Optimal Transport and Control

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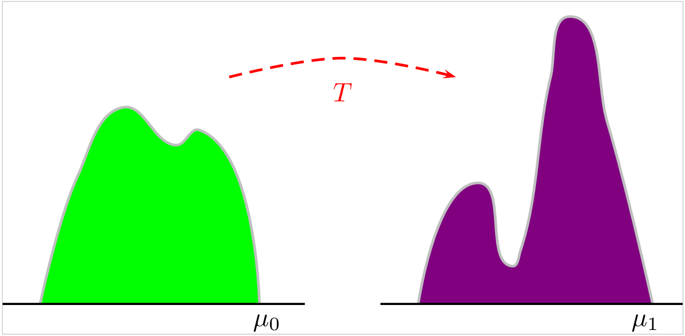
Transport maps

Let M be a **smooth manifold** and μ_0 and μ_1 be **probability measures** on M. We call **transport map** from μ_0 to μ_1 any measurable map $T : M \to M$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset M.$$



The Monge Optimal Transport Problem

Let $c : M \times M \longrightarrow \mathbb{R}$ be a **cost** and μ_0, μ_1 two probabilities measures on M, find a transport map $T : M \longrightarrow M$ from μ_0 to μ_1 which minimizes the **transportation cost**

$$\int_M c(x, T(x)) d\mu_0(x).$$

Existence ?

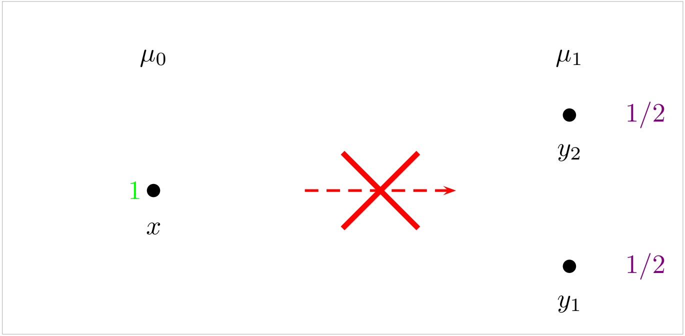
Uniqueness ?

Regularity ?

Example 1: Atomic measures

Let $x, y_1, y_2 \in M$ with $y_1 \neq y_2$. Then there is **no transport map** from

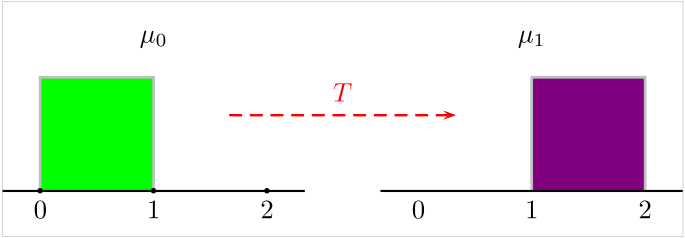
$$\mu_0 = \delta_x \qquad \text{to} \qquad \mu_1 = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta y_2.$$



Example 2: The original Monge problem in $\mathbb{R}$

Given two probability measures μ_0, μ_1 in $\mathbb{R}$, we are concerned with transport maps $T : \mathbb{R} \to \mathbb{R}$ from μ_0 to μ_1 which minimize the transportation cost

$$\int_{\mathbb{R}} |T(x) - x| \, d\mu_0(x).$$



$$T(x) = x + 1 \quad \text{and} \quad T(x) = 2 - x$$

both optimal with the same transportation cost.

Example 3: The quadratic Monge problem in $\mathbb{R}^n$

Given two probability measures μ_0, μ_1 in $\mathbb{R}^n$, we are concerned with transport maps $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ from μ_0 to μ_1 which minimize the transportation cost

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, d\mu_0(x).$$

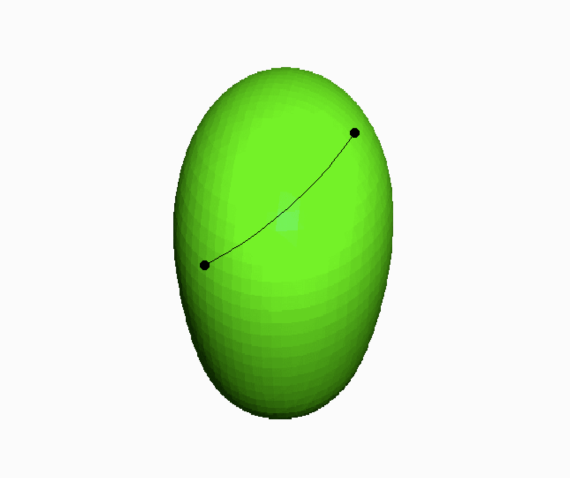
Theorem (Brenier '91)

If μ_0 is compactly supported and absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** *function $\psi : M \rightarrow \mathbb{R}$ such that*

$$T(x) = \nabla \psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Example 4: The McCann Theorem

Let (M, g) be a smooth compact Riemannian manifold equipped with its geodesic distance d_g.



Example 4: The McCann Theorem

Let (M, g) be a smooth compact Riemannian manifold equipped with its geodesic distance d_g. Given two probability measures μ_0, μ_1 on M, we are concerned with transport maps $T : M \rightarrow M$ which minimize the transportation cost

$$\int_M d_g(x, T(x))^2 d\mu_0(x).$$

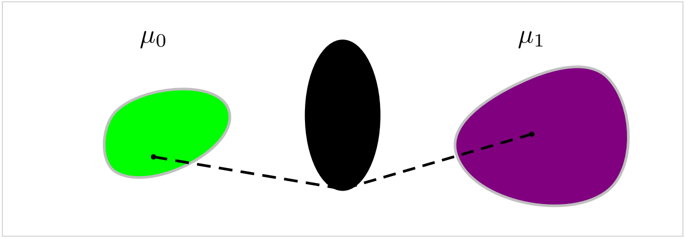
Theorem (McCann '01)

*If μ_0 is absolutely continuous w.r.t. Lebesgue, there exists a unique optimal transport map from μ_0 to μ_1. In fact, there is a c-**convex** function $\varphi : M \rightarrow \mathbb{R}$ satisfying*

$$T(x) = \exp_x\left(\nabla\varphi(x)\right) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Example 5: A Monge problem with obstacle

Let C be a smooth convex body in $\mathbb{R}^n$ and d the geodesic distance on $\Omega := \mathbb{R}^n \setminus C$.



Given two probability measures μ_0, μ_1 in $\mathbb{R}^n$, we are concerned with transport maps $T : \mathbb{R}^n \to \mathbb{R}^n$ from μ_0 to μ_1 which minimize the transportation cost $\int_{\mathbb{R}^n} d(x, T(x)^2 d\mu_0(x)$.

Theorem (Cardaliaguet-Jimenez '11)

Existence (but not uniqueness in general).

Optimal Transport and Control

The purpose of this talk is to study optimal transport problems with **costs coming from optimal control problems**. Two types of costs:

- LQR costs
- Quadratic sub-Riemannian distances.

The Kantorovitch Optimal Transport Problem

Given M, a cost $c : M \times M \to \mathbb{R}$ and two probability measures μ_0, μ_1 on M, we want to find a probability measure γ on $M \times M$ having **marginals** μ_0 and μ_1, i.e.

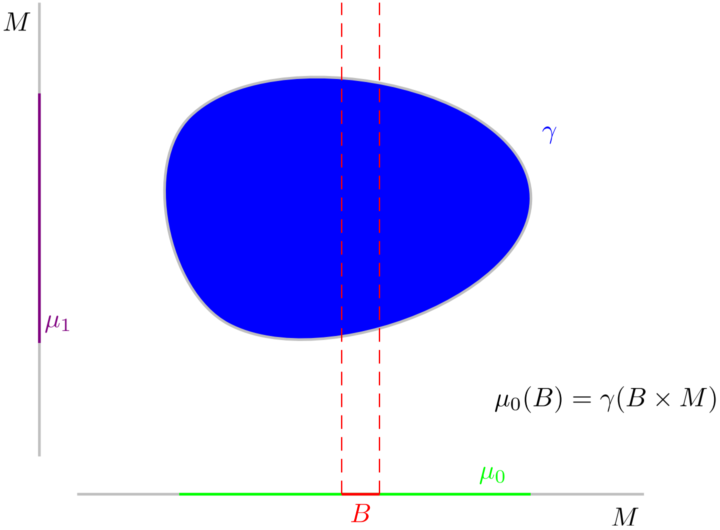
$$(\pi_1)_\sharp \gamma = \mu_0 \qquad \text{and} \qquad (\pi_2)_\sharp \gamma = \mu_1,$$

(where $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ are the canonical projections), which minimizes the transportation cost given by

$$\int_{M \times M} c(x, y) d\gamma(x, y).$$

When the transport condition $(\pi_1)_\sharp \gamma = \mu, (\pi_2)_\sharp \gamma = \nu$ is satisfied, we say that γ is a **transport plan**, and if γ minimizes also the cost we call it an **optimal transport plan**.

Kantorovitch allows splitting



Monge vs. Kantorovitch

Let M be a smooth manifold, $c : M \times M \longrightarrow \mathbb{R}$ be a cost function, and μ_0, μ_1 two probability measures on M.

Monge's Problem

Minimize

$$\int_M c(x, T(x)) d\mu_0(x)$$

among all transport maps T, that is $T_\sharp \mu_0 = \mu_1$.

Kantorovitch's Problem

Minimize

$$\int_M c(x, y) d\gamma(x, y)$$

among all transport plans γ, that is $(\pi_1)_\sharp \gamma = \mu_0, (\pi_2)_\sharp \gamma = \mu_1$.

Reminder: The Brenier-McCann Theorem

Theorem (Brenier '91)

If μ_0 is compactly supported and absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** *function $\psi : M \longrightarrow \mathbb{R}$ such that*

$$T(x) = \nabla\psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Theorem (McCann '01)

*If μ_0 is absolutely continuous w.r.t. Lebesgue, there exists a unique optimal transport map from μ_0 to μ_1. In fact, there is a c-***convex** *function $\varphi : M \to \mathbb{R}$ satisfying*

$$T(x) = \exp_x\left(\nabla\varphi(x)\right) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Kantorovitch's Duality

Theorem

There are two continuous function $\psi_1, \psi_2 : M \rightarrow \mathbb{R}$ *satisfying*

$$\psi_1(x) = \max_{y \in M} \{\psi_2(y) - c(x,y)\} \qquad \forall x \in M,$$

$$\psi_2(y) = \min_{x \in M} \{\psi_1(x) + c(x,y)\} \qquad \forall y \in M.$$

such that the following holds: a transport plan γ *is optimal if and only if one has*

$$\psi_2(x) - \psi_1(y) = c(x,y) \qquad \text{for } \gamma \text{ a.e. } (x,y) \in M \times M.$$

As a consequence, to obtain that an optimal transport plan corresponds to a Monge's optimal transport map, we have to show that γ is concentrated on a graph.

Proof of Brenier-McCann's Theorem I

Returning to the Riemannian setting, let $\psi_1, \psi_2 : M \to \mathbb{R}$ be a pair of **Kantorovitch potentials** given by the previous result.

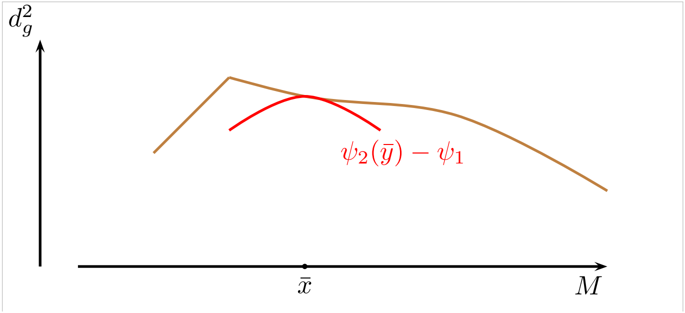
- The function $x \mapsto d_g(x,y)^2$ is locally Lipschitz on M.
- The function ψ_1 is locally Lipschitz on M. As a consequence, by Rademacher's Theorem, it is differentiable μ_0-a.e.
- Let $\bar{x} \in \text{supp}(\mu_0)$ be such that ψ_1 is differentiable at $\bar{x}$. Let $\bar{y}$ be such that

  $$\psi_1(\bar{x}) = \psi_2(\bar{y}) - d_g(\bar{x},\bar{y})^2.$$

  Then we have,

  $$d_g(x,\bar{y})^2 \geq \psi_2(\bar{y}) - \psi_1(x) \qquad \forall x \in M.$$

Proof of Brenier-McCann's Theorem II



Any Lipschitz curve $c : [0,1] \longrightarrow M$ with $c(1) = \bar{y}$ satisfies

$$\int_0^1 |\dot{c}(t)|_{c(t)}^2 \, dt \geq d_g\left(c(0), \bar{y}\right)^2 \geq \psi_2(\bar{y}) - \psi_1(c(0)),$$

with equality if c is the minimizing geodesic from $\bar{x}$ to $\bar{y}$.
$\Longrightarrow \bar{y} = \exp_{\bar{x}}\left(\nabla \psi_1(\bar{x})\right).$

Strategy of proof

TWO ISSUES

- (Regularity) Show that ψ_1 is differentiable μ-a.e.
- (Twist) Deduce that, if ψ_1 is differentiable at $\bar{x}$, then there is a unique $\bar{y}$ such that

$$\psi_1(\bar{x}) = \psi_2(\bar{y}) - c(\bar{x}, \bar{y}).$$

LQR costs

Let us consider the linear control system

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and A, B are $n \times n$ and $n \times m$ matrices. Cost

$$c(x,y) = \inf \left\{ \int_0^1 L(x(t), u(t))\, dt \,|\, u \in L^2, x_u(0) = x, x_u(1) = y \right\}$$

with (W sym. nonneg. and U sym. def. pos)

$$L(x,u) = \langle x, Wx \rangle + \langle u, Uu \rangle$$

Theorem (Hindawi-Pomet-R '11)

Existence, uniqueness, and regularity.

Quadratic sub-Riemannian distances I

Let (M, Δ, g) be a complete sub-Riemannian structure of dimension n and rank $m < n$, that is

- M a smooth connected manifold.
- Δ a totally nonholonomic distribution.
- g a smooth metric on Δ.

Let $d_{SR}(\cdot,\cdot)$ be the sub-Riemannian distance on $M \times M$, i.e.

$$d_{SR}(x,y) = \inf\Big\{ \text{length}_g(\gamma) \,|\, \gamma\, C^1, \gamma(0) = x, \gamma(1) = y, \dot{\gamma}(t) \in \Delta_{\gamma(t)} \Big\}.$$

From now on, we assume that the metric space (M, d_{SR}) is complete.

Quadratic sub-Riemannian distances II

Let μ, ν be two compactly supported probability measures on M. Find a measurable map $T : M \to M$ satisfying

$$T_\sharp \mu = \nu,$$

and in such a way that T minimizes the transportation cost given by

$$\int_M d_{SR}(x, T(x))^2 d\mu(x).$$

Theorem (Figalli-R '10)

Assume that there exists an open set $\Omega \subset M \times M$ such that supp$(\mu \times \nu) \subset \Omega$, and d_{SR}^2 is locally Lipschitz on $\Omega \setminus D$. Then, there is existence and uniqueness of an optimal transport map.

Examples

- Example 1: Two generating distributions

Proposition (A. Agrachev, P. Lee, 2008)

If Δ is two-generating on M, then the squared sub-Riemannian distance function is locally Lipschitz on $M \times M$.

- Example 2: Generic sub-Riemannian structures

Proposition (Y. Chitour, F. Jean, E. Trélat, 2006)

Let (M, g) be a complete Riemannian manifold of dim ≥ 4. Then, for any generic distribution of rank ≥ 3, the squared sub-Riemannian distance function is locally semiconcave (hence locally Lipschitz) on $M \times M \setminus D$.

Examples

- Example 3: Rank-two distributions in dimension 4
  Consider the distribution Δ in $\mathbb{R}^4$ spanned by

  $$f_1 = \partial_{x_1}, \qquad f_2 = \partial_{x_2} + x_1 \partial_{x_3} + x_3 \partial_{x_4}.$$

  A horizontal path $\gamma : [0,1] \to \mathbb{R}^4$ is singular if and only if it satisfies (up to reparameterization by arc-length)

  $$\dot{\gamma}(t) = f_1(\gamma(t)), \qquad \forall t \in [0,1].$$

  Then, for any metric, the sub-Riemannian distance function d_{SR} is locally lipschitz on the set

  $$\Omega = \left\{ (x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid (y-x) \notin \mathrm{SPAN}\{e_1\} \right\}$$

  Consequently, we get existence and uniqueness of optimal transport maps.

Thank you for your attention !!