Mass Transportation on the Earth

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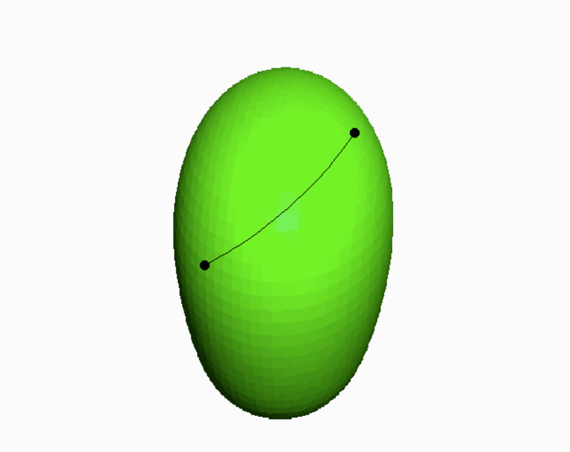
The framework

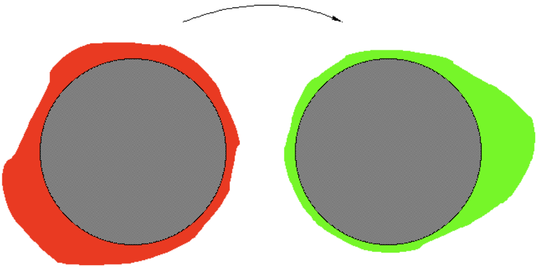
Let M be a **smooth connected compact surface** in $\mathbb{R}^n$. For any $x, y \in M$, we define the geodesic distance between x and y, denoted by $d(x, y)$, as the minimum of the lengths of the curves (drawn on M) joining x to y.



Mass transportation on surfaces

Let μ_0 and μ_1 be **probability measures** on M. We call **transport map** from μ_0 to μ_1 any measurable map $T : M \rightarrow M$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset M.$$



The McCann Theorem

Quadratic transport problem: Given two probability measures μ_0, μ_1 on M, find a measurable map $T : M \longrightarrow M$ with $T_\sharp \mu_0 = \mu_1$ which minimizes the quadratic transport cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

Theorem (McCann '01)

If μ_0 is absolutely continuous with respect to the Lebesgue measure, then there is a unique optimal transport map from μ_0 to μ_1 for the quadratic cost.
*There exists a c-**convex** function $\varphi : M \rightarrow \mathbb{R}$ such that*

$$T(x) = \exp_x(\nabla\varphi(x)) \qquad \mu_0 \text{ a.e. } x \in M.$$

More details

The potential φ is differentiable a.e. and satisfies a.e.

$$\nabla\varphi(x) \in \mathcal{I}(x) \quad \text{and} \quad T(x) = \exp_x\left(\nabla\varphi(x)\right).$$

Definition (Exponential mapping)

For every $v \in T_xM$, we define the **exponential** of v by $\exp_x(v) = \gamma_{x,v}(1)$, where $\gamma_{x,v} : [0,1] \to M$ is the unique geodesic starting at x with velocity v.

Definition (Injectivity domain)

We call **injectivity domain** of x, the subset of T_xM defined as

$$\mathcal{I}(x) := \left\{ v \in T_xM \,\middle|\, \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minim.} \\ \text{geod. between } x \text{ and } \exp_x(tv) \end{array} \right\}.$$

Problem

Regularity of T ?

Brenier's Theorem and Caffarelli's regularity theory

Theorem (Brenier '91)

Let μ_0, μ_1 be two probability measures with compact support in $\mathbb{R}^n$ such that μ_0 is absolutely continuous w.r.t. Lebesgue. Then there exists a unique optimal transport map for the quadratic cost $c(x,y) = |x-y|^2$ from μ_0 to μ_1. There is a convex function $\psi : M \to \mathbb{R}$ such that

$$T(x) = \nabla\psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected bounded open subsets of $\mathbb{R}^n$ and f_0, f_1 be probabilities densities on Ω_0 and Ω_1 respectively, with f_0 and f_1 bounded from above and below. Assume that μ_0 and μ_1 have respectively densities f_0 and f_1 with respect to the Lebesgue measure and that Ω_1 is convex. Then the quadratic optimal transport map from μ_0 to μ_1 is **continuous**.

References

- Cordero-Erausquin (1999)
- Ma, Trudinger, Wang (2005)
- Loeper (2006)
- Kim, McCann
- Delanoë, Ge
- Loeper, Villani
- Figalli, Rifford
- Loeper, Figalli
- Figalli, Rifford, Villani
- Figalli, Kim, McCann

Characterization of TCP on surfaces

We say that the surface $M \subset \mathbb{R}^n$ satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:
For any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \mathrm{vol}_g, \quad \mu_1 = \rho_1 \mathrm{vol}_g,$$

the optimal transport map from μ_0 to μ_1 is **continuous**.

Theorem (Figalli-R-Villani '10)

Let M be a surface in $\mathbb{R}^n$. It satisfies **TCP** *if and only if the following properties hold:*

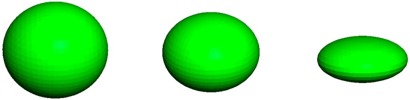
- *all the injectivity domains are convex,*
- *the cost $c = d^2/2$ is regular.*

Convexity of injectivity domains (examples)

- Flat tori: all the injectivity domains are convex
- Spheres: all the injectivity domains are open discs
- Ellipsoids of revolution (oblate case):

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1 \quad \mu \in (0,1].$$

The injectivity domains of an oblate ellipsoid of revolution are all convex if and only if and only if the ratio between the minor and the major axis is greater or equal to $1/\sqrt{3}\,(\simeq 0.58)$.



Regular costs

Definition

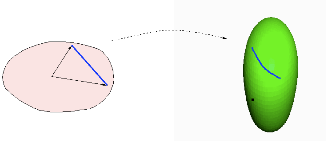
Assume that all the injectivity domains of M are convex. The cost $c = d^2/2 : M \times M \to \mathbb{R}$ is called **regular**, if for every $x, x' \in M$ the function

$$F_{x,x'} \; : \; v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

is **quasiconvex**, that is for every $v_0, v_1 \in \mathcal{I}(x)$, there holds

$$F_{x,x'}\big(v_t\big) \leq \max\Big(F_{x,x'}\big(v_0\big), F_{x,x'}\big(v_1\big)\Big) \qquad \forall t \in [0,1],$$

where $y_t := \exp_x v_t$ and $v_t := (1-t)v_0 + tv_1 (\in \mathcal{I}(x))$.



An easy lemma

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle > 0.$$

Then F is quasiconvex.

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$. Define $h : [0,1] \rightarrow \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t\in[0,1]} h(t) \; > \; \max\{h(0), h(1)\}.$$

There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.
Contradiction !! □

Exercises

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2.

False Lemma

Assume that for every $v \in U$ and every $w \in \mathbb{R}^n$, the following property holds:

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F\, w, w \rangle \geq 0.$$

Then F is quasiconvex.

True Lemma

Assume that there is a constant $C > 0$ such that

$$\langle \nabla_v^2 F\, w, w \rangle \geq -C \left| \langle \nabla_v F, w \rangle \right| |w| \qquad \forall v \in U, \forall w \in \mathbb{R}^n.$$

Then F is quasiconvex.

Back to our problem

Assume that all the injectivity domains are convex and fix $x, x' \in M$. Recall that

$$F_{x,x'}(v) = F(v) = c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big).$$

- F is not smooth.
- For generic segments, $t \mapsto F(v_t)$ is smooth outside a finite set of "convex" times.
- If it is smooth at v, then $\nabla_v^2 F$ has the form

$$\nabla_v^2 F(h,h) = -\int_0^1 (1-t) \frac{\partial^4 c}{\partial^2 x \partial^2 y}(*)(*)\, dt$$

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\bigg|_{s=0}\,\frac{d^2}{dt^2}\bigg|_{t=0} c\left(\exp_x(t\xi),\exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$.

Proposition (Villani '09, Figalli-R-Villani '10)

Let M be a surface all of whose injectivity domains are convex. Then the following properties are equivalent:

- *The cost $c = d^2/2$ is regular.*
- *The* **MTW** *tensor is $\succeq 0$, that is for any $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$,*

$$\langle \xi,\eta\rangle_x = 0 \Longrightarrow \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

Caracterization of **TCP** on surfaces

Theorem (Figalli-R-Villani '10)

Let M be a surface in $\mathbb{R}^n$. It satisfies **TCP** *if and only if the two following properties holds:*

- *all the injectivity domains are convex,*
- $\mathfrak{S} \succeq 0$.

Loeper noticed that for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_x M$, there holds

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = \sigma_x,$$

where σ_x denote the gaussian curvature of M at x. As a consequence,

$$\textbf{TCP} \implies \sigma \geq 0.$$

Therefore, if $M \subset \mathbb{R}^3$ satisfies **TCP**, then it is **convex**.

Spheres

Theorem (Loeper '06)

The **MTW** *tensor of the round unit sphere* $\mathbb{S}^2$ *satisfies* $\mathfrak{S} \succeq 1$, *that is for any* $x \in \mathbb{S}^2, v \in \mathcal{I}(x)$ *and* $\xi, \eta \in T_x\mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$

In particular, the round sphere $\mathbb{S}^2$ *satisifes* **TCP**.

Is this result stable under small perturbation ?

Two issues

- Stability of the convexity of all injectivity domains.
- Stability of $\mathfrak{S} \succeq K$.
  On $\mathbb{S}^2$, the **MTW** tensor is given by

$$\begin{aligned}\mathfrak{S}_{(x,v)}\left(\xi,\xi^{\perp}\right) &= 3\left[\frac{1}{r^2}-\frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)}-\frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4 \\ &\quad + \frac{3}{2}\left[-\frac{6}{r^2}+\frac{\cos(r)}{r\sin(r)}+\frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,\end{aligned}$$

with
$x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := |v|, \xi = (\xi_1,\xi_2), \xi^{\perp} = (-\xi_2,\xi_1)$.

Extended MTW tensor

Let $x \in M$ and $v \in T_xM$ be such that $\exp_x$ is a **local diffeomorphism** in a neighborhood of v and set $y := \exp_x v$. By the Inverse Function Theorem there are an open neighborhood $\mathcal{V}$ of (x,v) in TM, and an open neighborhood $\mathcal{W}$ of (x,y) in $M \times M$, such that

$$\begin{array}{rcl} \Psi_{(x,v)} : \mathcal{V} \subset TM & \longrightarrow & \mathcal{W} \subset M \times M \\ (x',v') & \longmapsto & \left(x', \exp_{x'}(v')\right) \end{array}$$

is a smooth diffeomorphism from $\mathcal{V}$ to $\mathcal{W}$. Then we define $\hat{c}_{(x,v)} : \mathcal{W} \to \mathbb{R}$ by

$$\hat{c}_{(x,v)}(x',y') := \frac{1}{2}\left|\Psi^{-1}_{(x,v)}(x',y')\right|^2_{x'} \qquad \forall (x',y') \in \mathcal{W}.$$

If $v \in \mathcal{I}(x)$ then for y' close to $\exp_x v$ and x' close to x we have $\hat{c}_{(x,v)}(x',y') = c(x',y') := d(x',y')^2/2$.

Extended **MTW** tensor...

Definition (Nonfocal domain)

We call **nonfocal domain** of x, the subset of T_xM defined as

$$\mathcal{NF}(x) := \left\{ v \in T_xM \;\middle|\; \begin{array}{c} \exp_x \text{ is a local diffeo. at} \\ tv \text{ for any } t \in [0,1] \end{array} \right\}.$$

Definition (Extended **MTW** tensor)

The extended **MTW** tensor $\mathfrak{S}$ is defined as

$$\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = -\frac{3}{2} \frac{d^2}{ds^2}\bigg|_{s=0} \frac{d^2}{dt^2}\bigg|_{t=0} \hat{c}_{(x,v)}\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{NF}(x)$, and $\xi, \eta \in T_xM$.

Why $\overline{\mathfrak{S}}$?

Proposition

On surfaces, the boundary of $\mathcal{NF}(x)$ is a smooth curve which depends "smoothly" on the surface.
Injectivity domains of small perturbations of the round sphere in C^5 topology are uniformly convex.

Proposition

Let $M \subset \mathbb{R}^n$ be a surface. Assume that all the nonfocal domains are convex and that for any $x \in M, v \in \mathcal{NF}(x)$, and $\xi, \eta \in T_x M$,

$$\langle \xi, \eta \rangle_x = 0 \Longrightarrow \overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) \geq 0.$$

Then all the injectivity domains are convex.

Mass Transportation on the Earth

Theorem (Figalli-R '09)

Any small deformation of $\mathbb{S}^2$ *in* C^5 *topology satisfies* $\overline{\mathfrak{S}} \succeq 1/2$, *has convex injectivity domains and satisfies* **TCP**.

Thank you for your attention !!

Greater dimensions

Let (M, g) be a smooth connected compact Riemannian manifold of dimension $n \geq 2$.

Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies **(TCP)**. *Then*

- *all its injectivity domains are convex,*
- *the* **MTW** *tensor is $\succeq 0$.*

Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies the two following properties:

- *all its injectivity domains are strictly convex,*
- *the* **MTW** *tensor is $\succ 0$,*

Then, it satisfies **TCP**.

Greater dimensions...

Theorem (Figalli-R-Villani '09)

Any small deformation of the round metric on $\mathbb{S}^n$ in C^4 topology satisfies $\overline{\mathfrak{S}} \succeq 1/2$, has uniformly convex injectivity domains and satisfies **TCP**.