Sard conjectures and measures contraction properties in sub-Riemannian geometry

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Setting

Throughout all the talk, M is a smooth connected manifold of dimension n equipped with a **sub-Riemannian structure of rank** m in M given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally by

  $$\Delta(x) = \mathrm{Span}\Big\{X^1(x), \ldots, X^m(x)\Big\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ is a family of m linearly independent smooth vector fields satisfying the **Hörmander condition**.
- g_x is a **scalar product** over $\Delta(x)$.

The sub-Riemannian geodesic distance on M is denoted by d_{SR} and the metric space (M, d_{SR}) is always assumed to be complete.

Some open problems in SR geometry

- The Sard Conjecture.
- The minimizing Sard Conjecture.
- Validity of Measure Contractions Properties.

The End-Point mapping

Assume that Δ is globally spanned by k smooth vector fields $X^1, \ldots, X^k$, that is $\Delta(x) = \text{Span}\left\{X^1(x), \ldots, X^k(x)\right\}$ for all $x \in M$. For every $x \in M$ and every $u \in L^2\left([0,1];\mathbb{R}^k\right)$, denote by $\gamma_{x,u} : [0,1] \to M$ the solution to the Cauchy problem

$$\dot{\gamma}(t) = \sum_{i=1}^{k} u_i(t) X^i(\gamma(t)) \text{ for a.e. t } \in [0,1], \quad \gamma(0) = x.$$

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Definition

Given a point $x \in M$, the **End-Point mapping**

$$E^{x,1} \; : \; L^2 \in \left([0,1]; \mathbb{R}^k\right) \longrightarrow M$$

is defined by

$$E^{x,1}(u) := \gamma_{x,u}(1) \qquad \forall u \in L^2 \in \left([0,1]; \mathbb{R}^k\right).$$

Singular horizontal paths and Examples

Definition

An horizontal path is called **singular** if it is, through the correspondence $\gamma \leftrightarrow u$, a critical point of the End-Point mapping $E^{x,1} : L^2 \to M$ (with $x = \gamma(0)$).

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Example 1: Riemannian case
Let $\Delta(x) = T_x M$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

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Example 2: Heisenberg, fat distributions
In $\mathbb{R}^3$, Δ given by $X^1 = \partial_x, X^2 = \partial_y + x\partial_z$ does not admin nontrivial singular horizontal paths.

Examples

Example 3: Martinet distributions

In $\mathbb{R}^3$, let $\Delta = \mathrm{Vect}\{X^1, X^2\}$ with X^1, X^2 given by

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \partial_{x_2} + x_1^2 \partial_{x_3}.$$

The singular horizontal paths are the horizontal paths which are contained in the set $\{x_1 = 0\}$.

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Example 4: Rank-2 distributions in dimension 3
In this case, the singular horizontal paths are those horizontal paths which are contained in the Martinet surface

$$\Sigma_\Delta = \Big\{ x \in M \,|\, \Delta(x) + [\Delta, \Delta](x) \neq T_x M \Big\}.$$

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Example 5: Rank-2 distributions in dimension 4
In this case, at least for a generic germ, the singular horizontal paths are given by the orbits of a smooth vector field in Δ.

The Sard Conjecture

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed. Set

$$\mathcal{S}_{\Delta}^{x} = \Big\{ \gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing.} \Big\}.$$

Conjecture (Sard Conjecture)

The set $\mathcal{S}_{\Delta}^{x}$ has Lebesgue measure zero.

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Remark

By the Brown-Morse-Sard Theorem (1935-42), we know that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function of class C^k with

$$k \geq \max\{1, n - m + 1\},$$

then the set $f(C_f)$ of critical values of f has Lebesgue measure zero.

Infinite dimension (Bates-Moreira, 2001)

The Sard Theorem is false in infinite dimension. Let $f : \ell^2 \to \mathbb{R}$ be defined by

$$f\left(\sum_{n=1}^{\infty} x_n\, e_n\right) = \sum_{n=1}^{\infty} \left(3 \cdot 2^{-n/3} x_n^2 - 2x_n^3\right).$$

The function f is polynomial ($f^{(4)} \equiv 0$) with critical set

$$C_f = \left\{ \sum_{n=1}^{\infty} x_n\, e_n \,\middle|\, x_n \in \left\{0, 2^{-n/3}\right\} \right\},$$

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and critical values

$$f(C_f) = \left\{ \sum_{n=1}^{\infty} \delta_n\, 2^{-n} \,|\, \delta_n \in \{0,1\} \right\} = [0,1].$$

The case of Martinet surfaces

Let M be a smooth manifold of dimension 3 and Δ be a totally nonholonomic distribution of rank 2 on M. We define the **Martinet surface** by

$$\Sigma_\Delta = \Big\{ x \in M \,|\, \Delta(x) + [\Delta,\Delta](x) \neq T_x M \Big\}$$

If Δ is generic, Σ_Δ is a surface in M. If Δ is analytic then Σ_Δ is analytic of dimension ≤ 2. If Δ is smooth, Σ_Δ is countably 2-rectifiable.

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Proposition

The singular horizontal paths are the orbits of the trace of Δ on Σ_Δ.

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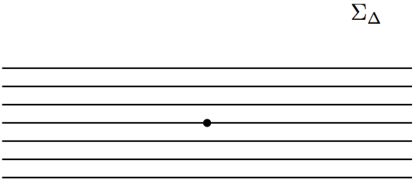
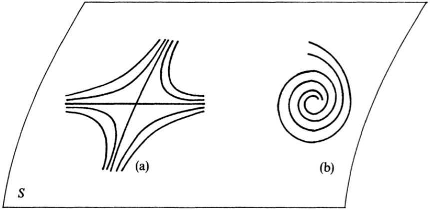
As a consequence, the Sard conjecture holds!!! In fact, we expect a stronger result...

Transverse case



Generic tangent case

(Zelenko-Zhitomirskii, 1995)



The Sard Conjecture on Martinet surfaces

Let M be of dimension 3 and Δ of rank 2.

$$\mathcal{S}_\Delta^x = \left\{ \gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing.} \right\}.$$

Conjecture (Strong Sard Conjecture)

The set $\mathcal{S}_\Delta^x$ has vanishing $\mathcal{H}^2$-measure.

Theorem (Belotto-R, 2016)

The above conjecture holds true under one of the following assumptions:

- *The Martinet surface is smooth;*
- *All datas are analytic and*

$$\Delta(x) \cap T_x Sing(\Sigma_\Delta) = T_x Sing(\Sigma_\Delta) \quad \forall x \in Sing(\Sigma_\Delta).$$

Sketch of proof

Ingredients of the proof

- Control of the divergence of vector fields which generates the trace of Δ over Σ_Δ of the form

$$\left|\mathrm{div}\mathcal{Z}\right| \leq C \left|\mathcal{Z}\right|.$$

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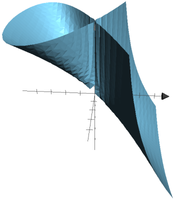
- Resolution of singularities.

An example

In $\mathbb{R}^3$,

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[\frac{y^3}{3} - x^2 y (x+z) \right] \partial_z .$$



Martinet Surface: $\Sigma_\Delta = \left\{ y^2 - x^2(x+z) = 0 \right\}.$

The minimizing Sard Conjecture

Let (Δ, g) be a SR structure on M and $x \in M$ fixed.

$$\mathcal{S}^x_{\Delta,min^g} = \Big\{ \gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.} \Big\}.$$

Conjecture (SR or Minimizing Sard Conjecture)

The set $\mathcal{S}^x_{\Delta,min^g}$ has Lebesgue measure zero.

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Remark

We know since the 90's that there are examples of sub-Riemannian structures with (strictly) singular minimizing curves (cf. Montgomery '94, Liu-Sussmann '95).

The minimizing Sard Conjecture: State of the art

Theorem (Agrachev, 2009)

The set $\mathcal{S}^x_{\Delta,min^g}$ is closed with empty interior.

Proposition

The Sard minimizing Conjecture holds true in the following cases:

- *Medium-fat distributions, that is for every $x \in M$ and all smooth section X of Δ with $X(x) \neq 0$,*

$$T_X M = \Delta(x) + [\Delta, \Delta](x) + [X, [\Delta, \Delta]](x).$$

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- *Generic distributions of rank $m \geq 3$.*

Minimizing Sard Conjecture vs. regularity of d_{SR}

Following Agrachev, we introduce the following definition:

Definition

We call **smooth point** of the function $y \mapsto d_{SR}(x,y)$ any $y \in M$ for which there is $p \in T_x^*M$ which is not a critical point of $\exp_x$ and such that the projection of the normal extremal starting at (x,p) is the unique minimizing geodesic from x to y.

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Proposition

Let $x \in M$ be fixed, the following properties are equivalent:

- *The Minimizing Sard conjecture is satisfied at x.*
- *The function $y \mapsto d_{SR}(x,y)$ is differentiable almost everywhere in M.*
- *The set of smooth points $\mathcal{O}_x$ is an open set with full measure in M ($d_{SR}(x,\cdot)$ is smooth on $\mathcal{O}_x$).*

Geodesic interpolation

Let (Δ, g) be a SR structure on M and $x \in M$ such that there is a measurable set $\mathcal{C}(x) \subset M$ with Lebesgue measure zero and a measurable map $\gamma_x : (M \setminus \mathcal{C}(x)) \times [0,1] \longrightarrow M$ such that for every $y \in M \setminus \mathcal{C}(x)$, the curve

$$s \in [0,1] \longmapsto \gamma_x(s,y)$$

is the unique minimizing horizontal path from x to y.

Definition

Let $A \subset M$ be a measurable set, for every $s \in [0,1]$, the s-interpolation of A from x is defined by

$$A_s := \left\{ \gamma_x(s,y) \,|\, y \in A \setminus \mathcal{C}(x) \right\} \qquad \forall s \in [0,1].$$

Measure Contraction Properties

Let μ a measure absolutely continuous with respect to $\mathcal{L}^n$ and $K \in \mathbb{R}, N > 1$ be fixed. The measure contraction property MCP(K, N) at x consists in comparing the contraction of volumes along minimizing geodesics from x with respect to the classical model space of Riemannian geometry of curvature K in dimension N.

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Definition

The property MCP(K, N) holds at x if for every measurable set $A \subset M \setminus \mathcal{C}(x)$ (provided that $A \subset B_{SR}(x, \pi\sqrt{N-1/K})$ if $K > 0$) with $0 < \mu(A) < \infty$,

$$\mu\left(A_s\right) \geq \int_A s \left[\frac{s_K\left(s d_{SR}(x,z)/\sqrt{N-1}\right)}{s_K\left(d_{SR}(x,z)/\sqrt{N-1}\right)} \right]^{N-1} ds,$$

for all $s \in [0, 1]$.

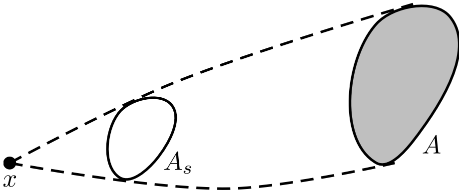
MCP(0, N)

In particular, we have:

Definition

The sub-Riemannian structure (Δ, g) equipped with μ satisfies MCP(0, N) at x if for every measurable set $A \subset M \setminus \mathcal{C}(x)$ with $0 < \mu(A) < \infty$,

$$\mu(A_s) \geq s^N \mu(A) \qquad \forall s \in [0,1].$$



Two qualitative results

Recall that a distribution Δ is two-step if

$$[\Delta, \Delta](x) = T_x M \qquad \forall x \in M.$$

Theorem (Badreddine-R, 2017)

Every two-step sub-Riemannian structure on a compact manifold equipped with a smooth measure satisfies $MCP(0, N)$ for some $N > 0$.

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Every two-step sub-Riemannian structure on a compact manifold equipped with a smooth measure satisfies $MCP(0, N)$ for some $N > 0$.

Theorem (Badreddine-R, 2017)

Every medium-fat Carnot group with the Haar measure satisfies $MCP(0, N)$ for some $N > 0$.

Sketch of proof

Ingredients of the proof

- Let $f := d_{SR}(x,\cdot)^2/2$ and $\nabla^h f$ its horizontal gradient. The control of $\mu(A_s)$ from below, $\mu(A_s) \geq s^N \mu(A)$, is equivalent to a control on the divergence of $\nabla^h f$ from above:
$$\mathrm{div}^{\mu}\left(\nabla^h f\right) \leq N$$

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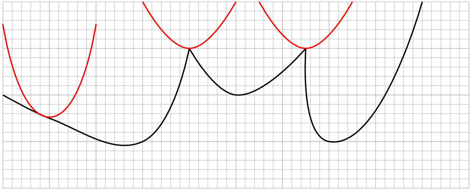
$$\mathrm{div}^{\mu}\left(\nabla^h f\right) \leq N$$
- The function f is nearly horizontally semiconcave.
- f is globally Lipschitz (Agrachev-Lee, 2009).

Semiconcave functions

Recall that a function $f : U \to \mathbb{R}$ is C-semiconcave in an open set $U \subset \mathbb{R}^n$ if for every $x \in U$ there is a function $\psi : U \to \mathbb{R}$ of class C^2 with $\|\psi\|_{C^2} \leq C$ such that

$$f(y) \leq \psi(y) \quad \forall y \in U.$$



Nearly horizontally semiconcave functions

Definition

A function $f : U \to \mathbb{R}$ in an open set $U \subset M$ is said to be C-nearly horizontally semiconcave with respect to (Δ, g) if for every $y \in U$, there are an open neighborhood V^y of 0 in $\mathbb{R}^m$, a function $\varphi^y : V^y \subset \mathbb{R}^m \to U$ of class C^2 and a function $\psi^y : V^y \subset \mathbb{R}^m \to \mathbb{R}$ of class C^2 such that

$$\varphi^y(0) = y, \, \psi^y(0) = f(y), \, f\left(\varphi^y(v)\right) \leq \psi^y(v) \quad \forall v \in V^y,$$

$$\left\{ d_0\varphi^y(e_1), \ldots, d_0\varphi^y(e_m) \right\} \text{ is orthonormal in } \Delta(y),$$

$$\text{and} \quad \|\varphi^y\|_{C^2}, \, \|\psi^y\|_{C^2} \leq C.$$

End of the proof

Proposition (Badreddine-R, 2017)

If M is compact and Δ is two-step then there is $C > 0$ such that all functions $f^x = d_{SR}(x, \cdot)^2/2$ are C-nearly horizontally semiconcave in M.

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Lemma

There is $B > 0$ such that for every $x \in M$ the following property holds: there is locally a orthonormal family of smooth vector fields $X^1, \ldots, X^m$ which parametrize Δ such that $\|X^i\|_{C^1} \leq B$ for $i = 1, \ldots, m$ and

$$X^i \cdot (X^i \cdot f^x) \leq B \left|\nabla_z f^x\right| + B \quad \forall i = 1, \ldots, m.$$

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$\rightsquigarrow$ We note that $\mathrm{div}_y^{\mu}\left(\nabla^h f^x\right) = \sum_{i=1}^m \left(X^i \cdot f^x\right)(y)\, \mathrm{div}_y^{\mu}\left(X^i\right) + \sum_{i=1}^m \left[X^i \cdot (X^i \cdot f)\right](y).$

Thank you for your attention !!