi_A := \Psi$ satisfies assertion (iii) on $B_{r_A}$, with $r_A := \bar r/4$. □

We are now ready to prove our constrained version of the Mai Lemma. We assume that a countable family of ellipsoids $\{E_i\}$ in $\mathbb{R}^k$ is given, and that $A\subset\mathbb{R}^k$ is a compact set having the origin as a cluster point. If $r_A>0$ and $\Psi_A:\Pi\cap B_{r_A}\to\Pi^\perp$ are given by the previous lemma, we set

$$\Gamma_A := \mathrm{graph}(\Psi_A) = \Big\{h+\Psi_A(h)\mid h\in\Pi\cap B_{r_A}\Big\}.$$

Recall that $L_A:[0,r_A)\to[0,+\infty)$ denotes the Lipschitz constant of $\Psi_A|_{B_r}$, and that $L_A(r)\to 0$ as $r\to 0$. The following constrained version of the Mai lemma holds:

**Lemma 3.4.** *Let $\hat{N} \geq 2$ be an integer. There exist a real number $\hat{\rho} \geq 3$, an integer $\eta \geq 3$, and a radius $\hat{r} \in (0, r_A)$, depending on the family $\{E_i\}$, on $\hat{N}$ and on the function $L_A$ only, such that the following property holds: For every $r \in (0, \hat{r})$ and every finite ordered set $Y = \{y_1, \ldots, y_J\} \subset \mathbb{R}^k$ such that $Y \subset A$ and $Y \cap B_r$ contains at least two points, there are $\eta$ points $\hat{y}_1, \ldots, \hat{y}_\eta$ in $\mathbb{R}^k$ and $(\eta - 1)$ positive real numbers $\hat{r}_1, \ldots, \hat{r}_{\eta-1}$ satisfying:*

*(i) there exist $j, l \in \{1, \ldots, J\}$, with $j > l$, such that $\hat{y}_1 = y_j$ and $\hat{y}_\eta = y_l$;*

*(ii) $\forall\, i \in \{1, \ldots, \eta\}$, $\hat{y}_i \in \Gamma_A \cap B_{\hat{\rho} r}$;*

*(iii) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $E_i\big(\hat{y}_i, \hat{r}_i\big) \subset B_{\hat{\rho} r}$;*

*(iv) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $E_i\big(\hat{y}_i, \hat{r}_i\big) \cap \big(Y \setminus \{y_j, y_l\}\big) = \emptyset$;*

*(v) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $\hat{y}_{i+1} \in E_i\big(\hat{y}_i, \hat{r}_i/\hat{N}\big)$;*

*(vi) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $\hat{r}_i \geq dist_i\big(\hat{y}_i, Y\big)/4$.*

*Proof.* For every $i$, let $P_i : \mathbb{R}^k \to \mathbb{R}^k$ be the linear map associated to the ellipsoid $E_i$, and let $\bar{P}_i : \Pi \to P_i(\Pi)$ be the restriction of $P_i$ to $\Pi$. Since $P_i$ is invertible, $\bar{P}_i$ is an invertible linear map from $\Pi \simeq \mathbb{R}^d$ into $P_i(\Pi) \simeq \mathbb{R}^d$. Define the countable family of ellipsoids $\{\bar{E}_i\}$ in $\Pi \simeq \mathbb{R}^d$ by

$$\bar{E}_i := \Big\{ h \in P \mid \big|\bar{P}_i(h)\big| < \big\|\bar{P}_i\big\| \Big\},$$

where $\big\|\bar{P}_i\big\|$ denotes the operator norm of $\bar{P}_i$. Let us apply the refined Mai Lemma 3.2 in $\Pi \simeq \mathbb{R}^d$ with the family $\{\bar{E}_i\}$ and $N := 4\hat{N}$. Then, there exist a real number $\bar{\rho} \geq 3$ and an integer $\eta \geq 3$ such that all properties of Lemma 3.2 are satisfied. Set

$$\hat{\rho} := \max \Big\{ \big(2 + \|P_i^{-1}\| \|\bar{P}_i\|\big)\, \bar{\rho} \mid i = 1, \ldots, \eta - 1 \Big\}.$$

We want to show that if $\hat{r} \in (0, r_A)$ is small enough, then assertions (i)-(vi) above hold.

Let $Y = \{y_1, \ldots, y_J\}$ be a finite set in $\mathbb{R}^k$ such that $Y \subset A$, and $Y \cap B_r$ contains at least two points for some $r \in (0, \hat{r})$, where $\hat{r}$ will be chosen later. For every $j = 1, \ldots, J$ we set

$$x_j := \mathrm{Proj}_\Pi(y_j).$$

Then the set $X = \{x_1, \ldots, x_J\}$ is a finite subset of $\Pi$ such that $X \cap (\Pi \cap B_r)$ contains at least two points. Hence we can apply Lemma 3.2 to find $\eta$ points $z_1, \ldots, z_\eta \in \Pi$ and $(\eta - 1)$ positive real numbers $\bar{r}_1, \ldots, \bar{r}_{\eta-1}$ satisfying:

(a) there exist $j, l \in \{1, \ldots, J\}$, with $j > l$, such that $z_1 = x_j$ and $z_\eta = x_l$;

(b) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $\bar{E}_i\big(z_i, \bar{r}_i\big) \subset B_{\bar{\rho} r}$;

(c) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $\bar{E}_i\big(z_i, \bar{r}_i\big) \cap \big(X \setminus \{x_j, x_l\}\big) = \emptyset$;

(d) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $z_{i+1} \in \bar{E}_i\big(z_i, \bar{r}_i/(4\hat{N})\big)$;

(e) $\forall\, i \in \{1, \ldots, \eta - 1\}$, $\bar{r}_i \geq \overline{\mathrm{dist}}_i\big(z_i, X\big)$.

Here $\overline{\mathrm{dist}}_i : \mathbb{R}^d \to \mathbb{R}$ denotes the distance function with respect to $\bar{E}_i$ (see (3.1)). Note that property (b) implies

$$|z_i|, \bar{r}_i \leq \bar{\rho} r \qquad \forall\, i = 1, \ldots, \eta - 1. \tag{3.5}$$

For every $i \in \{1, \ldots, \eta\}$ we set

$$\hat{y}_i := z_i + \Psi_A(z_i) \quad \text{and} \quad \hat{r}_i := \frac{\bar{r}_i \big\|\bar{P}_i\big\|}{2\|P_i\|}.$$

We now show that, with these choices, all assertions (i)-(vi) hold true.

First, if $\hat{r}$ is such that $\bar{\rho} r \le \bar{\rho}\hat{r} < r_A$, then each $\hat{y}_i$ belongs to $\Gamma_A$, so that (i) and (ii) are satisfied. Moreover, taking $\hat{r}$ smaller if necessary, we can assume that $\Psi_A$ is 1-Lipschitz on $B_{\bar{\rho}\hat{r}}$. Hence, if $y \in \mathbb{R}^k$ belongs to $E_i\big(\hat{y}_i, \hat{r}_i\big)$ for some $i \in \{1, \ldots, \eta - 1\}$, using (3.5) we get

$$\begin{aligned} |y| &\le \big|y - \hat{y}_i\big| + \big|\hat{y}_i\big| \\ &< \big\|P_i^{-1}\big\|\big|P_i\big(y - \hat{y}_i\big)\big| + 2|z_i| \\ &\le \hat{r}_i\big\|P_i^{-1}\big\|\big\|P_i\big\| + 2\bar{\rho}r \\ &\le \frac{\bar{r}_i}{2}\big\|P_i^{-1}\big\|\big\|\bar{P}_i\big\| + 2\bar{\rho}r \\ &\le \big(\big\|P_i^{-1}\big\|\big\|\bar{P}_i\big\|\bar{\rho} + 2\bar{\rho}\big)\, r \le \hat{\rho} r. \end{aligned}$$

so that also (iii) holds true.

Let us now prove (iv). We argue by contradiction and we assume that there exists a point $y_m$, with $m \notin \{j, l\}$, which belongs to $E_i\big(\hat{y}_i, \hat{r}_i\big)$ for some $i \in \{1, \ldots, \eta - 1\}$, that is,

$$\big|P_i\big(y_m - \hat{y}_i\big)\big| < \hat{r}_i\|P_i\|. \tag{3.6}$$

We now observe that the points $\hat{y}_i$ and $y_m$ can be written as

$$\hat{y}_i = z_i + \Psi_A(z_i) \quad \text{and} \quad y_m = x_m + \Psi_A(x_m),$$

for some $z_i, x_m \in \Pi$ satisfying

$$|z_i| \le \bar{\rho} r \le \hat{\rho}\hat{r} \quad \text{and} \quad |x_m| \le \hat{\rho} r \le \hat{\rho}\hat{r}.$$

Therefore (3.6) gives

$$\begin{aligned} \big|\bar{P}_i\big(x_m - z_i\big)\big| &= \big|P_i\big(x_m - z_i\big)\big| \\ &= \big|P_i\big(y_m - \hat{y}_i\big) - P_i\big(\Psi_A(x_m) - \Psi_A(z_i)\big)\big| \\ &< \hat{r}_i\|P_i\| + \|P_i\|\,|\Psi_A(x_m) - \Psi_A(z_i)| \\ &\le \hat{r}_i\|P_i\| + \|P_i\|L\big(\hat{\rho}\hat{r}\big)\big|x_m - z_i\big| \\ &\le \frac{\bar{r}_i}{2}\big\|\bar{P}_i\big\| + \|P_i\|L\big(\hat{\rho}\hat{r}\big)\big\|\bar{P}_i^{-1}\big\|\big|\bar{P}_i\big(x_m - z_i\big)\big|, \end{aligned}$$

which implies

$$\big|\bar{P}_i\big(x_m - z_i\big)\big| \le \frac{\bar{r}_i}{2\big(1 - L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big\|\bar{P}_i^{-1}\big\|\big)}\big\|\bar{P}_i\big\|.$$

Consequently, if $\hat{r} > 0$ is chosen sufficiently small so that

$$L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big\|\bar{P}_i^{-1}\big\| < 1/3 \qquad \forall\, i = 1, \ldots, \eta - 1, \tag{3.7}$$

then $\big|\bar{P}_i\big(x_m - z_i\big)\big| \le \big(3\bar{r}_i/4\big)\big\|\bar{P}_i\big\|$, which means that the set $X \setminus \{x_j, x_l\}$ intersects the ellipsoid $\bar{E}_i\big(z_i, 3\bar{r}_i/4\big)$, a contradiction to (c). This proves that if $\hat{r}$ is small enough, then assertion (iv) is satisfied.

We now observe that, due (d) and the fact that $\|\bar{P}_i\| \le \|P_i\|$, for every $i = 1, \ldots, \eta - 1$ we have

$$
\begin{aligned}
\big|P_i\big(\hat{y}_{i+1}-\hat{y}_i\big)\big| &= \big|P_i\big(z_{i+1}+\Psi_A(z_{i+1})\big)-P_i\big(z_i+\Psi_A(z_i)\big)\big| \\
&\le \big|\bar{P}_i\big(z_{i+1}-z_i\big)\big| + \big|P_i\big(\Psi_A(z_{i+1})-\Psi_A(z_i)\big)\big| \\
&\le \frac{\bar{r}_i}{4\hat{N}}\big\|\bar{P}_i\big\| + \|P_i\|L\big(\hat{\rho}\hat{r}\big)\big|z_{i+1}-z_i\big| \\
&\le \frac{\bar{r}_i}{4\hat{N}}\big\|\bar{P}_i\big\| + \|P_i\|L\big(\hat{\rho}\hat{r}\big)\big\|\bar{P}_i^{-1}\big\|\big|\bar{P}_i\big(z_{i+1}-z_i\big)\big| \\
&\le \frac{\bar{r}_i}{4\hat{N}}\big\|\bar{P}_i\big\| + \|P_i\|L\big(\hat{\rho}\hat{r}\big)\big\|\bar{P}_i^{-1}\big\|\big\|\bar{P}_i\big\|\frac{\bar{r}_i}{4\hat{N}} \\
&\le \frac{\hat{r}_i}{2\hat{N}}\Big(1+L\big(\hat{\rho}\hat{r}\big)\big\|\bar{P}_i^{-1}\big\|\|P_i\|\Big)\big\|P_i\big\|.
\end{aligned}
$$

Hence, by (3.7) we get

$$
\big|P_i\big(\hat{y}_{i+1}-\hat{y}_i\big)\big| \le \frac{2}{3\hat{N}}\hat{r}_i\|P_i\| < \frac{\hat{r}_i}{\hat{N}}\|P_i\|,
$$

that is, the point $\hat{y}_{i+1}$ belongs to the ellipsoid $E_i\big(\hat{y}_i, \hat{r}_i/\hat{N}\big)$ for every $i = 1, \ldots, \eta - 1$, which proves (v).

Finally, fix $i \in \{1, \ldots, \eta - 1\}$ and choose $x_m = x_{m(i)} \in X$ such that

$$
\bar{d}_i := \overline{\mathrm{dist}}_i(z_i, X) = \overline{\mathrm{dist}}_i(z_i, x_m) = \inf\big\{r \ge 0 \,|\, x_m \in \bar{E}_i(z, r)\big\}\,.
$$

Recall that $\hat{y}_i = z_i + \Psi_A(z_i)$ and $y_m := x_m + \Psi_A(x_m)$ belong to $Y$. In addition

$$
\begin{aligned}
\big|P_i\big(\hat{y}_i - y_m\big)\big| &= \big|P_i\big(z_i+\Psi_A(z_i)\big)-P_i\big(x_m+\Psi_A(x_m)\big)\big| \\
&\le \big|\bar{P}_i\big(z_i - x_m\big)\big| + \big|P_i\big(\Psi_A(z_i)-\Psi_A(x_m)\big)\big| \\
&\le \bar{d}_i\big\|\bar{P}_i\big\| + L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big|z_i - x_m\big| \\
&\le \bar{d}_i\big\|\bar{P}_i\big\| + L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big|\hat{y}_i - y_m\big|,
\end{aligned}
\tag{3.8}
$$

so that

$$
\begin{aligned}
\big|\hat{y}_i - y_m\big| &\le \big\|P_i^{-1}\big\|\big|P_i\big(\hat{y}_i - y_m\big)\big| \\
&\le \big\|P_i^{-1}\big\|\Big(\bar{d}_i\big\|\bar{P}_i\big\| + L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big|\hat{y}_i - y_m\big|\Big).
\end{aligned}
$$

Hence, if $\hat{r} > 0$ is small enough we get

$$
\big|\hat{y}_i - y_m\big| \le \frac{\bar{d}_i\big\|P_i^{-1}\big\|\big\|\bar{P}_i\big\|}{1 - L\big(\hat{\rho}\hat{r}\big)\big\|P_i^{-1}\big\|\|P_i\|} \le 2\bar{d}_i\big\|P_i^{-1}\big\|\big\|\bar{P}_i\big\|,
$$

which combined with (3.7) and (3.8) gives

$$
\big|P_i\big(\hat{y}_i - y_m\big)\big| \le \bar{d}_i\big\|\bar{P}_i\big\| + L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big|\hat{y}_i - y_m\big| \le \bar{d}_i\Big(1 + 2L\big(\hat{\rho}\hat{r}\big)\|P_i\|\big\|\bar{P}_i^{-1}\big\|\Big)\big\|\bar{P}_i\big\| < 2\bar{d}_i\big\|\bar{P}_i\big\|.
$$

Thus by (e) we obtain

$$
\mathrm{dist}_i\big(\hat{y}_i, Y\big) \le \frac{\big|P_i\big(\hat{y}_i - y\big)\big|}{\|P_i\|} \le \frac{2\bar{d}_i\big\|\bar{P}_i\big\|}{\|P_i\|} \le \frac{2\bar{r}_i\big\|\bar{P}_i\big\|}{\|P_i\|} = 4\hat{r}_i,
$$

which yields (vi) and concludes the proof. ∎

# 4 Proof of Theorem 1.1

## 4.1 Introduction

Let $H$ and $L$ be a Hamiltonian and its associated Lagrangian of class $C^k$, with $k \geq 4$, and let $\epsilon \in (0,1)$ be fixed. Without loss of generality, up to adding a constant to $H$ we may assume that $c[H] = 0$. We proceed as in the proof of [8, Theorems 2.1 and 2.4]: our goal is to find a potential $V : M \to \mathbb{R}$ of class $C^{k-1}$ with $\|V\|_{C^2} < \epsilon$, together with a $C^1$ function $v : M \to \mathbb{R}$ and a curve $\gamma : [0,T] \to M$ with $\gamma(0) = \gamma(T)$, such that the following properties are satisfied:

(P1) $H_V\big(x, dv(x)\big) \leq 0, \quad \forall\, x \in M;$

(P2) $\int_0^T L_V(\gamma(t), \dot\gamma(t))\, dt = 0.$

Indeed, as explained in [8, Subsection 5.1], these two properties imply that $c(H_V) = 0$ and that $\gamma([0,T])$ is contained in the projected Aubry set of $H_V$. Then, from this fact the statement of the theorem follows immediately by choosing as a potential $V - W$, where $W : M \to \mathbb{R}$ is any smooth function such that $W = 0$ on $\Gamma$, $W > 0$ outside $\Gamma$, and $\|W\|_{C^2} < \epsilon - \|V\|_{C^2}$.

As in the proof of [8, Theorems 2.1 and 2.4], we can assume that the Aubry set $\tilde{\mathcal{A}}(H)$ does not contain an equilibrium point or a periodic orbit (otherwise the proof is almost trivial, see [8, Subsection 5.1]), and we fix $\bar x \in \mathcal{A}(H)$ as in the statement of Theorem 1.1. By assumption, we know that there is a critical subsolution $u : M \to \mathbb{R}$ and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar x)$ such that $u$ is at least $C^{k+1}$ on $\mathcal{V}$. We set $\bar p := du(\bar x)$ and define the curve $\bar\gamma : \mathbb{R} \to M$ by

$$\bar\gamma(t) := \pi^*\Big(\phi_t^H\left(\bar x, \bar p\right)\Big) \qquad \forall\, t \in \mathbb{R}.$$

## 4.2 A review on how to close the Aubry set

In this subsection we briefly recall the construction performed in the proof of [8, Theorem 2.1], in particular the arguments in [8, Subsection 5.3].

Given $\epsilon > 0$ small, we fix a small neighborhood $\mathcal{U}_{\bar x} \subset M$ of $\bar x$, and a smooth diffeomorphism $\theta_{\bar x} : \mathcal{U}_{\bar x} \to B^n(0,1)$, such that

$$\theta_{\bar x}(\bar x) = 0_n \quad \text{and} \quad d\theta_{\bar x}(\bar x)\big(\dot{\bar\gamma}(0)\big) = e_1.$$

Then, we choose a point $\bar y = \bar\gamma(\bar t) \in \mathcal{A}(H)$, with $\bar t > 0$, such that, after a smooth diffeomorphism $\theta_{\bar y} : \mathcal{U}_{\bar y} \to B^n(0,2)$, $\theta_{\bar y}(\bar y) = (\bar\tau, 0_{n-1})$ and all assumptions (A1)-(A4) of Subsection 2.1 are satisfied at $(\bar\tau, 0_{n-1})$[4]. We denote by $\bar u : B^n(0,2) \to \mathbb{R}$ the $C^{k+1}$ function given by $\bar u(z) := u\big(\theta_{\bar y}^{-1}(z)\big)$ for $z \in B^n(0,2)$, and by $\bar H : B^n(0,2) \times \mathbb{R}^n \to \mathbb{R}$ the Hamiltonian of class $C^k$ associated with the Hamiltonian $H$ through $\theta$. Finally, we recall that $\Pi^0$ is the hyperplane passing through the origin which is orthogonal to the vector $e_1$ in $\mathbb{R}^n$, $\Pi^0_r := \Pi^0 \cap B^n(0,r)$ for every $r > 0$, and $\Pi^{\bar\tau} := \Pi^0 + \bar\tau e_1$, where $\bar\tau \in (0,1)$ is small but fixed.

We now fix $\bar r > 0$ small enough, and we use the recurrence assumption on $\bar x$ to find a time $T_{\bar r} > 0$ such that $\pi^*\big(\phi^H_{T_{\bar r}}(x, du(x))\big) \in \theta_{\bar x}^{-1}\left(\Pi^0_{\bar r}\right)$. Then, we look at the set of points

$$W := \Big\{ w_0 := \theta_{\bar x}(\bar x), w_1 := \theta_{\bar x}\big(\bar\gamma(t_1')\big), \ldots, w_J := \theta_{\bar x}\big(\bar\gamma(T_{\bar r})\big) \Big\} \subset \Pi^0 \cap \mathcal{A} \subset \Pi^0 \simeq \mathbb{R}^{n-1} \qquad (4.1)$$

(see [8, Equation (5.18)]) obtained by intersecting the curve

$$[0, T_{\bar r}] \ni t \mapsto \bar\gamma(t) := \pi^*\big(\phi_t^H\left(\bar x, du(\bar x)\right)\big)$$

[4] As shown in [8, Subsection 5.2] such a point always exists, see also Subsection 5.2 below.

with $\theta_{\bar{x}}^{-1}\big(\Pi^0_{\bar{\delta}/2}\big)$, where $\bar{r} \ll \bar{\delta} \ll 1$ (more precisely, $\bar{\delta} \in (0, 1/4)$ is provided by [8, Proposition 5.2]). We also consider the maps $\Phi_i : \Pi^0_{\delta_i} \to \Pi^0_{\bar{\delta}/2}$ corresponding to the $i$-th intersection of the curve $t \mapsto \pi^*\Big(\phi_t^H(\theta_{\bar{x}}^{-1}(w), du(\theta_{\bar{x}}^{-1}(w)))\Big)$ with $\theta_{\bar{y}}^{-1}(\Pi^0_{\bar{\delta}/2})$ (see [8, Equation (5.14)] and thereafter). Under our assumptions here, all the maps $\Phi_i$ are $C^1$. Hence, we define the ellipsoids $E_i$ associated to $P_i = D\Phi_i(0_{n-1})$, and we apply the classical Mai Lemma 3.1 to $X = W$ with $N \sim 1/\epsilon$. In this way, we get a sequence of points $\hat{w}_1, \ldots, \hat{w}_\eta$ in $\Pi^0_{\hat{\rho}\bar{r}}$ connecting $w_j$ to $w_l$ (see [8, Subsection 5.3, Properties (p5)-(p8)]), where $\hat{\rho} \geq 3$ is fixed and depends on $\epsilon$ but not on $\bar{r}$. Then, we use the flow map to send the points $\theta_{\bar{x}}^{-1}(\hat{w}_i)$ onto the "hyperplane" $S_{\bar{y}} := \theta_{\bar{y}}^{-1}\Big(\Pi^0_{\bar{\delta}/2}\Big)$ in the following way (see [8, Subsection 5.3, Figure 5]):

$$z_i^0 := \theta_{\bar{y}}\left(\Phi_i(\hat{w}_i)\right), \qquad z_i := \mathcal{P}(z_i^0), \qquad \tilde{z}_i^0 := \theta_{\bar{y}}\left(\Phi_i(\hat{w}_{i+1})\right), \qquad \tilde{z}_i := \mathcal{P}(\tilde{z}_i^0), \tag{4.2}$$

where $\mathcal{P}$ is the Poincaré mapping from $\Pi^0_{1/2}$ to $\Pi^{\bar{\tau}}_1$ (see [8, Lemma 5.1(ii)]).

Applying now [8, Proposition 5.2], we can find $C^2$-small potentials $V_i$, supported inside some suitable disjoints cylinders (see [8, Subsection 5.3, Property (p9)][5]), which allow to connect $z_i^0$ to $\tilde{z}_i$ with a control on the action like in Proposition 2.1(v), for some small constants $\sigma_i$ still to be chosen. Then the closed curve $\tilde{\gamma} : [0, t_f] \to M$ is obtained by concatenating $\gamma_1 : [0, \tilde{t}_\eta] \to M$ with $\gamma_2 : [\tilde{t}_\eta, t_f] \to M$, where

$$\gamma_2(t) := \pi^*\left(\phi^H_{t-\tilde{t}_\eta}\left(\theta_{\bar{y}}^{-1}(z^0_\eta), du\big(\theta_{\bar{y}}^{-1}(z^0_\eta)\big)\right)\right) \qquad \text{connects } \theta_{\bar{y}}^{-1}(z^0_\eta) \text{ to } x,$$

while $\gamma_1$ is obtained as a concatenation of $2\eta - 1$ pieces: for every $i = 1, \ldots, \eta - 1$, we use the flow $(t, z) \mapsto \pi^*\left(\phi_t^{H+V}(z, du(z))\right)$ to connect $\theta_{\bar{y}}^{-1}(z_i^0)$ to $\theta_{\bar{y}}^{-1}(\tilde{z}_i)$ on a time interval $[\tilde{t}_i, \tilde{t}_i + T_i^f]$, while on $[0, \tilde{t}_1]$ and on $[t_i + T_i^f, t_{i+1}]$ $(i = 1, \ldots, \eta - 1)$ we just use the original flow $(t, z) \mapsto \pi^*\left(\phi_t^H(z, du(z))\right)$ to send, respectively, $\theta_{\bar{x}}^{-1}(\hat{w}_1)$ onto $\theta_{\bar{y}}^{-1}(z_1^0)$ and $\theta_{\bar{y}}^{-1}(\tilde{z}_i)$ onto $\theta_{\bar{y}}^{-1}(z^0_{i+1})$. (See [8, Subsection 5.3] for more detail.) Moreover, as shown in [8, Subsection 5.4], one can choose the numbers $\sigma_i$ so that

$$|\sigma_i| \leq K_{\bar{u}}|\tilde{z}_i - z_i|^2 \leq 2K_{\bar{u}}|\tilde{z}_i^0 - z_i^0|^2 \tag{4.3}$$

(here $K_{\bar{u}} := \|\bar{u}\|_{C^2(B(0,2))}$, see [8, Equations (5.27) and (5.28)], and we used that $\mathcal{P}$ is 2-Lipschitz, see [8, Lemma 5.1(ii)]), and

$$\int_0^T L_V\big(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)\big)\, dt = 0$$

(which corresponds to property (P2) above).

Finally, using the characteristic theory for solutions to the Hamilton-Jacobi equation together with the estimates on the potential $V$, one can add another small potential, which vanishes together with its gradient on $\gamma$, so that one is able to construct a $C^{1,1}$ critical viscosity subsolution of $H_V \leq 0$ as in (P1) above, see [8, Subsection 5.5]. This concludes the argument in the proof of [8, Theorem 2.1].

## 4.3 Preliminary step

The above construction works well for instance when we have a critical subsolution which is a $C^2$ solution to the Hamilton-Jacobi equation in a neighborhood of the positive orbit $\mathcal{O}^+\big(\bar{x}\big)$ (see (1.2)), since we can control the action along all curves $t \mapsto \pi^*\big(\phi_t^H(x, du(x))\big)$ in terms of $u$ when $x$ is close to $\bar{x}$ (see [8, Paragraph 5.4]). However, since now we only have a smooth critical subsolution, we want to apply a refined version of the strategy used in [8, Theorem 2.4]: we define the nonnegative $C^k$-potential $V_0 : \mathcal{V} \to \mathbb{R}$ by

$$V_0(x) := -H\big(x, du(x)\big) \qquad \forall\, x \in \mathcal{V},$$

[5]This is the analogous of property ($\tilde{\pi}$12) in Subsection 4.4 below.

so that $u$ is a solution of

$$H\big(x, du(x)\big) + V_0(x) = 0 \qquad \forall\, x \in \mathcal{V}. \tag{4.4}$$

As in the proof of [8, Theorem 2.4] (see [8, Subsection 6.1]), the idea is to use the argument described in the previous subsection to find a small potential $V_\epsilon$ which allows to close the orbit $\mathcal{O}^+\big(\bar{x}\big)$ (since it belongs also to "the Aubry set of the Hamiltonian $H + V_0$"[6]), and then to replace $V_0$ by another potential $V_1 : M \to \mathbb{R}$, which has small $C^2$-norm and such that "the Aubry sets of $H + V_0 + V_\epsilon$ and $H + V_1 + V_\epsilon$ coincide" (see [8, Subsection 6.2]). In order to be able to apply this strategy in the current situation and to construct such a potential $V_1$, we will need to refine the argument described above in order to obtain finer properties on the "connecting" curve $\gamma_1$.

Let us recall that, by the proof of [8, Theorem 2.1] outlined above, fixed $\epsilon > 0$ small, for any small radius $\bar{r} > 0$ there exist an open set $\mathcal{U} = \mathcal{U}_{\bar{y}} \subset \mathcal{V}$, a potential $V_\epsilon : M \to \mathbb{R}$ of class $C^k$, a function $v : M \to \mathbb{R}$ of class $C^{1,1}$, and a closed curve $\gamma : [0, t_f] \to M$, such that $\gamma$ is obtained concatening two curves

$$\gamma_1 : \big[0, \tilde{t}_\eta\big] \longrightarrow M \quad \text{and} \quad \gamma_2 : \big[\tilde{t}_\eta, t_f\big] \longrightarrow M,$$

and, moreover, all the following properties are satisfied (recall that $\bar{\Gamma}_1 := \bar{\gamma}\big([0, \bar{t}_\eta]\big)$ for some suitable time $\bar{t}_\eta > 0$, see the proof of [8, Theorem 2.1] and [8, Subsection 6.1] for more details):

($\tilde{\pi}1$) $\big\|V_\epsilon\big\|_{C^2} < \epsilon/2$.

($\tilde{\pi}2$) Supp $(V_\epsilon) \subset \mathcal{U}$.

($\tilde{\pi}3$) $H\big(x, dv(x)\big) + V_0(x) = 0$ for every $x \in \mathcal{V} \setminus \mathcal{U}$.

($\tilde{\pi}4$) $H\big(x, dv(x)\big) + V_0(x) + V_\epsilon(x) \le 0$ for every $x \in \mathcal{U}$.

($\tilde{\pi}5$) $\int_0^{t_f} L\big(\gamma(t), \dot{\gamma}(t)\big) - V_0\big(\gamma(t)\big) - V_\epsilon\big(\gamma(t)\big)\, dt = 0$.

($\tilde{\pi}6$) For every $t \in \big[\tilde{t}_\eta, t_f\big], \gamma_2(t) \in \mathcal{A}(H)$.

($\tilde{\pi}7$) $\mathrm{dist}\big(\gamma_1(t), \bar{\Gamma}_1\big) \le K\bar{r}$ for all $t \in \big[0, \tilde{t}_\eta\big]$.

Assume for a moment that $V_0$ is defined everywhere on $M$. Then, as we explained above, this implies that the closed curve $\gamma : [0, t_f] \to M$ belongs to $\mathcal{A}(H + V_0 + V_\epsilon)$. However, although $V_\epsilon$ is small in $C^2$-norm, there is no reason for $\big\|V_0\big\|_{C^2(\mathcal{V})}$ to be small. This is why we have to replace it with a potential $V_1$ as described above. In [8, Subsections 6.2 and 6.3], the above properties ($\tilde{\pi}1$)-($\tilde{\pi}7$) were sufficient to construct such a $V_1$ when $M$ is two dimensional, but in this case they are not enough. Indeed, ($\tilde{\pi}7$) tells us that $\gamma_1$ is close to a set $\bar{\Gamma}_1$ which we know to be included in the projected Aubry set of $H$ (since $\bar{x}$, and so the whole curve $\bar{\gamma}$, are contained in $\mathcal{A}(H)$).

Since $\bar{x}$ is recurrent, in the two dimensional case this information allows to deduce that $V_0$ is very small, together with its derivative up to order 2, on $\gamma_1$ (see [8, Equation (6.2) and Remark 6.2]). Then, this fact together with the fact that $V_0$ vanishes with its gradient on $\gamma_2$ (since $\gamma_2$ in contained in $\mathcal{A}(H)$) allows to replace $V_0$ with a new potential $V_1$ as above.

Unfortunately, in higher dimension ($\tilde{\pi}7$) is not enough: even if we know that $\gamma_1$ is close to a set $\bar{\Gamma}_1$ where $V_0$ vanishes, this does not allow to get a control on all second derivatives, but only in the directions which are "tangent" to the Aubry set. Hence, it is important that the connecting curve $\gamma_1$ "almost" belongs to such directions.

For this reason, in the next subsection we will use our "constrained" results proved in Sections 2 and 3 to slightly modify the argument outlined in Subsection 4.2 and get a refined version of property ($\tilde{\pi}7$). Then, an improvement of [8, Lemma 6.1] (see Lemma 4.1 below) will allow to conclude as in [8, Theorem 2.4].

[6]Note that, since $V_0$ is well-defined only on $\mathcal{V}$, the Hamiltonian $H + V_0$ is not defined on $M$.

## 4.4 Refinement of connecting trajectories

The goal of this subsection is to use Proposition 2.1 and Lemma 3.4 to slightly modify the argument in the proof of [8, Theorem 2.1] and get a refined version of ($\tilde{\pi}$7).

With the same notation as in Subsection 4.2, set

$$A := \theta_{\bar{x}}\Big(\mathcal{A}(H) \cap \mathcal{U}_{\bar{x}}\Big) \cap \Pi^0_1 \subset \Pi^0.$$

By assumption, the origin is a cluster point of $A$. As in Section 3.3, we define the paratingent space to $A$ at the origin as

$$\Pi := \Pi_0(A) = \mathrm{Span}\left\{\lim_{i\to\infty} \frac{v_i - w_i}{|w_i - w_i|} \mid \lim_{i\to\infty} v_i = \lim_{i\to\infty} w_i = 0,\ v_i \in A,\ w_i \in A,\ v_i \neq w_i\, \forall\, i\right\}.$$

The vector space $\Pi$ has dimension $d \geq 1$ and is contained in $\Pi^0 \simeq \mathbb{R}^{n-1}$. Hence, from Lemma 3.3 there exist a radius $r_A > 0$ and a Lipschitz function $\Psi_A : \Pi \cap \bar{B}_{r_A} \to \Pi^\perp \subset \Pi_0$ such that, if we denote by $\Gamma_A$ the graph of $\Psi_A$, then

(p$_A$1) $A \cap \big\{(h, v) \in \Pi \times \Pi^\perp \mid |h| < r_A, |v| < r_A\big\} \subset \Gamma_A$.

(p$_A$2) For any $r \in (0, r_A)$, the Lipschitz constant $L_A(r)$ of $\Psi_A|_{B_r}$ satisfies $\lim_{r\downarrow 0} L_A(r) = 0$.

Hence, if we first choose $r_A \ll \bar{\delta}$ (with $\bar{\delta}$ to be chosen below, and which will be given by Proposition 2.1) and then $\bar{r} \ll r_0/\hat{\rho} := \min\{r_A, \hat{r}\}/\hat{\rho}$, with $\hat{r}, \hat{\rho}$ as in the constrainted Mai Lemma 3.4, since the set $W$ defined in (4.1) is contained inside $A$ we can apply Lemma 3.4 to obtain that all points $\hat{w}_1, \ldots, \hat{w}_\eta$ connecting $\hat{w}_1 = w_j$ to $\hat{w}_\eta = w_l$ belong to the graph $\Gamma_A$.

As a consequence, if for $r > 0$ small we denote, respectively, by $\mathcal{W}(r)$ and $\mathcal{W}_A(r)$ the image under the flow of $\Pi^0_{\hat{\rho}r}$ and $\Gamma_A \cap \Pi^0_{\hat{\rho}r}$ for $\eta$ laps, that is[7],

$$\mathcal{W}(r) := \left\{\pi^*\Big(\phi^H_t\Big(\theta^{-1}_{\bar{x}}(w), du\big(\theta^{-1}_{\bar{x}}(w)\big)\Big)\Big) \mid w \in \Pi^0_{\hat{\rho}r},\ t \in \big[0, \tau_\eta(w)\big]\right\},$$

$$\mathcal{W}_A(r) := \left\{\pi^*\Big(\phi^H_t\Big(\theta^{-1}_{\bar{x}}(w), du\big(\theta^{-1}_{\bar{x}}(w)\big)\Big)\Big) \mid w \in \Gamma_A \cap \Pi^0_{\hat{\rho}r},\ t \in \big[0, \tau_\eta(w)\big]\right\},$$

then, by to the construction of $\gamma$ outlined in Subsection 4.2 and ($\tilde{\pi}$6),

($\tilde{\pi}$8) $\gamma_1(t) \in \mathcal{W}_A(\bar{r})$ for every $t \in \big[0, \tilde{t}_1\big] \cup \big[\tilde{t}_1 + T^f_1, \tilde{t}_2\big] \cup \ldots \cup \big[\tilde{t}_{\eta-1} + T^f_{\eta-1}, \tilde{t}_\eta\big]$.

($\tilde{\pi}$9) For every $t \in \big[\tilde{t}_\eta, t_f\big]$,

$$\gamma_2(t) \in \mathcal{W}(r_0) \quad \Longrightarrow \quad \gamma_2(t) \in \mathcal{W}_A(r_0).$$

(Recall that $r_0 = \min\{r_A, \hat{r}\}$).

Let us recall that the points $z^0_i$ and $\tilde{z}_i$ are defined in (4.2). In particular, since $\theta^1_{\bar{x}}(w_1), \theta^1_{\bar{x}}(w_\eta) \in \mathcal{A}(H)$, there holds:

($\tilde{\pi}$10) $\theta^1_{\bar{y}}(z^0_1), \theta^1_{\bar{y}}(z^0_\eta) \in \mathcal{A}(H)$.

As explained in Subsection 4.2, in the proof of [8, Theorem 2.1] the two states $\big(z^0_i, \nabla\bar{u}(z^0_i)\big)$ and $\big(\tilde{z}_i, \nabla\bar{u}(\tilde{z}_i)\big)$ are connected using [8, Proposition 5.2] for every $i = 1, \ldots, \eta - 1$. Here, we need the connecting trajectories (seen in $\mathbb{R}^n$) to stay very close to the graph $\mathcal{W}_A(\bar{r})$. To this aim,

[7] Here, according to the notations of the proof of Theorem [8, Theorem 2.1] (see in particular [8, Equation (5.14)]), $\tau_\eta$ denotes the $\eta$-th Poincaré time mapping $\tau_\eta : \Pi^0_{\delta_\eta} \to (0, +\infty)$, i.e., $\tau_\eta(0)$ is the $\eta$-th time when the curve $t \mapsto \phi^H_t\big(\bar{x}, du(\bar{x})\big)$ intersects the hyperplane $\theta^{-1}_{\bar{y}}(\Pi^0_{\bar{\delta}/2})$ (recall that $\bar{x} = \theta^{-1}_{\bar{x}}(0)$), and

$$\phi^H_{\tau_\eta(z)}\big(z, du(z)\big) \in \theta^{-1}_{\bar{y}}\left(\Pi^0_{\bar{\delta}/2}\right) \qquad \forall\, z \in \theta^{-1}_{\bar{x}}\left(\Pi^0_{\hat{\rho}\bar{r}}\right).$$

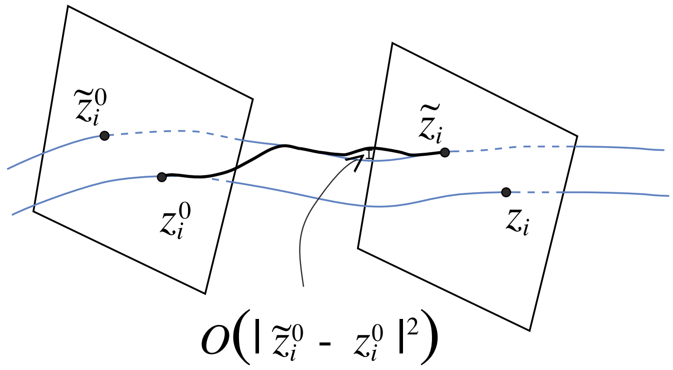


Figure 1: By using Proposition 2.1, we can connect $z_i^0$ to $\tilde{z}_i$ using a trajectory which almost belongs to the "surface" spanned by the original trajectories, and then we compensated the action by remaining in a $O(|\tilde{z}_i^0 - z_i^0|^2)$-neighborhood of the second trajectory.

since $H + V_0$ is of class $C^k$ with $k \geq 4$, we can apply Proposition 2.1: for every $i = 1, \dots, \eta - 1$ we denote by $c_i : [\tilde{t}_i, \tilde{t}_i + T_i^f] \to B^n(0, 2)$ the connecting trajectories of class $C^{k-1}$provided by Proposition 2.1[8], and we set

$$\gamma_1(t) := \theta_{\bar{y}}^{-1}(c_i(t)) \qquad \forall\, t \in [\tilde{t}_i, \tilde{t}_i + T_i^f].$$

Thanks to Proposition 2.1(vi) and the bound on the constants $\sigma_i$ provided by (4.3) we get:

($\tilde{\pi}$11) There exists a uniform constant $K > 0$ such that, for every $i = 1, \dots, \eta - 1$,

$$c_i(t) \in \mathcal{R}_i^1 \cup \mathcal{B}_i^2 \qquad \forall\, t \in [\tilde{t}_i, \tilde{t}_i + T_i^f],$$

where $\mathcal{R}_i^1$ and $\mathcal{B}_i^2$ are defined as

$$\mathcal{R}_i^1 := \mathcal{R}\Big((z_i^0, d\bar{u}(z_i^0)); (\tilde{z}_i^0, d\bar{u}(\tilde{z}_i^0)); K\,|\tilde{z}_i^0 - z_i^0|^2\Big) \cap \mathcal{E}^1,$$

$$\mathcal{B}_i^2 := \mathcal{B}\Big((\tilde{z}_i^0, d\bar{u}(\tilde{z}_i^0)); K\,|\tilde{z}_i^0 - z_i^0|^2\Big) \cap \mathcal{E}^2,$$

with $\mathcal{R}, \mathcal{B}, \mathcal{E}^1, \mathcal{E}^2$ as in (2.6)-(2.11).

Moreover, for $\bar{r}$ small enough, Proposition 2.1(i) gives that the $C^{k-1}$-potential $V_\epsilon : M \to \mathbb{R}$ used to connect the trajectories satisfies:

($\tilde{\pi}$12) $\mathrm{Supp}(V_\epsilon \circ \theta_{\bar{y}}^{-1}) \subset \bigcup_{i=1}^{\eta-1} \mathcal{C}^i \subset \mathcal{W}(r_0)$, where

$$\mathcal{C}^i := \mathcal{C}\Big((z_i^0, \nabla\bar{u}(z_i^0)); \tau(z_i^0, \nabla\bar{u}(z_i^0)), \hat{r}_i\Big),$$

with $\hat{r}_i \in (0, \hat{\rho}\bar{r})$ the radii provided by Lemma 3.4.

Recall now that $\hat{N} \sim 1/\epsilon$. More precisely, we can choose $\hat{N} \in (K/\epsilon, K\epsilon + 1) \cap \mathbb{N}$ with $K$ a sufficiently large constant (see [8, Equation (5.16)]) so that, thanks to Lemma 3.4(v)-(vi), each radius $\hat{r}_i$ satisfies:

($\tilde{\pi}$13) $|\tilde{z}_i^0 - z_i^0| < \hat{r}_i\,\epsilon$ and $\hat{r}_i \geq \mathrm{dist}(z_i^0, Z)/8$, where $Z := \Big\{z = (0, \hat{z}) \in \Pi^0_{\hat{\rho}\hat{r}} \,|\, \theta_{\bar{y}}^{-1}(z) \in \mathcal{A}(H)\Big\}$

We also recall that, by construction of $\gamma_1$ and $\gamma_2$, there holds (see [8, Subsection 5.3, Claim 2]):

($\tilde{\pi}$14) $\Big(\gamma_2\left([\tilde{t}_\eta, t_f]\right) \cup \gamma_1\left([0, \tilde{t}_1]\right) \cup \bigcup_{i=1}^{\eta-1} \gamma_1\left([\tilde{t}_i + T_i^f, \tilde{t}_{i+1}]\right)\Big) \cap \mathcal{C}^i = \emptyset$ for all $i = 1, \dots, \eta - 1$.

[8]When applying Proposition 2.1 we get a potential $V_\epsilon$ of class $C^{k-1}$, so the Hamiltonian trajectories are of class $C^{k-1}$ as well.

## 4.5 Modification of the potential $V_0$ and conclusion

In the previous subsection we found a potential $V_\epsilon$ and a closed curve $\gamma : [0, t_f] \to M$ such that ($\tilde\pi$1)-($\tilde\pi$14) hold, and $\gamma$ is "contained in the projected Aubry set for $H + V_0 + V_\epsilon$" (this is a bit informal, since $V_0$ is only defined on $\mathcal{V}$). As we already said before, although $V_\epsilon$ is small in $C^2$-norm, there is no reason for $\|V_0\|_{C^2(\mathcal{V})}$ to be small. The idea is therefore the following: first of all, by choosing $\bar r$ sufficiently small we can ensure that the curve $\gamma$ is as close as we want to $\mathcal{A}(H)$ (see ($\tilde\pi$7)). Now, since $V_0 \geq 0$ vanishes on $\mathcal{A}(H)$, we can ensure that both $V_0$ and $\nabla V_0$ are small in a neighborhood of $\gamma$. Moreover, since $V_0$ attains its minimum on $\mathcal{A}(H)$, the negative part of $\nabla^2 V_0$ can also be made as small as we wish. The strategy is then to find a new potential $V_1 : M \to \mathbb{R}$ of class $C^{k-1}$ such that the following properties are satisfied:

($\tilde\pi$15) $\|V_1\|_{C^2} < \epsilon/2$.

($\tilde\pi$16) $V_1(x) \leq V_0(x)$, for every $x \in \mathcal{V}$.

($\tilde\pi$17) $V_1(x) \leq 0$, for every $x \in M \setminus \mathcal{V}$.

($\tilde\pi$18) $V_1\big(\gamma(t)\big) = V_0\big(\gamma(t)\big)$, for every $t \in [0, t_f]$.

Assuming that we are able to do this, Theorem 1.2 follows as in [8, Subsection 6.2]. Hence we are left with the construction of $V_1$, which we perform in the next subsection.

## 4.6 Construction of the potential $V_1$

In this section, since the construction is already very involved, in order to avoid notational complications which may obscure the ideas behind the construction of $V_1$, we perform some change of coordinates. Since $H$ is of class at least $C^k$, $u$ of class at least $C^{k+1}$, and $V_0$ is of class $C^k$, the flow map $(t, z) \mapsto \phi_t^H\big(z, du(z)\big)$ is of class $C^{k-1}$. Hence, fixed a small radius $r_0 > 0$, we can construct a $C^{k-1}$ diffeomorphism $\Phi$ from $\mathcal{W} := \{z \,|\, \mathrm{dist}(z, \bar\Gamma_1) \leq r_0\}$ to $(0, \eta) \times B_{r_0}^{n-1} \subset \mathbb{R}^n$, depending on $\epsilon$ and $r_0$, so that we reduce to the following simplified situation[9]:

($\hat\pi$1) $\bar\gamma_1$ is a segment of the form

$$\bar\Gamma_1 := \left\{ \big(t, 0_{n-1}\big) \,|\, t \in [0, \eta] \right\}.$$

($\hat\pi$2) $\mathcal{W}$ is a long thin cylinder along $\bar\Gamma_1$, that is, $\mathcal{W} := (0, T) \times B_{r_0}$.

[9] The diffeomorphism $\Phi$ has to:

- transform a finite number of integral curves into straight lines, see ($\hat\pi$1);
- for $t \in [i, i + 1/20]$, let the connecting trajectory from $z_i^0$ to $\tilde z_i$ lie in the rectangle $[i, i + 1/20] \times [v_i, v_{i+1}]$, see (4) (compare with (xi) above, where the trajectory lies in $\mathcal{R}_i$).

This can be done, for instance, by first straightening the trajectories of $(t, z) \mapsto \phi_t^H\big(z, du(z)\big)$ with the inverse of the flow map (which will be of class $C^{k-1}$) and then modifying the connecting trajectory to make it lie in a plane (though the construction is a bit tedious).

There is, however, a simple possible way to avoid this construction, which has as the only drawback the need for $H$ to be of class $C^k$ with $k \geq 5$, and to finally produce a potential of class $C^{k-2}$, instead of $C^{k-1}$ (which, however, is irrelevant for the purpose of proving the Mañé conjecture in $C^1$-topology). This amounts to repeat the argument in Proposition 2.1 by:

- First, use a $C^k$ diffeomorphism $\Phi$ to transform the integral trajectories starting from $x^1$ and $x^2$ into straight segments.
- Then, connect the straigthened trajectories using the same formula as in the proof of Proposition 2.1 (i.e., considering a convex combination of them as in (2.18), and then reparameterizing the time as in (2.20)).

Observe however that, since $\Phi$ is $C^k$, the transformed Hamiltonian will only be $C^{k-1}$ (since the $p$-variable transforms through $d\Phi$), and so the potential provided by Proposition 2.1 will be only of class $C^{k-2}$.

($\hat\pi$3) The points $z_i^0$, $\tilde z_i^0$ and $\tilde z_i$ from the previous subsections are given by

$$z_i^0 = \big(i, v_i\big), \qquad \tilde z_i^0 = \big(i, v_{i+1}\big), \qquad \tilde z_i = \big(i+1/10, v_{i+1}\big),$$

for some sequence $v_1, \ldots, v_\eta \in B^{n-1}_{\hat\rho\bar r} \subset \mathbb{R}^{n-1}$.

($\hat\pi$4) The set $\Gamma_1 := \gamma_1\left([0, \tilde t_\eta]\right)$ has the form

$$\Gamma_1 = I_0 \cup \Bigg(\bigcup_{i=1,\ldots,\eta-1} I_i\Bigg) \cup \Bigg(\bigcup_{i=1,\ldots,\eta-1} C_i\Bigg),$$

where

$$I_0 := [0,1]\times\{v_1\}, \quad I_i := \big[i+1/10, i+1\big]\times\{v_{i+1}\}, \qquad i = 1,\ldots,\eta-1,$$

$$\begin{aligned} C_i := c_i\big([i, i+1/10]\big) \subset \Big\{\big(s, v\big) \mid s \in \big[i, i+1/20\big],\, v \in \big[v_i, v_{i+1}\big]\Big\} \\ \cup \Bigg(\bigcup_{s\in[i+1/20, i+1/10]} B\left(\big(s, v_{i+1}\big), K\big|v_{i+1}-v_i\big|^2\right)\Bigg), \end{aligned}$$

where $c_i$ are the (constrained) connecting curves (which are of class $C^{k-1}$).

($\hat\pi$5) For any $r > 0$ small, the set $\mathcal{W}_A(r)$ has the form

$$\mathcal{W}_A(r) = \Big\{(t, w) \mid w \in \Gamma_A \cap \Pi^0_{\hat\rho s},\, t \in [0, \eta]\Big\}.$$

($\hat\pi$6) $z_1^0, z_\eta^0 \in \mathcal{S}(\bar r) \cap \{V_0 = 0\}$ (thanks to ($\tilde\pi$10)).

($\hat\pi$7) There exist radii $r_i > 0$ such that, for all $i = 1, \ldots, \eta-1$,

$$B^{n-1}(v_i, r_i) \subset B^{n-1}_{\hat\rho\bar r}, \qquad r_i \geq \operatorname{dist}\big(v_i, \mathcal{Z}\big)/8, \qquad |v_{i+1}-v_i| \leq \frac{r_i}{32},$$

where $\mathcal{Z} := B^{n-1}_{r_0} \cap \{V_0 = 0\}$ (by ($\tilde\pi$13), for $\epsilon$ small enough).

($\hat\pi$8) The curve $\gamma_2$ never intersects the set

$$\mathcal{Q} := \bigcup_{i=1}^{\eta-1}\Bigg(\bigcup_{s\in[i,i+1]} B^n\Big(\big(s, v_i\big), r_i\Big)\Bigg)$$

(as a consequence of ($\tilde\pi$14)).

We further remark that, since the function $g := V_0 \circ \Phi^{-1}$ is of class $C^{k-1}$ (with $k-1 \geq 3$), nonnegative, vanishes on $\bar\Gamma_1$, it satisfies:

($\hat\pi$9) For all $(t, w) \in \mathcal{W}$, there holds:

$$\big|g(t,w)\big| \leq K'|w|^2, \quad \big|\nabla g(t,w)\big| \leq K'|w|, \quad \text{Hess}\, g(t,w) \geq -K'|w| I_n,$$

for some constant $K' > 0$, which depends on $\epsilon$ but not on $\bar r$.

Moreover, by definition of $\Pi$ as the paratingent space of the projected Aubry set at $\bar x$ in the direction orthogonal to $\bar\Gamma_1$, thanks to ($\hat\pi$5) above it is easily seen that the paratingent space of $\mathcal{A}(H)$ at every point of $z \in \bar\Gamma_1$ in the direction orthogonal $\bar\Gamma_1$ coincides with $\Pi$. Hence, since $\nabla V_0$ is at least $C^2$ and vanishes on the Aubry set, Hess $g$ vanishes in the directions of $\Pi$ along $\bar\Gamma_1$, which gives

($\hat{\pi}10$) $\left\|\text{Hess } g(t,w)_{|\Pi}\right\| \le K'|w|$ for all $(t,w) \in \mathcal{W}$.

As observed above, the diffeomorphism $\Phi$ depends on $\epsilon$ and $r_0$, which are small but fixed constants. Hence, if now we are able to construct a compactly supported potential $G : (0,\eta) \times B^{n-1}_{r_0} \to \mathbb{R}$ which satisfies the analogous of ($\tilde{\pi}16$)-($\tilde{\pi}18$) above and with a $C^2$-norm as small as we wish, then $V_1 := G \circ \Phi$ will satisfy ($\tilde{\pi}15$)-($\tilde{\pi}18$).

Now, if $\Pi = \mathbb{R}^{n-1}$, then ($\hat{\pi}10$) corresponds to [8, Equation (6.2)] with $g$ in place of $V_0$ and $\omega(r) = K'r$. So, the construction of $V_1$ becomes the same as in the two dimensional case and we easily conclude by [8, Lemma 6.1]. On the other hand, if $\dim(\Pi) \in \{1,\dots,n-2\}$ then the construction of $V_1$ relies on the following result, whose proof is postponed to Appendix A:

**Lemma 4.1.** *Given $n \ge 3$, real numbers $r_0, \bar{K} > 0$, and an integer $\eta \ge 3$, consider the cylinder $\mathcal{C} := [0,\eta] \times B^{n-1}_{r_0}$. Let $g : \mathcal{C} \to \mathbb{R}$ be a nonnegative function of class $C^l$ with $l \ge 3$, $\Pi \subset \mathbb{R}^{n-1}$ be a vector space of dimension $d \in \{1,\dots,n-2\}$, $\Pi^\perp$ its orthogonal in $\mathbb{R}^{n-1}$. Let $\Psi : \Pi \cap B^d_{r_0} \to \Pi^\perp$ be a Lipschitz function whose graph is denoted by $\Gamma$, and whose Lipschitz constant on $\Pi \cap \bar{B}^d_r$ is denoted by $L(r)$, for every $r \in (0,r_0)$. Assume that $L(r) \to 0$ as $r \to 0$, and there exists a positive constant $\bar{K}$ such that*

$$\left\|\text{Hess } g(x)_{|\Pi}\right\| \le \bar{K}r \qquad \forall\, x \in [0,\eta] \times B^{n-1}_r,\ r \in (0,r_0) \tag{4.5}$$

$$\text{Hess } g(x) \ge -\bar{K}rI_n \qquad \forall\, x \in [0,\eta] \times B^{n-1}_r,\ r \in (0,r_0). \tag{4.6}$$

*Define the set $\mathcal{Z} \in B^{n-1}_{r_0}$ as*

$$\mathcal{Z} := \Big\{ v \in B^{n-1}_{r_0} \cap \Gamma \,|\, g(s,v) = 0 \quad \forall\, s \in [0,\eta] \Big\}.$$

*Then, for every $\epsilon', \bar{K}' > 0$ there exists a (small) radius $r > 0$ such that the following holds: For every set of points $v_1,\dots,v_\eta \in \mathbb{R}^{n-1}$ such that*

$$v_1, v_\eta \in \Gamma \cap \mathcal{Z} \cap B^{n-1}_r, \qquad v_2,\dots,v_{\eta-1} \in \Gamma \cap B^{n-1}_r, \tag{4.7}$$

*and every set of real numbers $r_1,\dots,r_{\eta-1} \in (0,r)$ such that*

$$B^{n-1}\big(v_i,r_i\big) \subset B^{n-1}_r, \tag{4.8}$$

$$r_i \ge \operatorname{dist}\big(v_i,\mathcal{Z}\big)/8, \tag{4.9}$$

$$\big|v_{i+1} - v_i\big| \le r_i/32, \tag{4.10}$$

*and every set of $C^l$ curves $\{c_1(\cdot),\dots,c_{\eta-1}(\cdot)\}$ with*

$$c_i(\cdot) \,:\, \big[i, i+1/10\big] \longrightarrow \mathcal{C}$$

*satisfying*

$$\big|\ddot{c}_i(s)\big| \le \bar{K}' \qquad \forall\, s \in \big[i,i+1/10\big], \tag{4.11}$$

$$c_i(i) = (i,v_i), \quad c_i(i+1/10) = \big(i+1/10, v_{i+1}\big), \tag{4.12}$$

*and*

$$C_i := c_i\big([i,i+1/10]\big) \subset \Big\{ \big(s,v\big) \,|\, s \in \big[i,i+1/20\big],\, v \in \big[v_i,v_{i+1}\big] \Big\} \bigcup \Bigg( \bigcup_{s \in [i+1/20,i+1/10]} B\left( \big(s,v_{i+1}\big), \bar{K}\big|v_{i+1}-v_i\big|^2 \right) \Bigg), \tag{4.13}$$

*there exists a function $G : \mathcal{C} \to \mathbb{R}$ of class $C^l$ such that:*

*(a)* *Supp* $(G) \subset \mathcal{C}$*;*

*(b)* $\|G\|_{C^2(\mathcal{C})} < \epsilon'$*;*

*(c)* $G(x) \leq g(x)$ *for every* $x \in \mathcal{C}$*;*

*(d)* $G(x) = g(x)$ *for every* $x \in S \subset \mathcal{C}$*, with*

$$S := I_0 \cup \Bigg( \bigcup_{i=1,\ldots,\eta-1} I_i \Bigg) \cup \Bigg( \bigcup_{i=1,\ldots,\eta-1} C_i \Bigg),$$

*where* $I_0 := [0,1] \times \{v_1\}$ *and* $I_i := \big[i + 1/10, i+1\big) \times \{v_{i+1}\}$ *for every* $i = 1, \ldots, \eta - 1$*;*

*(e)* $G(x) = 0$ *for every* $x \in ([0,\eta] \times \mathcal{Z}) \cap \big(\big([0,\eta] \times \Gamma\big) \setminus \mathcal{Q}\big)$*, with*

$$\mathcal{Q} := \bigcup_{i=1}^{\eta-1} \Bigg( \bigcup_{s \in [0,\eta]} B^n\Big( (s, v_i), r_i \Big) \Bigg).$$

Let us remark that properties (c) and (d) imply that $\nabla G = \nabla g$ on $S$, while (e) ensures that $G = g = 0$ on $\gamma_2$ (compare with ($\hat{\pi}7$) above). Hence, by choosing $\epsilon'$ in (b) sufficiently small and defining $V_1 := G \circ \Phi$, we obtain a potential which satisfies ($\tilde{\pi}15$)-($\tilde{\pi}18$). This concludes the proof of Theorem 1.1.

# 5 Proof of Theorem 1.2

As already explained at the end of [8, Section 2], the rough idea of choosing a time $T \gg 1$ such that $\pi^*\left(\phi_T^H\left(\bar{x}, du(\bar{x})\right)\right)$ is sufficiently close to $\bar{x}$, and then "closing" the trajectory in one step, does not work if one want to use a potential which is small in $C^2$ topology. However, since now we only want the $C^1$ norm of $V$ to be small, we can use this strategy. Hence, as we will see, the "connecting part" of the construction becomes much easier (in particular, we do not need to use Mai Lemma). On the other hand, the construction of a critical subsolution becomes more involved. Indeed, since now we do not control the $C^2$ norm of the potential $V$ that we use to connect the orbit, the Hamilton-Jacobi equation associated to $H + V$ may have conjugate points along the connecting trajectory. In order to prevent this, we will add a second potential which has the feature to make the characteristics fall apart, so that regularity of solutions to the Hamilton-Jacobi equations propagates on some uniform time interval, see Lemma 5.5(vi). Thanks to this result, we will be able to construct a global critical visocsity subsolution, which will allow to conclude the proof.

## 5.1 Introduction

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$, with $k \geq 4$, and let $\epsilon \in (0,1)$ be fixed. Without loss of generality, up to adding a constant to $H$, we can assume that $c[H] = 0$. Let $L$ denote the Lagrangian associated to $H$. As in the proof of Theorem 1.1 (see Subsection 4.1 and [8, Subsection 5.1]), it suffices to find a potential $V : M \to \mathbb{R}$ of class $C^{k-2}$ with $\|V\|_{C^1} < \epsilon$, a continuous function $v : M \to \mathbb{R}$, and a curve $\gamma : [0,T] \to M$ with $\gamma(0) = \gamma(T)$, such that the following properties are satisfied:

(P1) $v$ is a viscosity subsolution of $H_V\big(x, dv(x)\big) = 0 \quad \forall\, x \in M$.

(P2) $\int_0^T L_V(\gamma(t), \dot{\gamma}(t))\, dt = 0$.

From now on, we assume that the Aubry set $\tilde{\mathcal{A}}(H)$ does not contain an equilibrium point or a periodic orbit, and we fix $\bar{x}$ a recurrent point of the projected Aubry set[10]. We also fix, thanks to Bernard's Theorem [2] (see also [8, Subsection 1.2]), a critical subsolution $u : M \to \mathbb{R}$ of class $C^{1,1}$. Moreover, we set $\bar{p} := du(\bar{x})$ and define the curve $\bar{\gamma} : \mathbb{R} \to M$ by

$$\bar{\gamma}(t) := \pi^* \Big( \phi_t^H (\bar{x}, \bar{p}) \Big) \qquad \forall\, t \in \mathbb{R}.$$

## 5.2 Preliminary step

As in [8, Subsection 5.2], we claim that there is a time $\bar{t} > 0$ such that

$$\frac{d}{dt} \Big\{ u\Big( \phi_t^H (\bar{x}, \bar{p}) \Big) \Big\}_{|t=\bar{t}} = \big\langle du\big(\bar{\gamma}(\bar{t})\big), \dot{\bar{\gamma}}(\bar{t}) \big\rangle \geq 0. \tag{5.1}$$

Indeed, arguing by contradiction and assuming that

$$\frac{d}{dt} \Big\{ u\Big( \phi_t^H (\bar{x}, \bar{p}) \Big) \Big\} = \big\langle du\big(\bar{\gamma}(t)\big), \dot{\bar{\gamma}}(t) \big\rangle < 0 \qquad \forall\, t > 0,$$

one obtains

$$u\big(\bar{\gamma}(T)\big) - u(\bar{x}) = u\big(\bar{\gamma}(T)\big) - u\big(\bar{\gamma}(0)\big) = \int_0^T \big\langle du\big(\bar{\gamma}(t)\big), \dot{\bar{\gamma}}(t) \big\rangle \, dt \leq -c_0 < 0 \qquad \forall\, T \geq 1,$$

which is absurd since $\bar{x}$ is recurrent.

We now proceed as in the proof of [8, Theorem 2.1]. We set $\bar{y} := \bar{\gamma}(\bar{t})$ and fix $\bar{\tau} \in (0, 1)$. Note that $\bar{y}$ is a recurrent point of $\mathcal{A}(H)$, and there exist an open neighborhood $\mathcal{U}_{\bar{y}}$ of $\bar{y}$ in $M$ with $\bar{x} \notin \mathcal{U}_{\bar{y}}$, and a smooth diffeomorphism

$$\theta_{\bar{y}} : \mathcal{U}_{\bar{y}} \to B^n(0, 2),$$

such that

$$\theta_{\bar{y}}(\bar{y}) = (\bar{\tau}, 0_{n-1}) \quad \text{and} \quad \big\langle d\theta_{\bar{y}}(\bar{y}), \dot{\bar{\gamma}}(\bar{t}) \big\rangle = e_1. \tag{5.2}$$

Denote by $\Pi^0$ the hyperplane passing through the origin which is orthogonal to the vector $e_1$ in $\mathbb{R}^n$ and set

$$\Pi^\tau := \tau e_1 + \Pi^0, \quad \Pi_r^0 := \Pi^0 \cap B^n(0, r), \quad \Pi_r^\tau := \Pi^\tau \cap B^n\big(\tau e_1, r\big) \qquad \forall\, \tau \in \mathbb{R},\, r > 0. \tag{5.3}$$

The Hamiltonian $H : T^*M \to \mathbb{R}$ is sent, via the smooth diffeomorphism $\theta_{\bar{y}}$, onto a Hamiltonian $\bar{H}$ of class $C^k$ on $B^n(0, 2) \times \mathbb{R}^n$, and the critical subsolution $u : M \to \mathbb{R}$ is sent via $\theta_{\bar{y}}$ onto the $C^{1,1}$ function $\bar{u} : B^n(0, 2) \to \mathbb{R}$,

$$\bar{u}(z) := u\big(\theta_{\bar{y}}^{-1}(z)\big) \qquad \forall\, z \in B^n(0, 2),$$

which is a $C^{1,1}$ subsolution of the Hamilton-Jacobi equation associated with $\bar{H}$, that is,

$$\bar{H}\big(z, \nabla \bar{u}(z)\big) \leq 0 \qquad \forall\, z \in B^n(0, 2). \tag{5.4}$$

We set

$$\bar{\mathcal{A}} := \theta_{\bar{y}} \Big( \mathcal{A}(H) \cap \mathcal{U}_{\bar{y}} \Big).$$

[10] Since the Aubry set is a compact invariant set which is invariant under the Lagrangian flow, it necessarily contains recurrent points. Indeed, one can use for instance Zorn Lemma to find a minimal invariant subset, and then minimality implies that all orbits are dense in such a subset.

We observe that the Hamiltonian $\bar{H}$ can be seen as the restriction of a Hamiltonian $\bar{H}$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying (H1)-(H3). For every $z^0 \in \Pi_1^0$, let us denote by

$$\Big(Z\big(\,\cdot\,;z^0\big),Q\big(\,\cdot\,;z^0\big)\Big)\,:\,[0,+\infty)\longrightarrow \mathbb{R}^n\times\mathbb{R}^n$$

the solution of the Hamiltonian system

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & \nabla_q \bar{H}\big(z(t),q(t)\big) \\ \dot{q}(t) & = & -\nabla_z \bar{H}\big(z(t),q(t)\big) \end{array}\right. \tag{5.5}$$

on $[0,+\infty)$ satisfying

$$z(0)=z^0 \quad \text{ and } \quad q(0)=\nabla\bar{u}(z^0).$$

Observe that, by (5.1) and (5.2), $Q\big(0,\theta_{\bar{y}}(\bar{y})\big) = e_1$ and (A4) holds. Moreover, by taking $\bar{\tau} \in (0,1/10)$ sufficiently small, we may assume that $Z\big(t;z^0\big)$ belongs to $B^n(0,2)$ for every $z^0 \in \Pi_1^0$ and $t \in \big[0,5\bar{\tau}\big]$, and that the Hamiltonian trajectory

$$\big(\bar{z}(\cdot),\bar{q}(\cdot)\big):=\Big(Z\big(\,\cdot\,;0_n\big),Q\big(\,\cdot\,;0_n\big)\Big)=\Big(\theta_{\bar{y}}\big(\bar{\gamma}(t-\bar{t}+\bar{\tau})\big),\big(d_{\theta_{\bar{y}}(\bar{\gamma}(t-\bar{t}+\bar{\tau}))}\theta_{\bar{y}}^{-1}\big)^*du\big(\bar{\gamma}(t-\bar{t}+\bar{\tau})\big)\Big) \tag{5.6}$$

satisfies (A1)-(A4) over the time intervals $[\bar{\tau},3\bar{\tau}/2]$ and $[3\bar{\tau}/2,2\bar{\tau}]$ (see (5.1)).

For every $0 \leq a < b$, let us denote by $\mathcal{H}_{[a,b]}$ the vertical slice defined by

$$\mathcal{H}_{[a,b]} := \Big\{z=\big(z_1,\hat{z}\big)\in\mathbb{R}^n \,|\, z_1\in[a,b]\Big\}.$$

Up to reducing again $\bar{\tau}$, we can also assume that the following holds[11]:

**Lemma 5.1.** *The following properties are satisfied:*

*(i) For every $\tau \in (0,5\bar{\tau}]$, the Poincaré time mapping $\mathcal{T}_\tau : \Pi_{1/2}^0 \to \mathbb{R}$*

$$Z\big(\mathcal{T}_\tau(z^0);z^0\big)=\pi^*\left(\phi^{\bar{H}}_{\mathcal{T}_\tau(z^0)}\big(z^0,\nabla\bar{u}(z^0)\big)\right)\in\Pi_1^\tau \qquad \forall\, z^0\in\Pi_{1/2}^0,$$

*is well-defined and of class $C^{k-1}$;*

*(ii) for every $\tau \in (0,5\bar{\tau}]$, the Poincaré mapping $\mathcal{P}_\tau$ defined by*

$$\begin{array}{rccl} \mathcal{P}: & \Pi_{1/2}^0 & \to & \Pi_1^\tau \\ & z^0 & \longmapsto & \mathcal{P}(z^0):=Z\big(\mathcal{T}_\tau(z^0);z^0\big) \end{array}$$

*is 2-Lipschitz;*

*(iii) the following inclusion holds for every $\tau \in (0,5\bar{\tau}]$:*

$$\Big\{Z\big(t;z^0\big)\,|\,z^0\in\Pi_{3/8}^0,\,t\in\big[0,\mathcal{T}_\tau(z^0)\big]\Big\}\subset[0,\tau]\times B^{n-1}(0_{n-1},1/2);$$

*(iv) the viscosity solution $\bar{u}_0$ to the Dirichlet problem*

$$\left\{\begin{array}{l} \bar{H}\big(z,\nabla\bar{u}_0(z)\big)=0 \quad \textit{in } \in B^n(0,1)\cap\mathcal{H}_{[0,5\bar{\tau}]}, \\ \bar{u}_0=\bar{u} \quad \textit{on } \Pi_1^0, \end{array}\right. \tag{5.7}$$

*is of class $C^{1,1}$.*

In the next section we state a series of lemmas which are crucial for the proof of Theorem 1.2. The proofs of some of them are postponed to Appendix C.

[11] Property (iv) is a consequence of the results proved in [4]: it states that the solution of (5.7) obtained by characteristics is locally of class $C^{1,1}$ (since $\bar{u}$ is itself of class $C^{1,1}$). We refer the reader to [3, 4, 13] for more details about the method of characteristics.

## 5.3 Preparatory lemmas

Our first lemma follows from [1, Remarque 6.3.3]. It states that, given a finite set of points, we can always find two of them which are "far enough" from the others. For the sake of completeness, we provide its proof in Appendix C.1.

**Lemma 5.2.** *Let $r > 0$ and $Y$ be a finite set in $\mathbb{R}^m$ such that $B_{r/12} \cap Y$ contains at least two points. Then, there are $y_1 \neq y_2 \in Y$ such that the cylinder $\mathrm{Cyl}_0^{1/3}\big(y_1; y_2\big)$ (see (2.31)) is included in $B_r$ and does not intersect $Y \setminus \{y_1, y_2\}$.*

As in Subsection 2.3, for every pair $z_1^0 = \big(0, \hat{z}_1^0\big), z_2^0 = \big(0, \hat{z}_1^0\big) \in \Pi_{1/8}^0$ and every $\lambda > 0, \tau \in (0, 5\bar{\tau}]$, we define the cylinders $\mathrm{Cyl}^{\lambda}_{[0,\mathcal{T}_\tau]}\big(z_1^0; z_2^0\big)$, $\mathcal{C}^{\lambda}_{[0,\mathcal{T}_\tau]}\big(z_1^0; z_2^0\big)$ along the trajectories $Z\big(\,\cdot\,; z_1^0\big), Z\big(\,\cdot\,; z_2^0\big)$ as

$$\mathrm{Cyl}^{\lambda}_{[0,\mathcal{T}_\tau]}\big(z_1^0; z_2^0\big) := \Big\{ Z\big(t; z^0\big) \mid z^0 = (0, \hat{z}^0),\ \hat{z}^0 \in \mathrm{Cyl}_0^{\lambda}\big(\hat{z}_1^0; \hat{z}_2^0\big),\ t \in \big[0, \mathcal{T}_\tau(z^0)\big] \Big\}.$$

and

$$\begin{aligned} &\mathcal{C}^{\lambda}_{[0,\mathcal{T}_\tau]}\big(z_1^0; z_2^0\big) \\ &\qquad := \left\{ Z\left(t; \left(\frac{z_1^0 + z_2^0}{2}\right)\right) + (0, \hat{z}) \,\middle|\, t \in \left[0, \mathcal{T}_\tau\left(\frac{z_1^0 + z_2^0}{2}\right)\right],\ \hat{z} \in \mathrm{Cyl}_0^{\lambda}\big(\hat{z}_1^0; \hat{z}_2^0\big) \right\}, \end{aligned} \tag{5.8}$$

where the convex set $\mathrm{Cyl}_0^{\lambda}\big(\hat{z}_1^0; \hat{z}_2^0\big)$ is defined as in (2.31). The proof of the following lemma is given in Appendix C.2.

**Lemma 5.3.** *Given $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, if $\bar{\tau} \in (0, 1/10)$ is sufficiently small, then for every pair $z_1^0 = \big(0, \hat{z}_1^0\big), z_2^0 = \big(0, \hat{z}_1^0\big) \in \Pi_{1/8}^0$ the following inclusions hold:*

$$\mathrm{Cyl}^{\lambda_1}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big) \subset \mathcal{C}^{\lambda_2}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big) \subset \mathcal{C}^{\lambda_3}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big) \subset \mathrm{Cyl}^{\lambda_4}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big).$$

Given a potential $\bar{V} : \mathbb{R}^n \to \mathbb{R}$ of class at least $C^2$ and $T > 0$, we denote by

$$\big(Z_{\bar{V}}\big(\,\cdot\,; z^0\big), Q_{\bar{V}}\big(\,\cdot\,; z^0\big)\big) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$$

the solution of the Hamiltonian system

$$\left\{ \begin{array}{rcl} \dot{z}(t) &=& \nabla_q \bar{H}_{\bar{V}}(z(t), q(t)) = \nabla_q \bar{H}(z(t), q(t)) \\ \dot{q}(t) &=& -\nabla_z \bar{H}_{\bar{V}}(z(t), q(t)) = -\nabla_z \bar{H}(z(t), q(t)) - \nabla \bar{V}(z(t)) \end{array} \right. \tag{5.9}$$

starting from $\big(z^0, \nabla \bar{u}(z^0)\big)$. Moreover, we denote by $\mathbb{A}_{\bar{V}}\big(z^0; T\big)$ the action of the curve $Z_{\bar{V}}\big(\,\cdot\,; z^0\big) : [0, T] \to \mathbb{R}^n$, that is,

$$\begin{aligned} \mathbb{A}_{\bar{V}}\big(z^0; T\big) &:= \int_0^T \bar{L}_{\bar{V}}\Big(Z_{\bar{V}}\big(t; z^0\big), \dot{Z}_{\bar{V}}\big(t; z^0\big)\Big)\, dt \\ &= \int_0^T \bar{L}\Big(Z_{\bar{V}}\big(t; z^0\big), \dot{Z}_{\bar{V}}\big(t; z^0\big)\Big) - \bar{V}\,\big(Z_{\bar{V}}\big(t; z^0\big)\big)\, dt. \end{aligned}$$

From Proposition 2.2 applied on $\big[0, 2\bar{\tau}\big]$, and Remark 2.4 applied with $\bar{\nu}_1 = \bar{\tau}$ and $\bar{\nu}_2 = 3\bar{\tau}/2$, we immediately get the following result:

**Lemma 5.4.** *There are $\bar{\delta} \in (0, 1/8)$ and $K > 0$ such that the following property holds: For every $z_1^0 = \big(0, \hat{z}_1^0\big), z_2^0 = \big(0, \hat{z}_1^0\big) \in \Pi_{\bar{\delta}}^0 \cap \bar{\mathcal{A}}$, there are $\bar{T}^f > 0$ and a potential $\bar{V}_0 : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$ such that:*

*(i)* *Supp* $\big(\bar{V}_0\big) \subset \mathrm{Cyl}^{1/9}_{[0,\mathcal{T}_{2\bar\tau}]}\big(z^0_1; z^0_2\big) \cap \mathcal{H}_{[\bar\tau,2\bar\tau]}$;

*(ii)* $\big\|\bar{V}_0\big\|_{C^1} < K\big|z^0_1 - z^0_2\big|$;

*(iii)* $\big\|\bar{V}_0\big\|_{C^2} < K$;

*(iv)* $\big|\bar{T}^f - \mathcal{T}_{2\bar\tau}(z^0_1)\big| < K\big|z^0_1 - z^0_2\big|$;

*(v)* $\phi^{\bar{H}_{\bar{V}_0}}_{\bar{T}^f}\big(z^0_1, \nabla\bar{u}(z^0_1)\big) = \phi^{\bar{H}}_{\mathcal{T}_{2\bar\tau}(z^0_2)}\big(z^0_2, \nabla\bar{u}(z^0_2)\big)$;

*(vi)* *for any* $\tau \in [0,\bar\tau]$, $t \in [0,\mathcal{T}_{2\bar\tau}(z^0_1)]$ *and* $t_{\bar{V}_0} \in [0,\bar{T}^f]$ *such that* $Z(t; z^0_1), Z_{\bar{V}_0}(t_{\bar{V}_0}; z^0_1) \in \Pi^\tau$, *it holds:* $\big|t_{\bar{V}_0} - t\big| \le K\left|z^0_1 - z^0_2\right|$ *and*

$$\left|\mathbb{A}_{\bar{V}_0}\left(z^0_1; t_{\bar{V}_0}\right) - \mathbb{A}\left(z^0_1; t\right) - \left\langle\nabla\bar{u}\big(Z(t; z^0_1)\big)\right), Z_{\bar{V}_0}(t_{\bar{V}_0}; z^0_1) - Z(t; z^0_1)\right\rangle\right| \le K\left|z^0_1 - z^0_2\right|^2;$$

*(vii)* $\bar{u}\Big(Z_{\bar{V}_0}\big(\mathcal{T}_{2\bar\tau}(z^0_2); z^0_2\big)\Big) = \bar{u}\big(z^0_1\big) + \mathbb{A}_{\bar{V}_0}\left(z^0_1; \bar{T}^f\right)$.

*In particular, thanks to (i),*

*(viii)* *for every* $t \in [0, \bar{T}^f - \mathcal{T}_{2\bar\tau}(z^0_2) + \mathcal{T}_{5\bar\tau}(z^0_2)]$,

$$Z_{\bar{V}_0}\big(t; z^0_1\big) \in \mathrm{Cyl}^{1/9}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z^0_1; z^0_2\big).$$

Unfortunately, though the above lemma is enough to connect trajectories, we will need a strengthened version of that result which ensures that the viscosity solution constructed by characteristics is of class $C^{1,1}$ in a neighborhood of the connecting trajectory (this fact will be needed in Subsection 5.5 to construct a global critical viscosity subsolution). The proof of the following result is given in Appendix C.3.

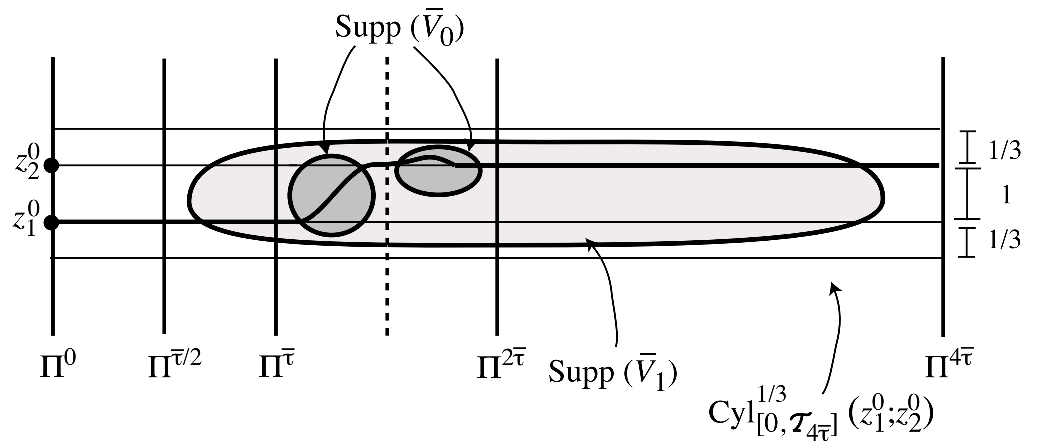


Figure 2: By adding to $V_0$ a non-positive potential $V_1$ which vanishes together with his gradient along the connecting trajectory $Z_{\bar{V}_0}\left(\,\cdot\,; z^0_1\right)$, we can ensure that the characteristics associated to $\bar{H}_{\bar{V}_0+\bar{V}_1}$ do not cross near $Z_{\bar{V}_0}\left(\,\cdot\,; z^0_1\right)$, so that the viscosity solution to the Dirichlet problem (5.10) is $C^{1,1}$ in a uniform neighborhood of $Z_{\bar{V}_0}\left(\,\cdot\,; z^0_1\right)$.

**Lemma 5.5.** *Taking $\bar\tau \in (0,1/10)$ smaller if necessary, there exist $\tilde\delta \in (0,\bar\delta)$ (with $\bar\delta$ given by Lemma 5.4) and $\tilde K > 0$ such that the following property holds: For every $z_1^0 = \big(0,\hat z_1^0\big), z_2^0 = \big(0,\hat z_2^0\big) \in \Pi^0_{\tilde\delta} \cap \bar{\mathcal{A}}$, let $\bar T^f > 0$ and $\bar V_0 : \mathbb{R}^n \to \mathbb{R}$ be as in Lemma 5.4. Then there exists a potential $\bar V_1 : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-2}$such that:*

*(i)* *Supp* $\big(\bar V_1\big) \subset \mathrm{Cyl}^{1/3}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0; z_2^0\big) \cap \mathcal{H}_{[\bar\tau/2,4\bar\tau]}$;

*(ii)* $\bar V_1 \le 0$;

*(iii)* $\bar V_1\big(Z_{\bar V_0}\big(t;z_1^0\big)\big) = 0$ *and* $\nabla \bar V_1\big(Z_{\bar V_0}\big(t;z_1^0\big)\big) = 0$ *for every* $t \in \big[0,5\bar\tau\big]$;

*(iv)* $\big\|\bar V_1\big\|_{C^1} < \tilde K \big|z_1^0 - z_2^0\big|$;

*(v)* $\big\|\bar V_1\big\|_{C^2} < \tilde K$;

*(vi)* *the viscosity solution* $\bar u_{\bar V_0+\bar V_1}$ *to the Dirichlet problem*

$$\begin{cases} \bar H\big(z, \nabla \bar u_{\bar V_0+\bar V_1}(z)\big) + \bar V_0(z) + \bar V_1(z) = 0 & \text{in } B^n(0,1)\cap \mathcal{H}_{[0,5\bar\tau]}, \\ \bar u_{\bar V_0+\bar V_1} = \bar u \quad \text{on } \Pi^0_1, \end{cases} \tag{5.10}$$

*is of class* $C^{1,1}$ *on* $\mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big)$, *with a* $C^{1,1}$*-norm bounded by* $\tilde K$;

*(vii)* $\big|\bar u_{\bar V_0+\bar V_1}\big(Z_{\bar V_0+\bar V_1}(t;z_1^0)\big) - \bar u\big(Z_{\bar V_0+\bar V_1}(t;z_1^0)\big)\big| \le \tilde K \big|z_1^0 - z_2^0|^2$ *for all* $t \in [0,\bar T^f]$.

*In particular, thanks to (i), (iii), and Lemma 5.4,*

*(viii)* $\phi^{\bar H_{\bar V_0+\bar V_1}}_{\bar T^f}\big(z_1^0, \nabla \bar u(z_1^0)\big) = \phi^{\bar H}_{\mathcal{T}_{2\bar\tau}(z_2^0)}\big(z_2^0, \nabla \bar u(z_2^0)\big)$;

*(ix)* $\bar u\Big(Z\big(\mathcal{T}_{2\bar\tau}(z_2^0); z_2^0\big)\Big) = \bar u\big(z_1^0\big) + \mathbb{A}_{\bar V_0+\bar V_1}\big(z_1^0; \bar T^f\big)$;

*(x)* *for every* $t \in [0, \bar T^f - \mathcal{T}_{2\bar\tau}(z_2^0) + \mathcal{T}_{5\bar\tau}(z_2^0)]$,

$$Z_{\bar V_0+\bar V_1}(t;z_1^0) = Z_{\bar V_0}\big(t;z_1^0\big) \in \mathrm{Cyl}^{1/9}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big).$$

## 5.4 Closing the Aubry set and the action

Let $\bar\delta > 0$ be as in Lemma 5.4, and set

$$S_{\bar y} := \theta_{\bar y}^{-1}\Big(\Pi^0_{\bar\delta/2}\Big).$$

Let $r \in \big(0,\bar\delta/2\big)$ be fixed, and set $z_0 := 0_n = \theta_{\bar y}\big(\bar\gamma\big(\bar t - \bar\tau\big)\big) \in \Pi^0_{r/12}$. Since $\bar y = (\bar\gamma(\bar t))$ is recurrent, there exists $t^0 > 0$ such that

$$z_0' := \theta_{\bar y}\big(\bar\gamma\big(\bar t - \bar\tau + t^0\big)\big) \in \Pi^0_{r/12}.$$

From Lemma 5.1(i), the set of nonnegative times

$$\mathcal{T} := \Big\{t \in \big[0,t^0\big] \,|\, \bar\gamma\big(\bar t - \bar\tau + t\big) \in S_{\bar y}\Big\}$$

is finite. Set

$$Y := \Big\{\theta_{\bar y}\Big(\bar\gamma\big(\bar t - \bar\tau + t\big)\Big) \,|\, t \in \mathcal{T}\Big\} \subset \bar{\mathcal{A}}.$$

Thanks to Lemma 5.2, there are $z_1^0 = \big(0,\hat z_1^0\big), z_2^0 = \big(0,\hat z_1^0\big) \in \Pi^0_r \cap \bar{\mathcal{A}} \subset \Pi^0_{\bar\delta/2} \cap \bar{\mathcal{A}}$ and $t_1, t_2 \ge 0$ with:

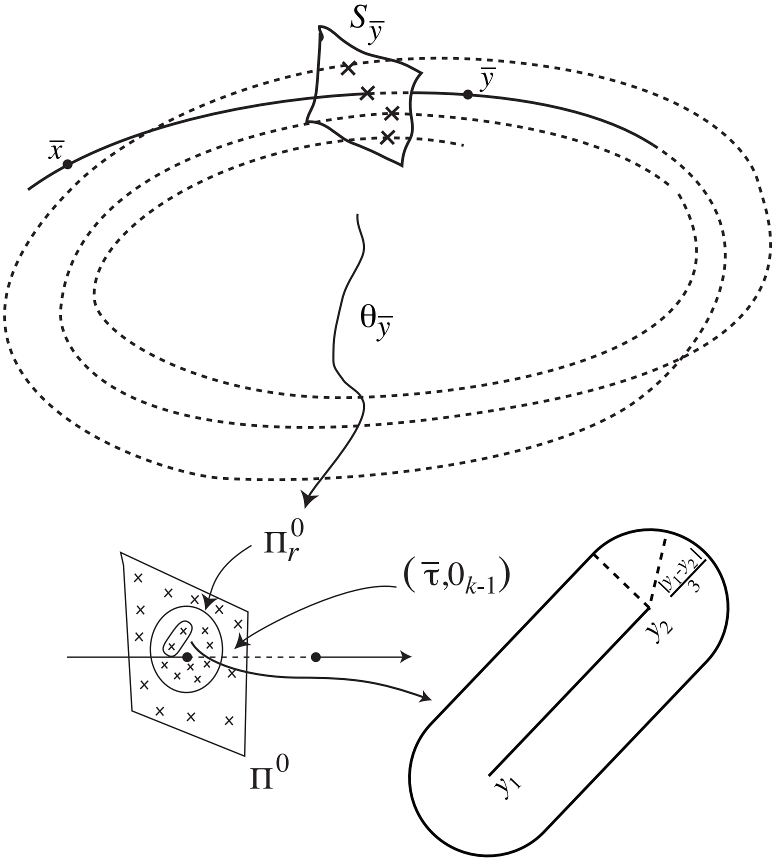


Figure 3: Using Lemma 5.2, we can find two points $z_1^0, z_2^0 \in \Pi_r^0 \cap \bar{\mathcal{A}}$ such that $\mathrm{Cyl}_0^{1/3}\left(\hat{z}_1^0; \hat{z}_2^0\right) \subset \Pi_r^0$ is disjoint from $Y \setminus \left\{\hat{z}_1^0, \hat{z}_2^0\right\}$.

(p1) $z_1^0 = \bar{\gamma}\big(\bar{t} - \bar{\tau} + t_1\big)$ and $z_2^0 = \bar{\gamma}\big(\bar{t} - \bar{\tau} + t_2\big)$.

(p2) $t_1 > t_2 + \bar{\tau}$.

(p3) $\mathrm{Cyl}_0^{1/3}\big(\hat{z}_1^0; \hat{z}_2^0\big) \subset \Pi_r^0$ and $\mathrm{Cyl}_0^{1/3}\big(\hat{z}_1^0; \hat{z}_2^0\big) \cap \Big(Y \setminus \big\{\hat{z}_1^0, \hat{z}_2^0\big\}\Big) = \emptyset$.

Note that the latter property, together with the definition of $\mathrm{Cyl}^{\lambda}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big)$, implies

(p4) for every $t \in \mathcal{T} \setminus \{t_1, t_2\}$ and every $s \in \Big[0, \mathcal{T}_{5\bar{\tau}}\Big(\theta_{\bar{y}}\big(\bar{\gamma}\big(\bar{t} - \bar{\tau} + t\big)\big)\Big)\Big]$,

$$\theta_{\bar{y}}\Big(\bar{\gamma}\big(\bar{t} - \bar{\tau} + t + s\big)\Big) \notin \mathrm{Cyl}^{1/3}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big).$$

By Lemmas 5.4 and 5.5, there exist a time $\bar{T}^f > 0$ and a potential $\bar{V} := \bar{V}_0 + \bar{V}_1 : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-2}$ such that:

(p5) $\mathrm{Supp}\ (\bar{V}) \subset \mathrm{Cyl}^{1/3}_{[0,\mathcal{T}_{4\bar{\tau}}]}\big(z_1^0; z_2^0\big) \cap \mathcal{H}_{[\bar{\tau}/2, 4\bar{\tau}]}$.

(p6) $\|\bar{V}\|_{C^1} < K\big|z_1^0 - z_2^0\big|$.

(p7) $\big|\bar{T}^f - \mathcal{T}_{2\bar{\tau}}(z_1^0)\big| < K\big|z_1^0 - z_2^0\big|$.

(p8) $\phi^{\bar H_{\bar V}}_{\bar T^f}\big(z_1^0,\nabla\bar u(z_1^0)\big)=\phi^{\bar H}_{\mathcal{T}_{2\bar\tau}(z_2^0)}\big(z_2^0,\nabla\bar u(z_2^0)\big)$.

(p9) $\big|\bar u_{\bar V}\big(Z_{\bar V}(t;z_1^0)\big)-\bar u\big(Z_{\bar V}(t;z_1^0)\big)\big|\leq\tilde K\big|z_1^0-z_2^0|^2$ for all $t\in[0,\bar T^f]$.

(p10) $\bar u\Big(Z\big(\mathcal{T}_{2\bar\tau}(z_2^0);z_2^0\big)\Big)-\bar u\big(z_1^0\big)=\mathbb{A}_{\bar V}\left(\big(z_1^0,\nabla\bar u(z_1^0)\big);\bar T^f\right)$.

Define the $C^{k-2}$ potential $V:M\to\mathbb{R}$ by

$$V(x):=\begin{cases}0 & \text{if } x\notin\mathcal{U}_{\bar y}\\ \bar V\big(\theta_{\bar y}(x)\big) & \text{if } x\in\mathcal{U}_{\bar y},\end{cases}$$

and the curve $\gamma:\Big[0,T:=\Big(t_1-\big(t_2+\bar\tau\big)\Big)+\bar T^f\Big]\to M$ by

$$\gamma(t):=\begin{cases}\theta_{\bar y}^{-1}\big(Z_{\bar V}\big(t;z_1^0\big)\big) & \text{if } t\in\big[0,\bar T^f\big]\\ \bar\gamma\big(t_2+\bar\tau+t-\bar T^f\big) & \text{if } t\in\big[\bar T^f,T\big].\end{cases}$$

By construction and the fact that $\bar\gamma\in\mathcal{A}(H)$, it is easily checked that $\gamma(T)=\gamma(0)$, $\|V\|_{C^1}<Kr$, and

$$\int_0^T L_V\big(\gamma(t),\dot\gamma(t)\big)\,dt=0.$$

This solves the problem (P2) by choosing $r$ small enough. It remains to construct a continuous function $v:M\to\mathbb{R}$ satisfying (P1).

## 5.5 Construction of a critical viscosity subsolution

The aim of this subsection is to modify the potential $V$ constructed above so that the action of $\gamma$ is still zero, and we can find a critical viscosity subsolution of $H_V(x,dv(x))=0$, see (P1).

Recall that $\bar u:B^n(0,2)\to\mathbb{R}$ is a function of class $C^{1,1}$ (obtained by looking at $u$ in the chart induced by $\theta_{\bar y}$) satisfying

$$\bar H\big(z,\nabla\bar u(z)\big)\leq 0\qquad\forall\, z\in B^n(0,2)\tag{5.11}$$

(see Lemma 5.1(iv)), and that $\bar u_{\bar V}:=\bar u_{\bar V_0+\bar V_1}:\mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big)\subset B^n(0,2)\to\mathbb{R}$ is a function of class $C^{1,1}$, with its $C^{1,1}$-norm bounded independently of $z_1^0$ and $z_2^0$, satisfying

$$\bar u_{\bar V}=\bar u\qquad\text{on }\Pi_1^0\tag{5.12}$$

and

$$\bar H\big(z,\nabla\bar u_{\bar V}(z)\big)+\bar V_0(z)+\bar V_1(z)=0\qquad\forall\, z\in\mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big)\tag{5.13}$$

(see Lemma 5.5). The idea is to "glue" these two functions together, to obtain a $C^{1,1}$-function $\hat u$ which coincides with $\bar u$ on $B^n(0,2)\setminus\mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big)$, and solves

$$\bar H_{\bar V+\bar W}(z,\nabla\hat u(z))\leq 0\qquad\text{on } B^n(0,2),$$

where $\bar W:B^n(0,2)\to\mathbb{R}$ is a potential of class $C^{k-1}$, small in $C^1$ topology, supported inside $\mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big)$, and which vanishes together with its gradient on the connecting trajectory $Z_{\bar V}(\,\cdot\,;z_1^0)=Z_{\bar V_0+\bar V_1}(\,\cdot\,;z_1^0)$ (see Figure 4 below). If we are able to do so, then it suffices to define

$$V(x):=\begin{cases}0 & \text{if } x\notin\mathcal{U}_{\bar y}\\ \bar V\big(\theta_{\bar y}(x)\big)+\bar W\big(\theta_{\bar y}(x)\big) & \text{if } x\in\mathcal{U}_{\bar y},\end{cases}$$

$$v := \begin{cases} u & \text{if } x \notin \mathcal{U}_{\bar{y}} \\ \hat{u}\big(\theta_{\bar{y}}(x)\big) & \text{if } x \in \mathcal{U}_{\bar{y}}, \end{cases}$$

to obtain that both (P1) and (P2) are satisfied, which will conclude the proof of Theorem 1.2.

To perform the above construction, we first apply a change of coordinates of class $C^{k-1}$ so that

($\pi$1) $Z(t; z^0_{1,2}) = z^0_{1,2} + te_1$ for all $z \in B^n(0,1)$, $t \in [0,1]$, where $z^0_{1,2} := \frac{z^0_1 + z^0_2}{2}$.

In particular, all cylinders $\mathcal{C}^\lambda_{[0,\mathcal{T}_\tau]}\big(z^0_1; z^0_2\big)$ have the form

$$\mathcal{C}^\lambda_{[0,\mathcal{T}_\tau]}\big(z^0_1; z^0_2\big) = [0,\tau] \times \mathrm{Cyl}^\lambda_0\big(\hat{z}^0_1; \hat{z}^0_2\big)$$

(see (5.8)). Moreover, since $\nabla \bar{u}_0$ is Lipschitz (see Lemma 5.1(iv)) and

$$\dot{Z}(t; z^0_i) = \nabla_q \bar{H}\big(Z(t; z^0_i), \nabla \bar{u}_0\big(Z(t; z^0_i)\big)\big), \qquad i = 1, 2,$$

using Gronwall's Lemma and ($\pi$1) we easily obtain the existence, a constant $K_0 > 0$, such that, if we denote by $\hat{Z}(t; z^1_i)$ the last $(n-1)$-coordinates of $Z(t; z^1_i)$, then

$$\big|\dot{\hat{Z}}(t; z^0_i)\big| \leq K_0 \big|z^0_1 - z^0_2\big| \qquad \forall\, t \in \big[0, 5\bar{\tau}\big],\ i = 1, 2. \tag{5.14}$$

Moreover, since the connecting trajectory $Z_{\bar{V}}(t; z^0_1)$ is constructed by interpolating between $Z(t, z^0_1)$ and $Z(t, z^0_2)$ (up to a quadratic term), it is not difficult to check that there exists a constant $K'_0$ such that

($\pi$2) $\big|\dot{\hat{Z}}_{\bar{V}}(t; z^0_1)\big| \leq K'_0 \big|z^0_1 - z^0_2\big|$ for all $t \in \big[0, 5\bar{\tau}\big]$.

We now assume that $\bar{\tau}$ is sufficiently small so that, using Lemma 5.3, we have

$$\mathrm{Cyl}^{1/9}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big) \subset \mathcal{C}^{1/8}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big) \subset \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big) \subset \mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big). \tag{5.15}$$

In particular, thanks to Lemma 5.4(viii) and ($\pi$1), we have

($\pi$3) $Z_{\bar{V}}(t; z^0_1) \in \mathcal{C}^{1/8}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big)$ for all $t \in \big[0, 5\bar{\tau}\big]$.

Furthermore, since $\mathrm{Supp}\big(\bar{V}_0\big) \subset \mathcal{H}_{[\bar{\tau},2\bar{\tau}]}$ and $z^0_1, z^0_2 \in \bar{\mathcal{A}}$, by (5.12), Lemma 5.4(i) and Lemma 5.5, the following holds:

($\pi$4) $\bar{u}\big(Z_{\bar{V}}(t; z^0_1)\big) = \bar{u}_{\bar{V}}\big(Z_{\bar{V}}(t; z^0_1)\big)$ and $\nabla \bar{u}\big(Z_{\bar{V}}(t; z^0_1)\big) = \nabla \bar{u}_{\bar{V}}\big(Z_{\bar{V}}(t; z^0_1)\big)$ on $\mathcal{H}_{[0,\bar{\tau}]} \cup \mathcal{H}_{[2\bar{\tau},5\bar{\tau}]}$.

Moreover, the fact that $\nabla \bar{u}(z^0_1) = \nabla \bar{u}_{\bar{V}}(z^0_1)$ together with (p6) and a simple Gronwall argument (see for instance [8, Lemma 5.5 and Equation (5.37)]) implies

$$\big|Z(t; z^0_1) - Z_{\bar{V}}(t; z^0_1)\big| + \big|\nabla \bar{u}\big(Z(t; z^0_1)\big) - \nabla \bar{u}_{\bar{V}}\big(Z_{\bar{V}}(t; z^0_1)\big)\big| \leq K\big|z^0_1 - z^0_2\big|,$$

which combined with the Lipschitz regularity of $\nabla \bar{u}$ and (p9), implies

($\pi$5) for all $t \in [0, T^f]$,

$$\big|\nabla \bar{u}\big(Z_{\bar{V}}(t; z^0_1)\big) - \nabla \bar{u}_{\bar{V}}\big(Z_{\bar{V}}(t; z^0_1)\big)\big| \leq K\big|z^0_1 - z^0_2\big|.$$

All in all, (p9), ($\pi$4), and ($\pi$5), together with the $C^{1,1}$-regularity of $\bar{u} - \bar{u}_{\bar{V}}$ on the set

$$\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big) \subset \mathrm{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z^0_1; z^0_2\big),$$

(see (5.15)) imply the following important estimate:

($\pi$6) there exists $\bar K > 0$ such that

$$\big|\bar u_{\bar V} - \bar u\big| \le \bar K \big|z_1^0 - z_2^0\big|^2, \quad \big|\nabla \bar u_{\bar V} - \nabla \bar u\big| \le \bar K \big|z_1^0 - z_2^0\big| \qquad \text{on } \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0; z_2^0\big).$$

Now, let $\Theta : B^n(0,2) \to [0,1]$ be a smooth function such that

$$\begin{cases} \Theta(z) = \Theta(z_1) = 1 & \text{if } z_1 \in [\bar\tau/2, 4\bar\tau], \\ \Theta(z) = \Theta(z_1) = 0 & \text{if } z_1 \in [0, \bar\tau/4] \cup [9\bar\tau/2, 5\bar\tau], \end{cases}$$

and define $\tilde u : \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0; z_2^0\big) \to \mathbb{R}$ by

$$\tilde u(z) := \Theta(z)\bar u_{\bar V}(z) + \big(1 - \Theta(z)\big)\bar u(z) \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0; z_2^0\big).$$

Observe that $\tilde u$ is of class $C^{1,1}$ on the cylinder $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0; z_2^0\big)$.

Let us define the set $\Gamma_1$ by

$$\Gamma_1 := \left\{ Z_{\bar V}(t; z_1^0) \,|\, t \in \big[0, 5\bar\tau\big] \right\},$$

and denote by $\mathrm{dist}(\,\cdot\,, \Gamma_1)$ the distance function to the curve $\Gamma_1$. The following result holds:

**Lemma 5.6.** *There exists a constant $\tilde K > 0$ such that*

$$\bar H_{\bar V}\big(z, \nabla \tilde u(z)\big) \le 0 \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0, z_2^0\big) \cap \big(\mathcal{H}_{[0,\bar\tau/4]} \cup \mathcal{H}_{[9\bar\tau/2,5\bar\tau]}\big),$$

$$\bar H_{\bar V}\big(z, \nabla \tilde u(z)\big) \le \tilde K \mathrm{dist}\big(z, \Gamma_1\big)^2 \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0, z_2^0\big) \cap \mathcal{H}_{[\bar\tau/4, 9\bar\tau/2]}.$$

*Proof.* Since $\bar V = 0$ on $B^n(0,2) \setminus \mathcal{H}_{\bar\tau_2, 4\bar\tau}$, we have $\tilde u = \bar u$ on $B^n(0,2) \cap \big(\mathcal{H}_{[0,\bar\tau/4]} \cup \mathcal{H}_{[9\bar\tau/2,5\bar\tau]}\big)$ and the first statement follows from (5.11).

Concerning the second part, observe that since $\tilde u = \bar u_{\bar V}$ on $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0, z_2^0\big) \cap \mathcal{H}_{[\bar\tau/2, 4\bar\tau]}$, by (5.13) we have

$$\bar H_{\bar V}\big(z, \nabla \tilde u(z)\big) \le 0 \le \tilde K \mathrm{dist}\big(z, \Gamma_1\big)^2 \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0, z_2^0\big) \cap \mathcal{H}_{[\bar\tau/2, 4\bar\tau]}.$$

Finally, since

$$\nabla \tilde u = \Theta \nabla \bar u_{\bar V} + (1 - \Theta)\nabla \bar u + \nabla\Theta(\bar u_{\bar V} - \bar u),$$

and $\bar V = 0$ on $B^n(0,2) \cap \big(\mathcal{H}_{[\bar\tau/4,\bar\tau]} \cup \mathcal{H}_{[4\bar\tau, 9\bar\tau/2]}\big)$, by (5.11), (5.13), and the convexity of $\bar H$ in the $q$ variable we get

$$\begin{aligned} \bar H_{\bar V}\big(z, \nabla \tilde u(z)\big) &\le \Theta(z)\bar H_{\bar V}\big(z, \nabla \bar u_{\bar V}(z)\big) + \big(1 - \Theta(z)\big)\bar H\big(z, \nabla \bar u(z)\big) + K'\big|\bar u_{\bar V}(z) - \bar u(z)\big| \\ &\le K'\big|\bar u_{\bar V}(z) - \bar u(z)\big| \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0, z_2^0\big) \cap \big(\mathcal{H}_{[\bar\tau/4, \bar\tau/2]} \cup \mathcal{H}_{[4\bar\tau, 9\bar\tau/2]}\big), \end{aligned}$$

where $K'$ is a constant depending only on $\frac{\partial \bar H}{\partial q}$ and $\nabla\Theta$. Thanks to ($\pi$1), ($\pi$3), ($\pi$4), and the $C^{1,1}$-regularity of both $\bar u$ and $\bar u_{\bar V}$, we have

$$\big|\bar u_{\bar V}(z) - \bar u(z)\big| \le K \mathrm{dist}\big(z, \Gamma_1\big)^2 \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0, z_2^0\big) \cap \big(\mathcal{H}_{[\bar\tau/4, \bar\tau/2]} \cup \mathcal{H}_{[4\bar\tau, 9\bar\tau/2]}\big),$$

which concludes the proof. □

We now consider $\Phi : B^n(0,2) \mapsto [0,1]$ a smooth cut-off function such that

$$\begin{cases} \Phi(z) = \Phi(\hat{z}) = 1 & \text{if } \hat{z} \in \mathrm{Cyl}_0^{1/7}\big(\hat{z}_1^0; \hat{z}_2^0\big), \\ \Phi(z) = \Phi(\hat{z}) = 0 & \text{if } \hat{z} \notin \mathrm{Cyl}_0^{1/6}\big(\hat{z}_1^0; \hat{z}_2^0\big), \end{cases}$$

and satisfying

$$\big|\nabla \Phi\big| = \big|\nabla_{\hat{z}} \Phi\big| \leq \frac{K_\Phi}{\big|z_1^0 - z_2^0\big|}, \qquad \big|D^2 \Phi\big| \leq \frac{K_\Phi}{\big|z_1^0 - z_2^0\big|^2}, \tag{5.16}$$

for some constant $K_\Phi$ independent of $z_1^0$ and $z_2^0$. Then, we define

$$\hat{u}(z) := \Phi(z)\tilde{u}(z) + \big(1 - \Phi(z)\big)\bar{u}(z) \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0; \hat{z}_2^0\big).$$

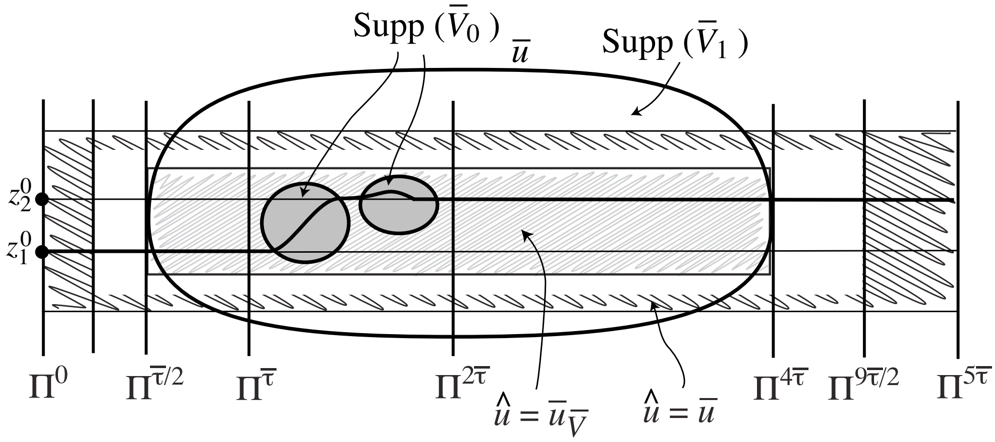


Figure 4: The function $\hat{u}$ is obtained by interpolating (using a cut-off function) between $\bar{u}$ (the critical viscosity for $\bar{H}$) and $\bar{u}_{\bar{V}}$ (the viscosity solution for $\bar{H}_{\bar{V}_0+\bar{V}_1}$) inside the cylinder $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0; \hat{z}_2^0\big)$. Since $V_1 \leq 0$, the function $\bar{u}$ is a viscosity subsolution to $\bar{H}_{\bar{V}_0+\bar{V}_1}(z, \nabla \bar{u}(z)) \leq 0$ outside $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0; \hat{z}_2^0\big)$. So, we can find a non-positive potential $\bar{W}$, small in $C^1$ topology and supported inside $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0; \hat{z}_2^0\big)$, such that $\bar{H}_{\bar{V}_0+\bar{V}_1+\bar{W}}(z, \nabla \hat{u}(z)) \leq 0$ on the whole ball $B^n(0,2)$.

Observe that $\hat{u} = \bar{u}$ outside $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0; \hat{z}_2^0\big)$. Moreover, thanks to (5.16), ($\pi$6), and the fact that both $\bar{u}$ and $\tilde{u}$ are of class $C^{1,1}$ on $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0, \hat{z}_2^0\big)$ (with a uniform bound on their $C^{1,1}$-norm), also the function $\hat{u}$ is of class $C^{1,1}$ on $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0, \hat{z}_2^0\big)$, with a bound on its $C^{1,1}$-norm independent of $z_1^0$ and $z_2^0$. Hence, similarly to Lemma 5.6 above, we can prove the following:

**Lemma 5.7.** *There exists a constant* $\hat{K} > 0$ *such that*

$$\bar{H}_{\bar{V}}\big(z, \nabla \hat{u}(z)\big) \leq 0 \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big) \cap \big(\mathcal{H}_{[0,\bar{\tau}/4]} \cup \mathcal{H}_{[9\bar{\tau}/2,5\bar{\tau}]}\big),$$

$$\bar{H}_{\bar{V}}\big(z, \nabla \hat{u}(z)\big) \leq \hat{K} \mathrm{dist}\big(z, \Gamma_1\big)^2 \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big) \cap \mathcal{H}_{[\bar{\tau}/4,9\bar{\tau}/2]}.$$

Before proving the above lemma, let us show how it allows to conclude the whole construction and to obtain (P1). Let us define $\bar{W} : \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(\hat{z}_1^0; \hat{z}_2^0\big) \to \mathbb{R}$ a non-positive potential of class $C^{k-1}$ such that

$$\bar{W}(z) = -\hat{K} \mathrm{dist}\big(x, \Gamma_1\big)^2 \qquad \forall\, z \in \mathcal{C}^{1/6}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0; z_2^0\big) \cap \mathcal{H}_{[\bar{\tau}/4,9\bar{\tau}/2]},$$

$$\mathrm{Supp}\big(\bar W\big) \subset \mathrm{Cyl}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(\hat z_1^0;\hat z_2^0\big), \qquad \text{and} \quad \|\bar W\|_{C^1} \leq \hat K'\big|z_1^0 - z_2^0\big|$$

for some universal constant $\hat K'$. (Observe that $\mathrm{dist}(\,\cdot\,,\Gamma_1)$ is of class $C^{k-1}$ on $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0;z_2^0\big)$, for $\big|z_1^0 - z_2^0\big|$ small enough.) Moreover, outside $\mathrm{Cyl}^{1/6}_{[0,\mathcal{T}_{5\bar\tau}]}\big(\hat z_1^0;\hat z_2^0\big)$ it holds $\bar V = \bar V_1 \leq 0$, $W \leq 0$, and $\hat u = \bar u$. So we clearly have

$$\bar H_{\bar V + \bar W}\big(z, \nabla \hat u(z)\big) \leq 0 \qquad \text{on } B^n(0,2).$$

Furthermore, $\bar W$ vanishes on $\Gamma_1$, and thanks to (p6)

$$\big\|\bar V + \bar W\big\|_{C^1} \leq (K + \hat K')\big|z_1^0 - z_2^0\big|.$$

This concludes the construction by choosing $\big|z_1^0 - z_2^0\big|$ sufficiently small.

*Proof of Lemma 5.7.* The first estimate is obvious, since $\hat u = \tilde u = \bar u$ on $\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big) \cap \big(\mathcal{H}_{[0,\bar\tau/4]} \cup \mathcal{H}_{[9\bar\tau/2,5\bar\tau]}\big)$.

For the second one, we observe that by ($\pi$3)

$$\big|z_1^0 - z_2^0\big| \leq 72\,\mathrm{dist}(z,\Gamma_1) \qquad \forall\, z \in \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big) \setminus \mathcal{C}^{1/7}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big).$$

Hence, since $\hat u = \tilde u$ on $\mathcal{C}^{1/7}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big)$, by Lemma 5.6 it suffices to prove

$$\bar H_{\bar V_0 + \bar V_1}\big(z, \nabla \hat u(z)\big) \leq \hat K\big|z_1^0 - z_2^0\big|^2 \qquad \forall\, z \in \Big(\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big) \setminus \mathcal{C}^{1/7}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big)\Big) \cap \mathcal{H}_{[\bar\tau/4,9\bar\tau/2]}.$$

As in the proof of Lemma 5.6 we observe that

$$\nabla \hat u = \Phi \nabla \tilde u + (1-\Phi)\nabla \bar u + \nabla\Phi(\tilde u - \bar u).$$

Moreover, by the convexity of $\bar H$ in the $q$ variable, Lemma 5.6, (5.11), Lemma 5.4(i), and Lemma 5.5(iii)-(v), for every $z \in \Big(\mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar\tau}]}\big(z_1^0,z_2^0\big) \setminus \mathcal{C}^{1/7}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0,z_2^0\big)\Big) \cap \mathcal{H}_{[\bar\tau/4,9\bar\tau/2]}$ we have

$$\begin{aligned}
\bar H_{\bar V}\,\big(z, \Phi(z)\nabla\tilde u(z) + \big(1-\Phi(z)\big)\nabla \bar u(z)\big) &\leq \Phi(z) H_{\bar V}\big(z,\nabla\tilde u(z)\big) + \big(1-\Phi(z)\big)H_{\bar V}\big(z,\nabla\bar u(z)\big)\\
&\leq \Phi(z)\tilde K \mathrm{dist}\big(z,\Gamma_1\big)^2 + \big(1-\Phi(z)\big)\bar V_1(z)\\
&\leq \tilde K \mathrm{dist}\big(z,\Gamma_1\big)^2.
\end{aligned}$$

Hence, we need to estimate

$$\begin{aligned}
&\big|\bar H_{\bar V}\big(z,\nabla\hat u(z)\big) - \bar H_{\bar V}\big(z,\Phi(z)\nabla\tilde u(z) + (1-\Phi(z))\nabla\bar u(z)\big)\big|\\
&\qquad\leq \big|\tilde u(z) - \bar u(z)\big|\left|\frac{\partial \bar H}{\partial q}\big(z,\nabla\hat u(z)\big)\cdot\nabla\Phi(z)\right| + K''\big|\nabla\Phi(z)\big|^2\big|\tilde u(z) - \bar u(z)\big|^2
\end{aligned}$$

on $\Big(\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big) \setminus \mathcal{C}^{1/7}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big)\Big) \cap \mathcal{H}_{[\bar\tau/4,9\bar\tau/2]}$, where the constant $K''$ depends only on $\frac{\partial^2 \bar H}{\partial q^2}$. Now, thanks to ($\pi$6) and (5.16), we have

$$\big|\nabla\Phi\big|^2|\tilde u - \bar u|^2 \leq \big(K_\Phi \bar K\big)^2\,\big|z_1^0 - z_2^0\big|^2 \qquad \text{on } \mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big) \setminus \mathcal{C}^{1/7}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big).$$

Concerning the term

$$\big|\tilde u(z) - \bar u(z)\big|\left|\frac{\partial \bar H}{\partial q}\big(z,\nabla\hat u(z)\big)\cdot\nabla\Phi(z)\right|,$$

by ($\pi$6) it suffices to prove that $\frac{\partial \bar H}{\partial q}\big(z,\nabla\hat u(z)\big)\cdot\nabla\Phi$ is bounded by a constant independent of $z_1^0$ and $z_2^0$.

Let us observe that

$$\frac{\partial \bar{H}}{\partial q}\Big(Z_{\bar{V}}(t,z_0^1),\nabla \bar{u}_{\bar{V}}\big(Z_{\bar{V}}(t,z_0^1)\big)\Big)=\dot{Z}_{\bar{V}}(t,z_0^1),$$

which by ($\pi$2), (5.16), and the fact that $\Phi$ depends only on $\hat{z}$, implies

$$\left|\frac{\partial \bar{H}}{\partial q}\Big(Z_{\bar{V}}(t,z_0^1),\nabla \bar{u}\big(Z_{\bar{V}}(t,z_0^1)\big)\Big)\cdot \nabla\Phi(z)\right|\leq K_0'K_\Phi \qquad \forall\, z,\, t\in\big[0,5\bar{\tau}\big].$$

Moreover, by ($\pi$6)

$$\left|\frac{\partial \bar{H}}{\partial q}\Big(Z_{\bar{V}}(t,z_0^1),\nabla \hat{u}\big(Z_{\bar{V}}(t,z_0^1)\big)\Big)-\frac{\partial \bar{H}}{\partial q}\Big(Z_{\bar{V}}(t,z_0^1),\nabla \bar{u}\big(Z_{\bar{V}}(t,z_0^1)\big)\Big)\right|\leq K'''\big|z_1^0-z_2^0\big|,$$

so by the $C^{1,1}$ regularity of $\hat{u}$ we get

$$\left|\frac{\partial \bar{H}}{\partial q}\big(z,\nabla \hat{u}(z)\big)-\frac{\partial \bar{H}}{\partial q}\Big(Z_{\bar{V}}(t,z_0^1),\nabla \bar{u}\big(Z_{\bar{V}}(t,z_0^1)\big)\Big)\right|\leq K_1\big|z_1^0-z_2^0\big|$$

for all $z\in\mathcal{C}^{1/5}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0,z_2^0\big)$ and $t\in\big[0,5\bar{\tau}\big]$, such that $\hat{Z}_{\bar{V}}(t,z_0^1)=\hat{z}$. Hence, combing all together we obtain

$$\left|\frac{\partial \bar{H}}{\partial q}\big(z,\nabla \hat{u}(z)\big)\cdot\nabla\Phi\right|\leq K_0'K_\Phi+K_1\big|z_1^0-z_2^0\big|\leq K_0'K_\Phi+K_1 \qquad \text{on } \mathcal{C}^{1/5}_{[0,\mathcal{T}_{4\bar{\tau}}]}\big(z_1^0,z_2^0\big),$$

which concludes the proof. $\square$

# A Proof of Lemma 4.1

The idea of the proof is the following: thanks to (4.5), (4.9), and using a Taylor expansion, we get $|g|\lesssim L(r_i)r_i^2$ and $|\nabla g|\lesssim L(r_i)r_i$ on $C_i$. Hence we can apply [8, Lemma 3.3] to find a function $f_i$ such that $\|f_i\|_{C^2}\lesssim L(r_i)$ and $f_i=g, \nabla f_i=\nabla g$ on $C_i$. Finally, subtracting to $f_i$ the function $K_0\mathrm{dist}(\,\cdot\,,C_i)^2$, with $K_0\gg 1$ and using a partition of unity argument, thanks to (4.6) we can construct a function $G$ with small $C^2$-norm and such that $G\leq g$ everywhere. We now perform the construction in details.

Define the curves $D_1,\cdots,D_{\eta-1}$ as

$$D_i:=\Big(\big[i-2/3,i+2/3\big]\times B_{r_0}^{n-1}\Big)\cap\Big(I_{i-1}\cup C_i\cup I_i\Big), \qquad i=1,\cdots,\eta-1,$$

and let $\{\psi_i\}_{i=1,\cdots,\eta-1}:\mathcal{C}\longrightarrow[0,1]$ be a family of smooth functions satisfying the following properties:

(A) for every $i=1,\cdots,\eta-1$, Supp $(\psi_i)\subset\big(\big[i-2/3,i+2/3\big]\times\mathbb{R}^{n-1}\big)\cap\mathcal{C}$;

(B) $\sum_{i=1}^{\eta-1}\psi_i(x)=1$, for every $x\in\mathcal{C}$.

Let $r>0$ be a small number to be fixed later. For every $i=1,\dots,\eta-1$ we define the function $\Phi_i:[-2/3,2/3]\times\mathbb{R}^{n-1}\to\mathbb{R}^n$ (of class $C^{l-1}$) by

$$\Phi_i(t,y):=\begin{cases}(i+t,v_i+y) & \text{if} \quad t\leq 0\\ c_i(i+t)+(0,y) & \text{if} \quad 0\leq t\leq 1/10\\ (i+t,v_{i+1}+y) & \text{if} \quad t\geq 1/10,\end{cases}$$

for every $(t,y)\in[2/3,i+2/3]\times\mathbb{R}^{n-1}$. Thanks to (4.8)-(4.13), $\Phi_i$ is a diffeomorphism of class $C^{l-1}$ from $[-2/3,2/3]\times\mathbb{R}^{n-1}$ into $[i-2/3,i+2/3]\times\mathbb{R}^{n-1}$which satisfies the following properties for $r>0$ small enough:

- $\Phi_i\left([-2/3,2/3]\times\{0_{n-1}\}\right)=D_i$;
- $\|\Phi_i\|_{C^2}\leq K'$, with $K'$ independent of both $r$ and $i$;
- $\Phi_i\left([2/3,2/3]\times B^{n-1}(0,r_i/2)\right)\subset\mathcal{Q}_i$, where $\mathcal{Q}_i$ is defined by

$$\mathcal{Q}_i:=\left(\bigcup_{s\in[0,\eta]}B^n\Big((s,v_i),r_i\Big)\right).$$

Let us define the functions $g_i:[-2/3,2/3]\times B^{n-1}_{r_i/2}\to\mathbb{R}^n$ (of class $C^{l-1}$) as

$$g_i(t,y):=\big(\psi_i g\big)\circ\Phi_i\big(t,0_{n-1}\big)\qquad\forall\,(t,y)\in[-2/3,2/3]\times B^{n-1}_{r_i/2}.$$

Thanks to (4.9), there exists $v_i^{\mathcal{Z}}\in\mathcal{Z}$ such that $r_i\geq|v_i-v_i^{\mathcal{Z}}|/8$. Hence, using a Taylor expansion we obtain

$$\begin{aligned}&\big|\nabla g\big(s,\lambda v_i+(1-\lambda)v_i^{\mathcal{Z}}\big)-\lambda\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)\big(v_i-v_i^{\mathcal{Z}}\big)\big|\\&=\big|\nabla g\big(s,\lambda v_i+(1-\lambda)v_i^{\mathcal{Z}}\big)-\nabla g\big(s,v_i^{\mathcal{Z}}\big)-\lambda\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)\big(v_i-v_i^{\mathcal{Z}}\big)\big|\leq K''\,\lambda^2\big|v_i-v_i^{\mathcal{Z}}\big|^2\end{aligned}\quad\text{(A.1)}$$

for every $s\in[0,\eta]$ and $\lambda\in[0,1]$, where $K''$ is any constant greater than $\|g\|_{C^2}$.

By (4.7) and definition of $v_i^{\mathcal{Z}}$, both $v_i$ and $v_i^{\mathcal{Z}}$ belong to $B^{n-1}_{2r}\cap\Gamma$. So, there are $h_i,h_i^{\mathcal{Z}}\in B^d_{2r}$ such that

$$v_i=h_i+\Psi(h_i)\qquad\text{and}\qquad v_i^{\mathcal{Z}}=h_i^{\mathcal{Z}}+\Psi(h_i^{\mathcal{Z}}).$$

Then, thanks to (4.5), recalling the definition of $L(r)$ and $K''$ we deduce that

$$\begin{aligned}\big|\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)\big(v_i-v_i^{\mathcal{Z}}\big)\big|&=\big|\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)\left[\big(h_i-h_i^{\mathcal{Z}}\big)+\big(\Psi(h_i)-\Psi(h_i^{\mathcal{Z}})\big)\right]\big|\\&=\Big|\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)_{|\Pi}\big(h_i-h_i^{\mathcal{Z}}\big)+\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)\big(\Psi(h_i)-\Psi(h_i^{\mathcal{Z}})\big)\Big|\\&\leq\Big|\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)_{|\Pi}\big(h_i-h_i^{\mathcal{Z}}\big)\Big|+\big|\text{Hess }g\big(s,v_i^{\mathcal{Z}}\big)\big(\Psi(h_i)-\Psi(h_i^{\mathcal{Z}})\big)\big|\\&\leq 2\bar{K}r\big|h_i-h_i^{\mathcal{Z}}\big|+K''\big|\Psi(h_i)-\Psi(h_i^{\mathcal{Z}})\big|\\&\leq 2\bar{K}r\big|h_i-h_i^{\mathcal{Z}}\big|+K''L(2r)\big|h_i-h_i^{\mathcal{Z}}\big|\\&\leq\big(2\bar{K}r+K''L(2r)\big)\left|v_i-v_i^{\mathcal{Z}}\right|.\end{aligned}$$

Hence, combining the above estimate with (A.1) and recalling that the points $v_i$ belong to $B^{n-1}_r$, we obtain

$$\begin{aligned}\big|\nabla g\big(s,\lambda v_i+(1-\lambda)v_i^{\mathcal{Z}}\big)\big|&\leq(2Kr+K''L(2r))\left|v_i-v_i^{\mathcal{Z}}\right|\lambda+K''\big|v_i-v_i^{\mathcal{Z}}\big|^2\lambda^2\\&\leq(2Kr+K''L(2r))\left|v_i-v_i^{\mathcal{Z}}\right|\lambda+2K''r\big|v_i-v_i^{\mathcal{Z}}\big|\lambda^2\end{aligned}$$

for every $s\in[0,\eta]$, $\lambda\in[0,1]$.

Since $g(s,v_i^{\mathcal{Z}})=0$, integrating the above inequality on $[0,1]$ yields, for $s\in[0,\eta]$,

$$g(s,v_i)=\int_0^1\langle\nabla g\big(s,\lambda v_i+(1-\lambda)v_i^{\mathcal{Z}}\big),v_i-v_i^{\mathcal{Z}}\rangle d\lambda\leq K'''(r+L(2r))\big|v_i-v_i^{\mathcal{Z}}\big|^2,$$

for some uniform constant $K'''>0$. Finally, taking $K''$ larger if necessary, by definition of $v_i^{\mathcal{Z}}$ and the fact that $\frac{\partial}{\partial s}(\nabla g)\big(s,0_{n-1}\big)=0$ we finally get

$$\left\{\begin{array}{l}|g(s,v_i)|\leq K''(r+L(2r))r_i^2\\|\nabla g(s,v_i)|\leq K''(r+L(2r))r_i\\\big|\frac{\partial}{\partial s}(\nabla g)(s,v_i)\big|\leq K''r\end{array}\right.\qquad\forall\,s\in[0,\eta].\quad\text{(A.2)}$$

(Recall that $g \in C^l$ with $l \geq 3$.) Combining this estimate with (4.10), (4.12), and (4.13), we easily deduce the existence of a constant $\hat{K} > 0$ such that, for every $i = 1, \dots, \eta - 1$,

$$\begin{cases} \left|g_i\big(t, 0_{n-1}\big)\right| \leq \hat{K}(r + L(2r))r_i^2 \\ \left|\nabla g_i\big(t, 0_{n-1}\big)\right| \leq \hat{K}(r + L(2r))r_i \\ \left|\frac{\partial}{\partial t}(\nabla g_i)\big(t, 0_{n-1}\big)\right| \leq K' r \end{cases} \qquad \forall\, t \in [-2/3, 2/3]. \tag{A.3}$$

The following result follows immediately from [8, Lemma 3.3] applied with $\tilde{v}(t) = \nabla f(t - 2/3, 0_{n-1})$ and $\bar{\tau} = 4/3$:

**Lemma A.1.** *Let $\delta, \rho \in (0,1)$ with $3\rho \leq \delta < 4/3$, and let $f : [-2/3, 2/3] \times B_\rho^{n-1} \to \mathbb{R}$ be a compactly supported function of class $C^m$, with $m \geq 2$, satisfying*

$$\nabla f(t, y) = 0_n \qquad \forall\, (t, y) \in \big([-2/3, -2/3 + \delta] \times \{0_{n-1}\}\big) \cup \big([2/3 - \delta, 2/3] \times \{0_{n-1}\}\big). \tag{A.4}$$

*Then there exist a universal constant $K$ depending only on the dimension, and a function $F : \mathbb{R}^n \to \mathbb{R}$ of class $C^m$, such that the following properties hold:*

*(i)* $\mathrm{Supp}(F) \subset \big[-2/3 + \delta/2, 2/3 - \delta/2\big] \times B_{2\rho/3}^{n-1}$;

*(ii)* $\|F\|_{C^2} \leq K \left(\frac{1}{\rho^2}\big\|f(\cdot, 0_{n-1})\big\|_\infty + \frac{1}{\rho}\big\|\nabla f(\cdot, 0_{n-1})\big\|_\infty + \big\|\frac{\partial}{\partial t}(\nabla f)(\cdot, 0_{n-1})\big\|_\infty\right)$;

*(iii)* $\nabla F\big(t, 0_{n-1}\big) = \nabla f\big(t, 0_{n-1}\big)$ *for every* $t \in [-2/3, 2/3]$.

Applying the above lemma to $f = g_i$ for $i = 1, \dots, \eta - 1$ and using (A.3) yields a function $G_i : \mathbb{R}^n \to \mathbb{R}$ of class $C^{l-1}$ satisfying:

(C) $\mathrm{Supp}(G_i) \subset \big[-2/3 + \delta/2, 2/3 - \delta/2\big] \times B_{r_i/3}^{n-1}$;

(D) $\|G_i\|_{C^2} \leq \tilde{K}\,(r + L(2r))$;

(E) $\nabla G_i\big(t, 0_{n-1}\big) = \nabla g_i\big(t, 0_{n-1}\big)$ for every $t \in [-2/3, 2/3]$.

Hence, thanks to (A)-(E) above, it is easily seen that the function $G_0 : \mathcal{C} \to \mathbb{R}$ defined by

$$G_0(x) := \sum_{i=1}^{\eta-1} G_i\left(\Phi_i^{-1}(x)\right) \qquad \forall\, x \in \mathcal{C}$$

is of class $C^{l-1}$ and satisfies (a), (b), (d), and (e) in the statement of the lemma for $r$ sufficiently small. However, assertion (c) does not necessary hold. To enforce it, fix $\mu : [0, +\infty) \to [0, 1]$ a smooth function satisfying

$$\mu(r) = 1 \quad \text{for } r \in [0, 1], \qquad \mu(r) = 0 \quad \text{for } r \geq 2,$$

and replace in the above formula each $G_i$ by $\tilde{G}_i : \mathbb{R}^n \to \mathbb{R}$, with

$$\tilde{G}_i(t) := \mu\left(\frac{2|y|}{r_i}\right)\Big(G_i(t, y) - M(r + L(2r))|y|^2\Big) \qquad \forall\, (t, y) \in \mathbb{R}^n,$$

where $M := (\bar{K} + \tilde{K})$, with $\bar{K}$ as in (4.6) and $\tilde{K}$ as in (D) above. We leave it to the reader to check that

$$G(x) := \sum_{i=1}^{\eta-1} \tilde{G}_i\left(\Phi_i^{-1}(x)\right) \qquad \forall\, x \in \mathcal{C}$$

satisfies all assumptions (a)-(e) for $r > 0$ small enough.

# B Proof of Lemma 2.3

Let $\phi : \mathbb{R} \to [0,1]$ be an even function of class $C^\infty$ satisfying the following properties:

(a) $\phi(s) = 1$ for $s \in [0, 1/3]$;

(b) $\phi(s) = 0$ for $s \geq 2/3$;

(c) $|\phi'(s)|, |\phi''(s)| \leq 10$ for any $s \in [0, +\infty)$.

Extend the function $\tilde{v}$ on $\mathbb{R}$ by $\tilde{v}(t) := 0$ for $t \leq 0$ and $t \geq \bar{\tau}$, and define the function $W : \big[0, \bar{\tau}\big] \times \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$W(t, \hat{z}) := \phi\left(\frac{\mathcal{N}(\hat{z})}{r}\right) \left[\sum_{i=1}^{n-1} \int_0^{\hat{z}_i} \tilde{v}_{i+1}(t+s)\, ds\right] \qquad \forall\, (t, \hat{z}) \in \big[0, \bar{\tau}\big] \times \mathbb{R}^{n-1}.$$

Since $\tilde{v}$ is $C^{k-2}$, $\mathcal{N}$ is smooth on $\mathbb{R}^{n-1} \setminus \{0\}$, and $\phi$ is smooth and equal 1 on $[0, 1/3]$, it is easy to check that $W$ is of class $C^{k-1}$. Then, since $\mathcal{N}$ is positively 1-homogeneous, assertions (i)-(iv) follow as in the proof of [8, Lemma 3.3]. We leave the details to the interested reader.

# C Proofs of Lemmas 5.2, 5.3, 5.5, 5.6

## C.1 Proof of Lemma 5.2

We claim that there are $y_1 \neq y_2 \in Y \cap B_r$ such that the following properties are satisfied:

(1) $B\big(y_1, 3|y_1 - y_2|\big) \subset B_r$;

(2) for every $y \in Y \cap \bar{B}\big(y_1, 3|y_1 - y_2|/2\big)$ and every $y' \in Y \setminus \{y\}$, there holds

$$|y' - y| > \frac{|y_1 - y_2|}{2}.$$

To prove the claim, we define the set $\Delta \subset Y \times Y$ as

$$\Delta := \Big\{(y, y') \in \big(Y \cap B_r\big) \times \big(Y \cap B_r\big) \,|\, B\big(y, 3|y - y'|\big) \subset B_r \text{ and } y \neq y'\Big\}.$$

Since $B_{r/12} \cap Y$ contains at least two points, $\Delta$ is nonempty. Moreover, since $Y$ is finite, $\Delta$ is finite too. Therefore, there are $(y_1, y_2) \in \Delta$ such that

$$|y_1 - y_2| \leq |y - y'| \qquad \forall\, (y, y') \in \Delta. \tag{C.1}$$

Since $(y_1, y_2) \in \Delta$, the points $y_1, y_2$ are distinct, contained in $Y \cap B_r$ and satisfy assertion (1) of the claim.

To show that (2) is satisfied, we argue by contradiction. Let $y \in Y \cap \bar{B}\big(y_1, 3|y_1 - y_2|/2\big)$ and $y' \in Y \setminus \{y\}$ be such that $|y - y'| \leq |y_1 - y_2|/2$. Thanks to (1) we have $|y_1| + 3|y_1 - y_2| < r$. Hence

$$|y| \leq |y_1| + \frac{3|y_1 - y_2|}{2} < r,$$

$$|y'| \leq |y| + |y' - y| \leq \left(|y_1| + \frac{3|y_1 - y_2|}{2}\right) + \left(\frac{|y_1 - y_2|}{2}\right) \leq |y_1| + 2|y_1 - y_2| < r,$$

$$B\big(y,3|y-y'|\big) \subset B\big(0,|y|+3|y-y'|\big)$$
$$\subset B\left(0,\left(|y_1|+\frac{3|y_1-y_2|}{2}\right)+\left(\frac{3|y_1-y_2|}{2}\right)\right) = B\Big(0,|y_1|+3|y_1-y_2|\Big) \subset B_r.$$

This shows that $(y,y')$ belongs to $\Delta$, which contradicts the definition of $(y_1,y_2)$ and proves the claim.
It remains to check that the pair $y_1,y_2$ satisfies the two properties given in the statement of Lemma 5.2, namely

$$\mathrm{Cyl}_0^{1/3}\big(y_1;y_2\big) \subset B_r \quad \text{and} \quad \mathrm{Cyl}_0^{1/3}\big(y_1;y_2\big) \cap \big(Y\setminus\{y_1,y_2\}\big) = \emptyset.$$

The inclusion follows from $\mathrm{Cyl}_0^{1/3}\big(y_1;y_2\big) \subset B\big(y_1,3|y_1-y_2|\big)$ together with (1). For the second property, by elementary geometry we have

$$\mathrm{Cyl}_0^{1/3}\big(y_1;y_2\big) \subset B\big(y_1,|y_1-y_2|\big) \cup B\left(y_2,\frac{|y_1-y_2|}{3}\right) \subset \bar B\left(y_1,\frac{3|y_1-y_2|}{2}\right).$$

Hence, if by contradiction $y \in \mathrm{Cyl}_0^{1/3}\big(y_1;y_2\big) \cap \big(Y\setminus\{y_1,y_2\}\big)$, then (C.1) implies that $y \notin B\big(y_1,|y_1-y_2|\big)$, while (2) applied with $y'=y_2$ gives $y \notin B\big(y_2,|y_1-y_2|/2\big)$, absurd.

## C.2 Proof of Lemma 5.3

Let us first consider the following general setting: let $\mathcal{N}$ be a Lipschitz norm on $\mathbb{R}^n$, and let $Y(t,y)$ denote the flow map of a Lipschitz vector field $W$:

$$\begin{cases} \dot Y = W(Y), \\ Y(0,y) = y. \end{cases}$$

Then, differentiating in time $\mathcal{N}\big(Y(t,y_2)-Y(t,y_1)\big)$ and using Gronwall's lemma, we get

$$e^{-Mt}\mathcal{N}\big(y_2-y_1\big) \le \mathcal{N}\big(Y(t,y_2)-Y(t,y_1)\big) \le e^{Mt}\mathcal{N}\big(y_2-y_1\big) \qquad \forall\, t\ge 0, \tag{C.2}$$

where $M=M(\mathcal{N},W)$ depends only on the Lipschitz constant of $N$ and $W$. Now, the proof of the lemma follows easily.

Indeed, let us show for instance the inclusion

$$\mathrm{Cyl}^{\lambda_1}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big) \subset \mathcal{C}^{\lambda_2}_{[0,\mathcal{T}_{5\bar\tau}]}\big(z_1^0;z_2^0\big) \tag{C.3}$$

(the other inclusion being analogous).

As in the proof of Proposition 2.2, let $\mathcal{N}^i$ denote the norm associated with $z_1^0$, $z_2^0$ and $\lambda_i$:

$$B_1^{\mathcal{N}^i} := \mathrm{Cyl}_0^{\lambda_i}\left(-\frac{v}{2};\frac{v}{2}\right), \qquad v := \frac{z_2^0-z_1^0}{|z_2^0-z_1^0|}.$$

Observe that the Lipschitz constant of $\mathcal{N}^i$ is independent of $z_1^0$, $z_2^0$. Moreover, since $\lambda_1<\lambda_2$, there exists $\mu>0$ such that

$$(1+\mu)\mathcal{N}^2 \le \mathcal{N}^1 \tag{C.4}$$

Now, the inclusion (C.3) is equivalent to show

$$\mathcal{N}^2\Big(Z\big(t,z\big)-Z\big(t,z_{1,2}^0\big)\Big) < 1 \qquad \forall\, z=(0,\hat z),\ \text{with } \mathcal{N}^1(z^0-z_{1,2}^0)<1,\ t\in[0,\mathcal{T}_{5\bar\tau}]. \tag{C.5}$$

where we set $z_{1,2}^0 := (z_1^0+z_2^0)/2$. Then, since $Z\big(t,z\big)$ is an integral curve of the Lipschitz vector field $z\mapsto \nabla_q \bar H(z,\nabla\bar u_0(z))$, (C.5) follows immediately from (C.4) and (C.2), by choosing $\bar\tau$ sufficiently small (the smallness being independent of $z_1^0$ and $z_2^0$).

## C.3 Proof of Lemma 5.5

Let us first prove that assertion (vii) follows from assertions (i), (iii), and (vi), together with Lemma 5.4(vi).

Indeed, by assumption (iii), we have that $Z_{\bar V_0+\bar V_1}(\,\cdot\,; z_1^0) = Z_{\bar V_0}(\,\cdot\,; z_1^0)$, the action of the curve $Z_{\bar V_0+\bar V_1}(\,\cdot\,; z_1^0)$ computed with respect to $L_{\bar V_0}$ is the same as the one with respect to $L_{\bar V_0+\bar V_1}$, and $Z_{\bar V_0+\bar V_1}(\mathcal{T}_{\bar\tau}(z_1^0); z_1^0) = Z(\mathcal{T}_{\bar\tau}(z_1^0); z_1^0)$ (since Supp $(\bar V_0) \subset \mathrm{Cyl}^{1/9}_{[0,\mathcal{T}_{2\bar\tau}]}(z_1^0, z_2^0) \cap \mathcal{H}_{[\bar\tau,2\bar\tau]}$, see Lemma 5.4(i)). Hence, since $z_1^0 \in \bar{\mathcal{A}}$ and by the theory of characteristics for Hamilton-Jacobi equations [3, 4, 12], we have (with obvious notation)

$$\begin{aligned}
\bar u_{\bar V_0+\bar V_1}\big(Z_{\bar V_0+\bar V_1}(t; z_1^0)\big) &= \bar u\left(Z\big(\mathcal{T}_{\bar\tau}(z_1^0); z_1^0\big)\right) + \mathbb{A}_{\bar V_0+\bar V_1}\left(Z\big(\mathcal{T}_{\bar\tau}(z_1^0); z_1^0\big); t - \mathcal{T}_{\bar\tau}(z_1^0)\right)\\
&= \bar u(z_1^0) + \mathbb{A}\big(z_1^0; \mathcal{T}_{\bar\tau}(z_1^0)\big) + \mathbb{A}_{\bar V_0}\left(Z\big(\mathcal{T}_{\bar\tau}(z_1^0); z_1^0\big); t - \mathcal{T}_{\bar\tau}(z_1^0)\right)\\
&= \bar u(z_1^0) + \mathbb{A}_{\bar V_0}\big(z_1^0; t\big).
\end{aligned}$$

Moreover,

$$\bar u\big(Z(t; z_1^0)\big) = \bar u(z_1^0) + \mathbb{A}\big(z_1^0; t\big).$$

Hence, Lemma 5.4 (vi) together with assertion (iii) imply that, for any $\tau \in [0, 2\bar\tau]$, $t \in [0, \mathcal{T}_{2\bar\tau}(z_1^0)]$ and $t_{\bar V_0} \in [0, \bar T^f]$ such that $Z(t; z_1^0), Z_{\bar V_0+\bar V_1}(t_{\bar V_0}; z_1^0) \in \Pi^\tau$, there holds

$$\Big|\bar u_{\bar V_0+\bar V_1}\left(Z_{\bar V_0+\bar V_1}(t_{\bar V_0}; z_1^0)\right) - \bar u\big(Z(t; z_1^0)\big) - \left\langle \nabla \bar u\big(Z(t; z_1^0)\big), Z_{\bar V_0+\bar V_1}(t_{\bar V_0}; z_1^0) - Z(t, z_1^0)\right\rangle\Big| \le K\big|z_1^0 - z_2^0\big|^2,$$

which by the $C^{1,1}$-regularity of $\bar u$ implies

$$\big|\bar u_{\bar V_0+\bar V_1}\left(Z_{\bar V_0+\bar V_1}(t_{\bar V_0}; z_1^0)\right) - \bar u\big(Z_{\bar V_0+\bar V_1}(t_{\bar V_0}; z_1^0)\big)\big| \le \tilde K\big|z_1^0 - z_2^0\big|^2.$$

By the arbitrariness of $t_{\bar V_0}$, this proves (vii). We now prove all the other assertions.

First, we notice that, without loss of generality, we can assume that

$$\bar H(z, 0) < 0 \qquad \forall\, z \in \mathcal{H}_+ \cap B^n\big(0, \bar\delta_0\big), \tag{C.6}$$

for some $\bar\delta_0 > 0$ small enough. Indeed, since $\bar\gamma(t)$ belongs to $\mathcal{A}(H)$ and $c[H] = 0$, by (5.2) we have

$$\bar H\big((\bar\tau, 0), \nabla \bar u(\bar\tau, 0)\big) = 0, \qquad \nabla_q \bar H\big((\bar\tau, 0), \nabla \bar u(\bar\tau, 0)\big) = e_1.$$

Then, there exists $\lambda < 1$, with $|\lambda - 1|$ small, such that $\bar H\big(z, \bar\lambda e_1\big) < 0$ for any $z \in \mathcal{H}_+ \cap B^n(0, \bar\delta)$, for some $\bar\delta_0 > 0$ small. Hence, if we replace $\bar H$ by the new Hamiltonian $\tilde H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$\tilde H(z, q) := \bar H\big(z, q + \bar\lambda e_1\big) \qquad \forall\, (z, q) \in \mathbb{R}^n \times \mathbb{R}^n,$$

then $\tilde H$ satisfies (C.6). Moreover, any solution to the Hamiltonian system

$$\begin{cases} \dot z(t) &= \nabla_q \tilde H\big(z(t), q(t)\big)\\ \dot q(t) &= -\nabla_z \tilde H\big(z(t), q(t)\big)\end{cases}$$

starting from $\big(z^0, q^0\big)$ satisfies

$$z(t) = Z_0\Big(t; \big(z^0, q^0\big)\Big) \quad \text{and} \quad q(t) = Q_0\Big(t; \big(z^0, q^0\big)\Big) - \bar\lambda e_1 \qquad \forall\, t \ge 0,$$

where $(Z_0, Q_0)$ is the Hamiltonian flow associated to $\bar H$. Moreover, the function $\tilde u : B^n(0,1) \cap \mathcal{H}_+ \to \mathbb{R}$ defined by

$$\tilde u(z) := \bar u(z) - \bar\lambda z_1 \qquad \forall\, z = \big(z_1, \hat z\big) \in B^n(0, 1) \cap \mathcal{H}_+,$$

is a subsolution of class $C^{1,1}$ of the Hamilton-Jacobi equation associated with $\tilde{H}$. Thanks to these facts, it is easy to check that if we can construct two potentials $\bar{V}_0, \bar{V}_1$ so that Lemma 5.5 holds with $\tilde{H}, \tilde{u}$ in place of $\bar{H}, \bar{u}$, then Lemma 5.5 will also be true for $\bar{H}, \bar{u}$. Hence, there is no loss of generality in assuming that (C.6) for some constant $\bar{\delta}_0 \in (0, 1/8)$.

Consider now $\tilde{\delta} \leq \min\{\bar{\delta}, \bar{\delta}_0, 1/K\}$ (to be fixed later), with $\bar{\delta}$ and $K$ as in Lemma 5.4. Then, given $z_1^0 = \big(0, \hat{z}_1^0\big), z_2^0 = \big(0, \hat{z}_1^0\big) \in \Pi^0_{\tilde{\delta}}$, there exist $\bar{T}^f > 0$, and a potential $\bar{V}_0 : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$, such that assertions (i)-(viii) of Lemma 5.4 are satisfied. In particular, since $\big|z_1^0 - z_2^0\big| \leq 2\tilde{\delta} \leq 2/K$, by Lemma 5.4(ii) we get

$$\big\|\bar{V}_0\big\|_{C^1} \leq 2. \tag{C.7}$$

To simplify the notation, we set $H^0(z, q) := \bar{H}_{\bar{V}_0}(z, q)$. Observe that, if $\tilde{\delta}$ is sufficiently small, then also the Hamiltonian $H^0$ (which is of class $C^{k-1}$) satisfies (C.6), that is,

$$H^0(z, 0) < 0 \qquad \forall\, z \in \mathcal{H}_+ \cap B^n\big(0, \bar{\delta}_0\big), \tag{C.8}$$

Define the trajectory $\big(Z(\cdot), Q(\cdot)\big) : [0, +\infty) \to \mathbb{R}^n \times \mathbb{R}^n$ associated to $H^0$ by

$$\big(Z(t), Q(t)\big) := \Big(Z_{\bar{V}_0}\Big(t; \big(z_1^0, \nabla \bar{u}(z_1^0)\big)\Big), Z_{\bar{V}_0}\Big(t; \big(z_1^0, \nabla \bar{u}(z_1^0)\big)\Big)\Big) \qquad \forall\, t \geq 0, \tag{C.9}$$

and let $T > 0$ be the first time such that $Z(T) \in \Pi^{5\bar{\tau}}$. The proof of the following result is postponed to the end of the section[12].

**Lemma C.1.** *Up to a change of coordinates $\Phi$ of class $C^{k-2}$ in an open neighborhood of $Z([0, T])$, with $\|\Phi\|_{C^1}$ and $\|\Phi^{-1}\|_{C^1}$ uniformly bounded (by a constant independent of $\bar{V}_0$, $z_1^0, z_2^0$), we can assume that the following properties are satisfied for every $t \in [0, T]$:*

*(i)* $Z(t) = te_1$;

*(ii)* $Q(t) = e_1$;

*(iii)* $\frac{\partial^2 H^0}{\partial z \partial q}\big(Z(t), Q(t)\big) = 0$;

*(iv)* $\frac{\partial^2 H^0}{\partial q_1 \partial \hat{q}}\big(Z(t), Q(t)\big) = 0$;

*(v)* $\frac{\partial^2 H^0}{\partial \hat{q}^2}\big(Z(t), Q(t)\big) = I_{n-1}$.

Now, the strategy is to add to $H^0$ a smooth nonpositive potential $W : \mathbb{R}^n \to \mathbb{R}$, such that

$$W\big(Z(t)\big) = 0 \quad \text{and} \quad \nabla W\big(Z(t)\big) = 0 \qquad \forall\, t \in \big[0, T\big] \tag{C.10}$$

and which is very "concave" along the curve $Z\big([0, T]\big)$ in the transversal directions, so that the "curvature" of the system is sufficiently negative along $\big(Z(t), Q(t)\big)$, and the characteristics associated to $H^0 + W$ will not cross.

Let us observe that, if $W$ satisfies (C.10) then $\big(Z(\cdot), Q(\cdot)\big)$ is still a trajectory of the Hamiltonian system

$$\begin{cases} \dot{z} &= \frac{\partial H^0_W}{\partial q}(z, q) \\ \dot{q} &= -\frac{\partial H^0_W}{\partial z}(z, q). \end{cases} \tag{C.11}$$

[12] As also noticed in the proof of the Lemma C.1, $\|\Phi\|_{C^1}$ and $\|\Phi^{-1}\|_{C^1}$ actually depend on $V_0$ only through its $C^1$-norm, which however is uniformly bounded by (C.7).

where the Hamiltonian $H^0_W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$H^0_W(p,q) := H^0(p,q) + W(z) = \bar{H}(p,q) + \bar{V}_0 + W(z) \qquad \forall\, (z,q) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Given $W$ as above, for every $z = (0,\hat{z}) \in \Pi^0_{\tilde{\delta}}$ (see (5.3)), let us denote by $\big(Z_W(\,\cdot\,;z), Q_W(\,\cdot\,;z)\big)$ the solution of (C.11) starting at $(z, \nabla\bar{u}(z) + \nu_W(z)e_1)$, where $\nu_W : \Pi^0_{\tilde{\delta}} \to \mathbb{R}$ is the Lipschitz function satisfying[13]

$$H^0_W\big(z, \nabla\bar{u}(z) + \nu_W(z)e_1\big) = 0 \qquad \forall\, z \in \Pi^0_{\tilde{\delta}}.$$

Then, consider the Lipschitz function $\exp_{\bar{V}_0+W} : \mathbb{R} \times \Pi^0_{\tilde{\delta}} \to \mathbb{R}^n$ defined by

$$\exp_{\bar{V}_0+W}\big(t, z^0\big) := Z_{\bar{V}_0+W}\Big(t; \big(z^0, \nabla\bar{u}(z^0) + \nu_{\bar{V}_0+W}(z^0)e_1\big)\Big) \qquad \forall\, t \in \mathbb{R},\ \forall\, z^0 \in \Pi^0_{\tilde{\delta}}.$$

We claim the following: Assume that there are $\rho, C > 0$ (with $\rho \leq \tilde{\delta}$) such that $\exp_{\bar{V}_0+W}$ is injective on the cylinder $\big[0, 5\bar{\tau}\big] \times \Pi^0_\rho$ and satisfies

$$\begin{aligned} &\big|\exp_{\bar{V}_0+W}\big(t, z^0\big) - \exp_{\bar{V}_0+W}\big(t', (z^0)'\big)\big| \\ &\qquad \geq C\left(|t-t'| + |z^0 - (z^0)'|\right) \qquad \forall\, (t,z^0), (t',(z^0)') \in \big[0,5\bar{\tau}\big] \times \Pi^0_\rho. \end{aligned} \tag{C.12}$$

Set $\Omega := \exp_{\bar{V}_0+W}\big(\big[0,5\bar{\tau}\big] \times \Pi^0_\rho\big)$. Then the viscosity solution of the Dirichlet problem

$$\begin{cases} \bar{H}\big(z, \nabla\tilde{u}(z)\big) + \bar{V}_0(z) + W(z) = 0 & \text{in } \Omega, \\ \tilde{u} = \bar{u} & \text{on } \Pi^0_\rho, \end{cases} \tag{C.13}$$

is of class $C^{1,1}$, with a $C^{1,1}$-norm bounded by some constant depending on the constant $C$ above.

Indeed, if $\exp_{\bar{V}_0+W}$ is injective, then the function $\bar{u}_{\bar{V}_0+W} : \Omega \to \mathbb{R}$ defined by

$$\begin{aligned} &\bar{u}_{\bar{V}_0+W}\Big(\exp_{\bar{V}_0+W}\big(t,z^0\big)\Big) := \bar{u}(z^0) \\ &+ \int_0^t \Big\langle Q_{\bar{V}_0+W}\Big(s; \big(z^0, \nabla\bar{u}(z^0) + \nu_{\bar{V}_0+W}(z^0)e_1\big)\Big), \dot{Z}_{\bar{V}_0+W}\Big(s; \big(z^0, \nabla\bar{u}(z^0) + \nu_{\bar{V}_0+W}(z^0)e_1\big)\Big)\Big\rangle\, ds \end{aligned}$$

is of class $C^{1,1}$ (see [4, 12]), solves (C.13), and its gradient is given by

$$\nabla\bar{u}_{\bar{V}_0+W}\Big(\exp_{\bar{V}_0+W}\big(t,z^0\big)\Big) = Q_{\bar{V}_0+W}\Big(t; \big(z^0, \nabla\bar{u}(z^0) + \nu_{\bar{V}_0+W}(z^0)e_1\big)\Big).$$

Thanks to Gronwall's Lemma, we deduce easily that if (C.12) holds, then we have a uniform bound on $\big\|\bar{u}_{\bar{V}_0+W}\big\|_{C^{1,1}}$.

Now, using Lemma C.1(iii), by linearizing (C.11) along $\big(Z(\cdot), Q(\cdot)\big) : [0,+\infty) \to \mathbb{R}^n \times \mathbb{R}^n$ we get

$$\begin{cases} \dot{\xi}(t) &=& \frac{\partial^2 H^0}{\partial q^2}\big(Z(t),Q(t)\big)\eta(t) \\ \dot{\eta}(t) &=& -\Big(\frac{\partial^2 H^0}{\partial z^2}\big(Z(t),Q(t)\big) + \text{Hess } W\big(Z(t)\big)\Big)\xi(t). \end{cases} \tag{C.14}$$

[13] Observe that $\nu_W$ is well-defined: indeed

$$\bar{H}\left((\bar{\tau}, 0_{n-1}), \nabla\bar{u}\big((\bar{\tau},0_{n-1})\big)\right) = 0, \qquad \nabla_q\bar{H}\left((\bar{\tau}, 0_{n-1}), \nabla\bar{u}\big((\bar{\tau},0_{n-1})\big)\right) = e_1,$$

and $\bar{V}_0 + W$ is small in $C^1$ topology on $\Pi^0_{\tilde{\delta}}$ for $\tilde{\delta}$ sufficiently small (the smallness depending on $W$), from which the existence of $\nu_W$ follows immediately from the Implicit Function Theorem. Moreover, $\nu_W$ satisfies $\nu_W\big((\bar{\tau}, 0_{n-1})\big) = 0$.

We define $W : \mathbb{R}^n \to \mathbb{R}$ as

$$W\big(t, \hat{z}\big) := -\frac{N\phi(t)}{2} \sum_{i=1}^{n-1} \hat{z}_i^2 \qquad \forall\, \big(t, \hat{z}\big) \in \mathbb{R} \times \mathbb{R}^{n-1}, \tag{C.15}$$

where $N$ is a large positive constant which will be chosen later, and $\phi : [0, 4\bar{\tau}] \to [0, 1]$ is a smooth function satisfying

$$\phi(t) = 0 \quad \forall\, t \in \big[0, \bar{\tau}/2\big] \cup \big[7\bar{\tau}/2, 4\bar{\tau}\big] \quad \text{and} \quad \phi(t) = 1 \quad \forall\, t \in \big[\bar{\tau}, 3\bar{\tau}\big]. \tag{C.16}$$

In this way, $W$ satisfies (C.10) and

$$\text{Hess}\, W\big(Z(t)\big) = -N\phi(t) \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}. \tag{C.17}$$

Now, the idea is to take $N$ sufficiently large so to impose a uniform lower bound on the eigenvalues of the symmetric matrix $\frac{\partial^2 (H^0+W)}{\partial \hat{z}^2}\big(Z(t), Q(t)\big)$ for $t \in \big[0, 5\bar{\tau}\big]$. Indeed, this makes the "curvature not too positive" in the directions transversal to the "geodesic" $Z(t)$, so that "characteristics fall apart" and there are no "conjugate points" along the curve $t \mapsto \big(Z(t), Q(t)\big)$ on the time interval $\big[0, 5\bar{\tau}\big]$, provided $\bar{\tau} > 0$ is sufficiently small.

Observe that, to ensure the absence of conjugate points, we cannot say simply that it suffices to choose $\bar{\tau}$ sufficiently small. Indeed, given $\bar{\tau} > 0$, we have constructed $\bar{V}_0$ in Lemma 5.4, and the $C^2$-norm of $\bar{V}_0$ depends on $\bar{\tau}$. So, we need to prove that we can choose $\bar{\tau} > 0$, small but universals, such that, for any $N$ sufficiently large, characteristics do not cross in a cylinder of size $\rho$ around $\big[0, 5\bar{\tau}\big] \ni t \mapsto \big(Z(t), Q(t)\big)$. Moreover, $\rho$ may depend on $\bar{V}_0$ but not on $N$. This is the aim of the next lemma.

**Lemma C.2.** *Let $K$ be as in Lemma 5.4. If $\bar{\tau} > 0$ is small enough then, for every $N \geq K$, there exists a radius $\bar{\rho}$ and $\bar{c} > 0$, depending only on $K$, such that the map*

$$\begin{array}{cccc} \Psi_N : & \big[0, 5\bar{\tau}\big] \times \Pi^0_{\bar{\rho}} & \longrightarrow & \mathbb{R}^n \\ & (t, z^0) & \longmapsto & Z_W\big(t; z^0\big) \end{array}$$

*is injective and satisfies*

$$\begin{aligned} &\big|\Psi_N\big(t, z^0\big) - \Psi_N\big(t', (z^0)'\big)\big| \\ &\qquad \geq \bar{c}\left(|t - t'| + |z^0 - (z^0)'|\right) \qquad \forall\, \big(t, z^0\big), \big(t', (z^0)'\big) \in \big[0, 5\bar{\tau}\big] \times \Pi^0_{\bar{\rho}}. \end{aligned} \tag{C.18}$$

Before proving the above result, let us see how it allows to conclude the proof of Lemma 5.5. Set $N := K$, and take $\tilde{\delta} \leq \min\{\bar{\rho}/6, \bar{\delta}, \bar{\delta}_0\}$ (to be chosen), where $\bar{\rho}$ is given by Lemma C.2, $\bar{\delta}$ is given by Lemma 5.4, and $\bar{\delta}_0$ is as in (C.6). Observe that, if $z_1^0, z_2^0 \in \Pi^0_{\tilde{\delta}}$, then $\text{Cyl}_0^{1/4}\big(z_1^0, z_2^0\big) \subset \Pi^0_{\bar{\rho}/4}$. In particular, if $\bar{\tau}$ sufficiently small (the smallness being independent of $z_0^1, z_0^2$ and $\bar{\rho}$), then $\text{Cyl}^{1/3}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0, z_2^0\big) \subset \big[0, 5\bar{\tau}\big] \times \Pi^0_{\bar{\rho}/2}$ (see, for instance, the proof of Lemma 5.3). Then, since $Z_{\bar{V}_0}(t; z_1^0)$ belongs to $\text{Cyl}^{1/9}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0, z_2^0\big)$ for $t \in [0, \mathcal{T}_{5\bar{\tau}}(z_1^0)]$ (by Lemma 5.4(viii)), it suffices to consider a cut-off function $\bar{\varphi}$ which is identically equal to 1 on $\text{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0, z_2^0\big) \cap \mathcal{H}_{[0,4\bar{\tau}]}$ and vanishes outside $\text{Cyl}^{1/3}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0, z_2^0\big) \cap \mathcal{H}_{[0,4\bar{\tau}]}$, and set $\bar{V}_1 := W\bar{\varphi}$. We leave the reader to check that, if $\tilde{\delta}$ and $\bar{\tau}$ are small enough, then $\text{Cyl}^{1/4}_{[0,\mathcal{T}_{5\bar{\tau}}]}\big(z_1^0, z_2^0\big) \subset \text{Cyl}^{1/3}_{[0,\mathcal{T}_{4\bar{\tau}}]}\big(z_1^0, z_2^0\big) \subset \exp_{\bar{V}_0+W}\big(\big[0, 5\bar{\tau}\big] \times \Pi^0_{\bar{\rho}}\big)$, and all the assumptions in the statement of Lemma 5.5 are satisfied.

*Proof of Lemma C.2.* We fix $N \geq K$.

Let us observe that, by Lemma C.1(i)-(ii), it holds

$$\frac{\partial^2 H^0}{\partial z_1 \partial z}\big(Z(t),Q(t)\big) = \frac{d}{dt}\frac{\partial H^0}{\partial z}\big(Z(t),Q(t)\big) = \dot{Q}(t) = 0 \qquad \text{on } [0,T].$$

Hence, thanks to this fact and Lemma C.1(iv)-(v), the linearized system (C.14) can be written as

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & S(t)\eta(t) \\ \dot{\eta}(t) & = & R(t)\xi(t). \end{array} \right. \tag{C.19}$$

where $S(t), R(t)$ are $n \times n$ symmetric matrices of the form

$$S(t) = \begin{pmatrix} s_1(t) & 0 \\ 0 & I_{n-1} \end{pmatrix} \quad \text{and} \quad R(t) := \begin{pmatrix} 0 & 0 \\ 0 & \hat{R}(t), \end{pmatrix}. \tag{C.20}$$

with $\hat{R}(t)$ a $(n-1)\times(n-1)$ symmetric matrices . Moreover, thanks to our choice of $N$ and recalling that $\mathrm{Supp}(\bar{V}_0) \subset \mathcal{H}_{[\bar{\tau},2\bar{\tau}]}$, we have $\hat{R}(t) \geq -C_0 I_{n-1}$ for any $t \in [0,T]$, with $C_0 := \left\| \frac{\partial^2 \bar{H}}{\partial \hat{z}^2}\big(Z(t),Q(t)\big)\right\|_{L^\infty([0,T])}$.

We want to prove that, if $\rho > 0$ is sufficiently small (the smallness being independent of both $\bar{V}_0$ and $N$), then $\Psi_N$ is injective on $[0,5\bar{\tau}] \times \Pi^0_\rho$, and satisfies (C.18) for some universal constant $\bar{c} > 0$. Let us first prove, arguing by contradiction and using a compactness argument, that we can find such $\rho, c > 0$ when $\bar{V}_0$ and $N$ are fixed (so, a priori $\rho$ and $c$ may depend on both). Then, we will explain how to remove this dependence.

Let us remark that, since $\dot{Z}(t)\cdot e_1 > 0$, there exists a universal constant $c_0 > 0$ such that

$$\big|\Psi_N(t) - \Psi_N(t')\big| \geq c_0|t-t'|. \tag{C.21}$$

Hence, if we assume that the statement is false, then there exist two sequences $\{(t^k_a,\hat{z}^k_a)\}, \{(t^k_b,\hat{z}^k_b)\}$ in $[0,5\bar{\tau}] \times \Pi^0_\rho$ converging to some point $(\bar{t},0) \in [0,5\bar{\tau}] \times \{0\}$ such that

$$\big|\Psi_N\big(t^k_a,\hat{z}^k_a\big) - \Psi_N\big(t^k_b,\hat{z}^k_b\big)\big| < \frac{1}{k}\left(\big|t^k_a - t^k_b\big| + \big|z^k_a - z^k_b\big|\right) \qquad \forall\, k.$$

Denote by $\bar{u}^N : B^n(0,1)\cap\mathcal{H}_{[\bar{\tau},+\infty)} \to \mathbb{R}$ the viscosity solution to the Dirichlet problem

$$\left\{ \begin{array}{l} H^0_W\big(z,\nabla\bar{u}^N(z)\big) = 0 \quad \text{in } B^n_1(0)\cap\mathcal{H}_{[0,+\infty)} \\ \bar{u}^N = \bar{u} \quad \text{on } \Pi^0_1. \end{array} \right.$$

By [4], we know that $\bar{u}^N$ can be extended to a function of class $C^{1,1}$ on a ball $B^n(0,r_N)$, for some $r_N > 0$. Moreover, since $\bar{u}$ is $C^{1,1}$, the restriction of $\bar{u}^N$ to $\Pi^0_1$ is also $C^{1,1}$, with a $C^{1,1}$-bound independent of $\bar{V}_0$. Concerning the "time regularity", since $\dot{Q} = -\frac{\partial \bar{H}}{\partial z} - \nabla V_0(Z)$ and $\|V_0\|_{C^1}$ is bounded by 2 (see (C.7)), up to choosing $r_N$ smaller the Lipschitz constant of $\nabla\bar{u}^N$ on $B^n(0,r_N)$ is bounded by a universal constant $K_{\bar{u}}$ independent of $\bar{V}_0$ and $N$.

Set $z^k_a := (0,\hat{z}^k_a)$, $z^k_b := (0,\hat{z}^k_b)$, assume with no loss of generality that $t^k_a \geq t^k_b$ for all $k$, and define $z^k_c := \pi\Big(\phi^{H^0_W}_{t^k_a - t^k_b}\big(z^k_a,\nabla\bar{u}^N(z^k_a)\big)\Big)$. Then the following holds: there exists a constant $K'$, independent of $\bar{V}_0$ and $N$, such that:

(a) $z^k_b, z^k_c \in B^n_1(0)\cap\mathcal{H}_{[\bar{\tau},+\infty)}$ both converge to $0_n$ as $k\to\infty$;

(b) $\big|t^k_a - t^k_b\big| + \big|z^k_a - z^k_b\big| \leq K'\big|z^k_b - z^k_c\big|$ (this follows from (C.21) and a simple geometric argument);

(c) the sequence $\left\{ k\frac{\left|Z_W\left(t^k_b;z^k_b\right) - Z_W\left(t^k_b;z^k_c\right)\right|}{\left|z^k_b - z^k_c\right|} \right\}$ is bounded;

(d) $t_k \to \bar t$ as $k \to \infty$;

(e) $\limsup_{k\to\infty} \frac{\left|\nabla \bar u^N\left(z_b^k\right)-\nabla \bar u^N\left(z_c^k\right)\right|}{\left|z_b^k-z_c^k\right|} \leq K' K_{\bar u}$.

Hence, by considering $\big(\xi(0),\eta(0)\big)$ an arbitrary cluster point of the sequence

$$\left\{\left(\frac{z_b^k-z_c^k}{\left|z_b^k-z_c^k\right|}, \frac{\nabla \bar u^N\left(z_b^k\right)-\nabla \bar u^N\left(z_c^k\right)}{\left|z_b^k-z_c^k\right|}\right)\right\}$$

we deduce the existence of a solution $\big(\xi(\cdot),\eta(\cdot)\big) : [0,\bar t] \to \mathbb{R}^n \times \mathbb{R}^n$ to the linearized system (C.19), satisfying

$$\left|\xi(0)\right| = 1, \quad \left|\eta(0)\right| \leq K' K_{\bar u} \quad \text{and} \quad \xi(\bar t) = 0_n. \tag{C.22}$$

We now show that the above situation is impossible. Write $\xi(0) = \big(\xi_1(0), \hat\xi(0)\big)$, $\eta(0) = \big(\eta_1(0), \hat\eta(0)\big)$. We distinguish two cases:

(1) $\left|\xi_1(0)\right| \geq \frac{1}{2}$;

(2) $\left|\hat\xi(0)\right| \geq \frac{1}{2}$.

In case (1), we observe that since $\dot\eta_1(t) = 0$ for any $t$, we have

$$\xi_1(t) = \xi_1(0) + \left(\int_0^t s_1(\sigma)\, d\sigma\right) \eta_1(0),$$

and since $|\eta_1(0)| \leq K' K_{\bar u}$ and $s_1(t) = \frac{\partial^2 \bar H}{\partial q_1^2}\big(Z(t),Q(t)\big)$ is bounded independently of $\bar V_0$, $\xi_1(t)$ cannot vanish on some interval $[0,\bar t]$, with $\bar t > 0$ universal. In particular, if we choose $\bar\tau \leq \bar t/4$, then we get a contradiction.
In case (2), we observe that $\hat\xi$ satisfies $\ddot{\hat\xi}(t) = \hat R(t)\hat\xi(t)$. So, since $\hat R \geq -C_0 I_{n-1}$ we get

$$\frac{d^2}{dt^2}\frac{\left|\hat\xi(t)\right|^2}{2} = \left|\dot{\hat\xi}(t)\right|^2 + \left\langle \hat R(t)\hat\xi(t), \hat\xi(t)\right\rangle \geq -C_0\left|\hat\xi(t)\right|^2,$$

that is, $t \mapsto \left|\hat\xi(t)\right|^2 e^{2C_0 t}$ is convex. Hence

$$\left|\hat\xi(t)\right|^2 e^{2C_0 t} \geq \left|\hat\xi(0)\right|^2 - 2\Big(\left|\dot{\hat\xi}(0)\right| \left|\hat\xi(0)\right| - C_0\left|\hat\xi(0)\right|^2\Big)t \geq \left|\hat\xi(0)\right|^2 - 2\big(K'K_{\bar u} + C_0\big)t,$$

which again implies that $\xi_1(t)$ cannot vanish on some interval $[0,\bar t]$, with $\bar t > 0$ universal.

This argument shows that there exists a radius $\bar\rho > 0$ and a constant $\bar c > 0$, which a priori may depend on both $N$ and $\bar V_0$, such that $\Psi_N$ is injective on $\big[0, 5\bar\tau\big] \times \Pi^0_{\bar\rho}$ and satisfies (C.18). To show that actually we can choose both $\bar\rho$ and $\bar c$ independently of both $N \geq K$ and $\bar V_0$ (but of course they will depend on the constant $K$ provided by Lemma 5.4), it suffices to observe that the compactness argument used above could be repeated with letting at the same time $\rho, c \to 0$, $\bar V_0$ varying inside the class of $C^2$ potentials whose $C^1$-norm is bounded by 2 (see (C.7)) and whose $C^2$-norm is bounded by $K$, and $N$ varying inside $[K,+\infty)$. Indeed, the change of coordinates provided by Lemma C.1 depends on $\bar V_0$ only through its $C^1$-norm, which is universally bounded (due to (C.7)), while in the compactness argument above the choice of $\bar\tau$ depended only on $K$.

This concludes the proof. □

*Proof of Lemma C.1.* Set $B^n\big(0,\bar\delta_0\big)_+ := \mathcal{H}_+ \cap B^n\big(0,\bar\delta_0\big)$. From (C.8), the uniform convexity of $H^0$, and its $C^{k-1}$ regularity, for every $z \in B^n\big(0,\bar\delta_0\big)_+$ the set

$$C_z := \big\{q \in \mathbb{R}^n \,|\, H^0(z,q) \leq 0\big\} = \big\{q \in \mathbb{R}^n \,|\, \bar H(z,q) \leq -\bar V_0(z)\big\} \tag{C.23}$$

is a bounded uniformly convex set containing $0_n$ of class $C^{k-1}$, and the $C^{k-1}$-norm of $\partial C_z$ is independent of $\bar V_0$.

For each $z \in B^n\big(0,\bar\delta_0\big)_+$, define the support function $\varphi(z,\cdot):\mathbb{R}^n \to \mathbb{R}$ as

$$\varphi(z,v) := \max\left\{\langle v,q\rangle \,|\, q \in C_z\right\} \qquad \forall\, v \in \mathbb{R}^n.$$

The function $\varphi$ is of class $C^{k-1}$ outside the origin, and homogeneous of degree 1 in the $v$ variable. Moreover, it is not difficult to check that the curve $Z(\cdot):[0,T]\to\mathbb{R}^n$ defined in (C.9) satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial\varphi}{\partial v}\big(Z(t),\dot Z(t)\big)\right) = \frac{\partial\varphi}{\partial z}\big(Z(t),\dot Z(t)\big) \qquad \forall\, t \in [0,T]. \tag{C.24}$$

Indeed, denote by $q_{\max}(z,v) \in \partial C_z$ the unique element such that

$$\varphi(z,v) = \langle v, q_{\max}(z,v)\rangle \qquad \forall\, z \in B^n\big(0,\bar\delta_0\big)_+, \forall\, v \in \mathbb{R}^n. \tag{C.25}$$

Then

$$\frac{\partial\varphi}{\partial v}(z,v) = q_{\max}(z,v) \quad \text{ and } \quad \frac{\partial\varphi}{\partial z}(z,v) = -\lambda(z,v)\frac{\partial H^0}{\partial z}\big(z,q_{\max}(z,v)\big), \tag{C.26}$$

where $\lambda(z,v) \in \mathbb{R}$ satisfies

$$v = \lambda(z,v)\frac{\partial H^0}{\partial q}\big(z,q_{\max}(z,v)\big) = \lambda(z,v)\frac{\partial \bar H}{\partial q}\big(z,q_{\max}(z,v)\big). \tag{C.27}$$

Furthermore, since by definition $\big(Z(\cdot),Q(\cdot)\big):[0,T]\to\mathbb{R}^n$ is a solution to the Hamiltonian system

$$\begin{cases} \dot Z(t) &=& \nabla_q H^0\big(Z(t),Q(t)\big) \\ \dot Q(t) &=& -\nabla_z H^0\big(Z(t),Q(t)\big) \end{cases} \qquad \forall\, t \in [0,T] \tag{C.28}$$

and satisfies

$$H^0\big(Z(t),Q(t)\big) = 0 \qquad \forall\, t \in [0,T],$$

(see (C.9)), for any $t \in [0,T]$ it holds

$$q_{\max}\big(Z(t),\dot Z(t)\big) = Q(t), \quad \lambda\big(Z(t),\dot Z(t)\big) = 1, \quad \varphi\big(Z(t),\dot Z(t)\big) = \langle \dot Z(t), Q(t)\rangle. \tag{C.29}$$

Then, by (C.26) we deduce

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\varphi}{\partial v}\big(Z(t),\dot Z(t)\big)\right) &= \frac{d}{dt}\Big(q_{\max}\big(Z(t),\dot Z(t)\big)\Big) = \dot Q(t) \\ &= -\frac{\partial H^0}{\partial z}\big(Z(t),Q(t)\big) = \frac{\partial\varphi}{\partial z}\big(Z(t),\dot Z(t)\big) \qquad \forall\, t \in [0,T]. \end{aligned}$$

which proves (C.24).

Hence, by applying [9, Lemma 3.1], up a change of variable $\Phi_0$ of class $C^{k-2}$, the following properties hold for any $t \in [0,T]$ (the fact that $\Phi \in C^{k-2}$ and both $\|\Phi\|_{C^1}$ and $\|\Phi^{-1}\|_{C^1}$ are bounded independently of $V_0$ will be discussed at the end of the proof ):

(a) $Z(t) = te_1$;

(b) $\dot{Z}(t) = e_1$;

(c) $\varphi\big(Z(t), \dot{Z}(t)\big) = 1$;

(d) $\frac{\partial \varphi}{\partial z}\big(Z(t), \dot{Z}(t)\big) = 0$;

(e) $\frac{\partial \varphi}{\partial \hat{v}}\big(Z(t), \dot{Z}(t)\big) = 0$;

(f) $\frac{\partial \varphi}{\partial v_1}\big(Z(t), \dot{Z}(t)\big) = 1$;

(g) $\frac{\partial^2 \varphi}{\partial z \partial v}\big(Z(t), \dot{Z}(t)\big) = 0$;

(h) $\frac{\partial^2 \varphi}{\partial v_1 \partial \hat{v}}\big(Z(t), \dot{Z}(t)\big) = 0$.

We leave the reader to check that, thanks to (C.25)-(C.29), this implies

$$Q(t) = q_{\max}\big(Z(t), \dot{Z}(t)\big) = e_1 \quad \text{and} \quad \frac{\partial^2 H^0}{\partial q \partial z}\big(Z(t), Q(t)\big) = 0 \qquad \forall\, t \in [0, T],$$

which combined with (a) concludes the proof of assertions (i)-(iv). It remains to perform a new change of coordinates to get (v). Observe that $\frac{\partial^2 H^0}{\partial \hat{q}^2} = \frac{\partial^2 \bar{H}}{\partial \hat{q}^2}$ is positive definite and of class $C^{k-2}$. Let $R : [0, T] \to M_{n-1}(\mathbb{R})$ be a function of class $C^{k-2}$ such that

$$R(t) \frac{\partial^2 \bar{H}}{\partial \hat{q}^2}\big(Z(t), Q(t)\big) R(t)^* = I_{n-1} \qquad \forall\, t \in [0, T],$$

and define $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi\big(t, \hat{z}\big) := (t, R(t)\hat{z}) \qquad \forall\, t \in [0, T], \forall \hat{z} \in \mathbb{R}^{n-1}.$$

We leave the reader to check that, in the new system of coordinates induced by $\Phi$, all assertions (i)-(v) are satisfied (see also [8, Subsection 4.3]).

Thus, to conclude the proof, we only need to check that the change of variable $\Phi_0$ provided by [9, Lemma 3.1] is of class $C^{k-2}$, and the $C^1$-norm of $\Phi_0$ and its inverse are both bounded independently of $V_0$.

Let us recall that $\Phi_0$ is obtained as a composition of four change of variables $\Phi_0^j$, $j = 1, 2, 3, 4$ (see the proof of [9, Lemma 3.1]):

$$\Phi_0^1(z) = z + te_1 - Z(t) \qquad \text{(so that } Z(t) \text{ becomes } te_1\text{);}$$

$$\Phi_0^2(z) := \Big(z_1 + \sum_{i=2}^n b_i(z_1) z_i, \hat{z}\Big), \qquad b_i(s) := -\int_0^s \frac{\partial \varphi}{\partial z_i}\big(\sigma e_1, e_1\big)\, d\sigma;$$

$$\Phi_0^3(z) := \Big(z_1 - \sum_{i=2}^n \frac{\partial \varphi}{\partial v_i}(0, e_n) z_i, \hat{z}\Big);$$

$$\Phi_0^4(z) := \Big(z_1 - \frac{1}{2}\langle M\hat{z}, \hat{z}\rangle, B(z_1)\hat{z}\Big),$$

where $M$ and $B$ are defined as follows: set $A(t) := \frac{\partial^2 \varphi}{\partial \hat{v}^2}(te_1, e_1)$, $E(t) := \frac{\partial^2 \varphi}{\partial \hat{z} \partial \hat{v}}(te_1, e_1)$, and let $X(t)$ be the solution to the equation

$$X(t)^* A(t) - A(t) X(t)^* = E(t)^* - E(t)$$

with least Euclidean norm (as proven in [9, Lemma B.2], the (affine) space of solutions to the above equation is $n(n-1)/2$, so $X(t)$ is well-defined and unique). Then, $B(t)$ is defined as the solution of

$$\begin{cases} \dot{B}(t) = X(t)B(t), \\ B(0) = I_{n-1}, \end{cases}$$

and $M(t)$ is the $(n-1) \times (n-1)$ matrix defined as

$$M(t) := B(t)^* E(t)^* B(t) + B(t)^* A(t) X(t) B(t).$$

Since, for any $z \in B^n(0, \bar{\delta})^+$, the function $\bar{V}_0$ enters in the definition of the convex sets $C_z$ only as an additive constant (see (C.23)), it is not difficult to check that $\frac{\partial^2 \varphi}{\partial \hat{v}^2}, \frac{\partial^2 \varphi}{\partial \hat{z} \partial \hat{v}}$ depends only on the $C^1$-norm of $\bar{V}_0$. Hence, since $\|\bar{V}_0\|_{C^1}$ is bounded by 2 (by (C.7)), all maps $\Phi_0^j$, $j = 1, 2, 3, 4$, are bounded in $C^1$ topology together with their inverse, with a bound independent of $\bar{V}_0$. This concludes the proof. □

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