Mass Transportation on surfaces

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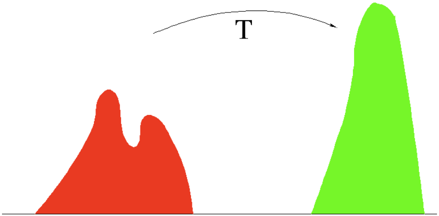
Monge quadratic transport in $\mathbb{R}^n$

Let μ_0 and μ_1 be **probability measures with compact support** in $\mathbb{R}^n$. We call **transport map** from μ_0 to μ_1 any measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset \mathbb{R}^n.$$



The Brenier Theorem

Monge quadratic problem : Study of transport maps $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which minimize the **quadratic** transport cost

$$\int_{\mathbb{R}^n} \left| T(x) - x \right|^2 d\mu_0(x).$$

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Assume that μ_0 is absolutely continuous with respect to the Lebesgue measure. Then there exists a unique optimal transport map for the quadratic cost from μ_0 to μ_1. There is a convex function $\psi : M \longrightarrow \mathbb{R}$ such that

$$T(x) = \nabla\psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

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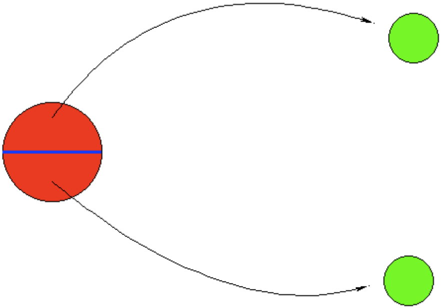
Regularity ?

Contre-exemple trivial



Caffarelli's regularity theory

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected bounded open subsets of $\mathbb{R}^n$ and f_0, f_1 be probabilities densities on Ω_0 and Ω_1 respectively, with f_0 and f_1 bounded from above and below. Assume that μ_0 and μ_1 have respectively densisites f_0 and f_1 with respect to the Lebesgue measure and that Ω_1 is convex. Then the quadratic optimal transport map from μ_0 to μ_1 is continuous.

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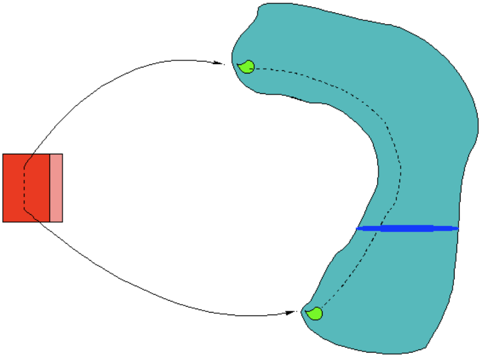
The convexity of the target is necessary.

Example of a nonconvex target



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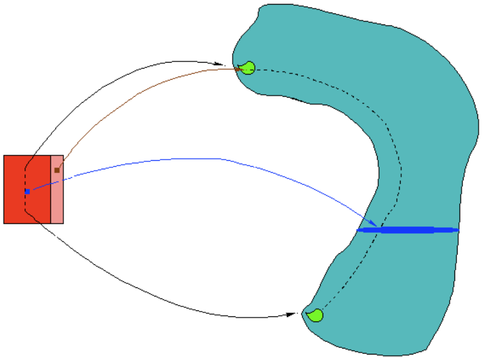

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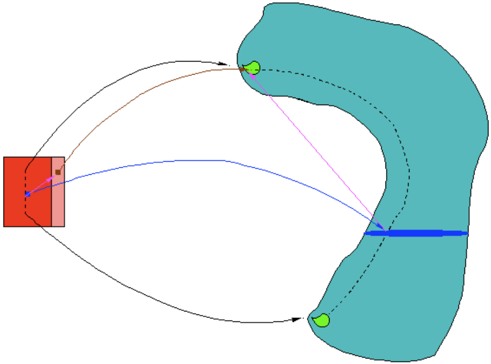

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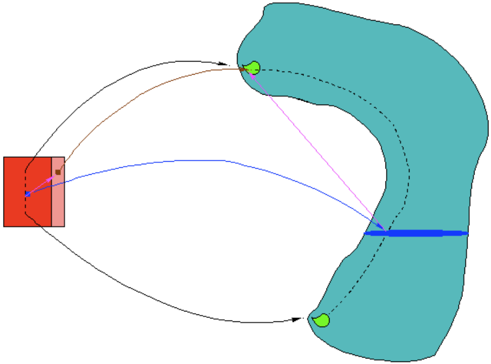

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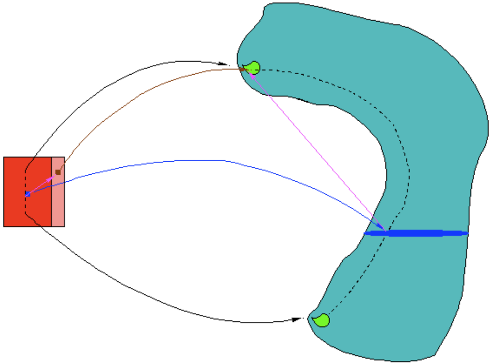

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Mass transportation on surfaces

Let M be a smooth connected compact surface in $\mathbb{R}^n$. For any $x, y \in M$, we define the geodesic distance between x and y, denoted by $d(x, y)$, as the minimum of the lengths of the curves (drawn on M) joining x to y.

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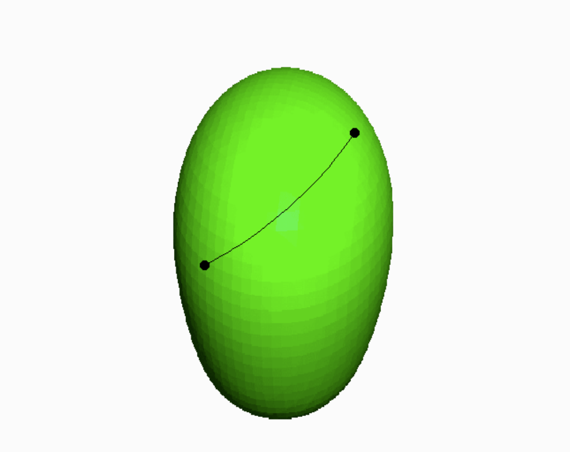

The McCann Theorem

Quadratic transport problem: Given two probability measures μ_0, μ_1 on M, find a measurable map $T : M \longrightarrow M$ with $T_\sharp \mu_0 = \mu_1$ which minimizes the quadratic transport cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

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Theorem (McCann '01)

If μ_0 is absolutely continuous with respect to the Lebesgue measure, then there is a unique optimal transport map from μ_0 to μ_1 for the quadratic cost.
There exists a c-convex function $\varphi : M \rightarrow \mathbb{R}$ such that

$$T(x) = \exp_x (\nabla \varphi(x)) \qquad \mu_0 \text{ a.e. } x \in M.$$

The TCP property

We say that the surface $M \subset \mathbb{R}^n$ satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:

For any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \mathrm{vol}_g, \quad \mu_1 = \rho_1 \mathrm{vol}_g,$$

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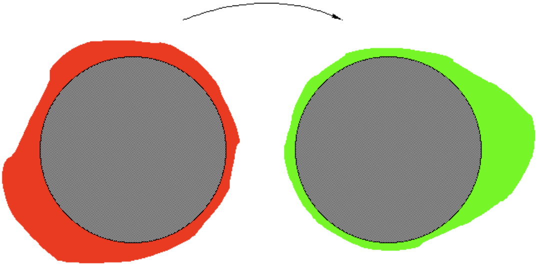

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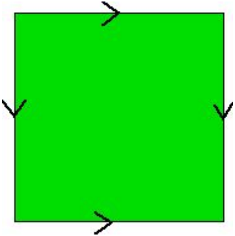
Flat tori



on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, there holds $\quad d(x,y) = \inf_{p\in\mathbb{Z}^2} |x-y+p|.$

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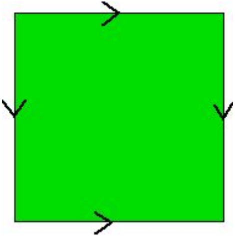

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We can "lift" a transport problem on $\mathbb{T}^2$ to $\mathbb{R}^2$ and apply the Caffarelli regularity theory.

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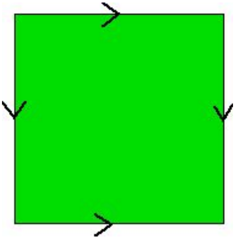

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Theorem (Cordero-Erausquin '99)

Flat tori satisfy **TCP**.

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- *the cost $c = \frac{1}{2}d^2$ is regular.*

Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_xM$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

where $\gamma_{x,v} : [0,1] \longrightarrow M$ is the unique geodesic starting at x with velocity v.

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- We call **injectivity domain** of x, the subset of T_xM defined as

$$\mathcal{I}(x) := \left\{ v \in T_xM \;\middle|\; \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minim.} \\ \text{geod. between } x \text{ and } \exp_x(tv) \end{array} \right\}.$$

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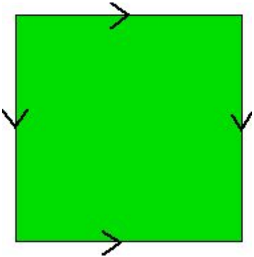
It is a star-shaped (w.r.t. $0 \in T_x M$) domain with Lipschitz boundary.

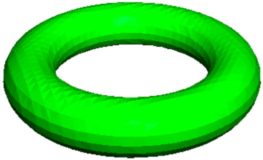
Injectivity domains : Examples...

Flat tori : all the injectivity domains are convex.



Injectivity domains : Examples...

Torus of revolution: the injectivity domains are not necessarily convex.



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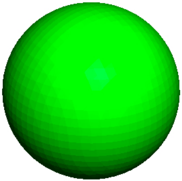
Spheres : all the injectivity domains are balls.

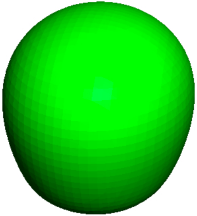


Injectivity domains : Examples...

C^4 perturbations of round spheres:

Theorem (Figalli-R '09)

Any small deformation of the round sphere $\mathbb{S}^2$ in C^4 topology has all its injectivity domains uniformly convex.



Injectivity domains : Examples...

Ellipsoids of revolution (oblate case):

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1 \qquad \mu \in (0, 1].$$

Theorem (Bonnard-Caillau-R '10)

The injectivity domains of an oblate ellipsoid of revolution are all convex if and only if and only if the ratio between the minor and the major axis is greater or equal to $1/\sqrt{3} (\simeq 0.58)$.

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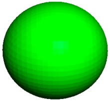

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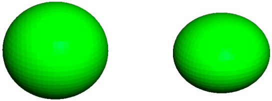

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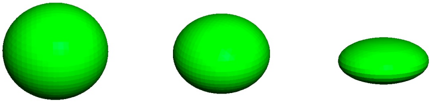

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- *the cost $c = \frac{1}{2}d^2$ is regular.*

Regular costs

The cost $c = \frac{1}{2}d^2/2 : M \times M \to \mathbb{R}$ is called **regular**, if for every $x \in M$ and every $v_0, v_1 \in \mathcal{I}(x)$, there holds

$$c\big(x, y_t\big) - c\big(x', y_t\big) \leq \max\Big(c\big(x, y_0\big) - c\big(x', y_0\big),\ c\big(x, y_1\big) - c\big(x', y_1\big)\Big),$$

for every $x' \in M$ and every $t \in [0, 1]$, where

$$y_t := \exp_x v_t \quad \text{and} \quad v_t := (1-t)v_0 + tv_1 \quad (\in \mathcal{I}(x)).$$

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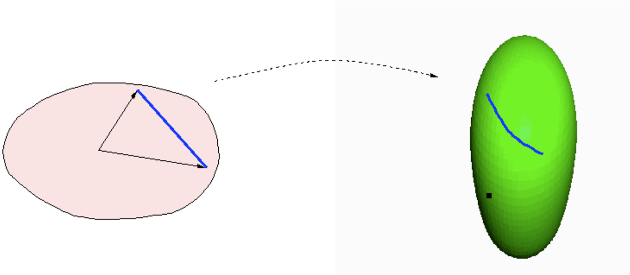

An obvious remark

Remark

Assume that all the injectivity domains of M are convex. Then the cost c is **regular** *if and only if for any $x, x' \in M$, the function*

$$F_{x,x'} \,:\, v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

is **quasiconvex** *(its sublevels sets are always convex).*

An easy lemma

Lemma

Let $U \subset \mathbb{R}^n$ *be an open convex set and* $F : U \to \mathbb{R}$ *be a function of class* C^2*. Assume that for every* $v \in U$ *and every* $w \in \mathbb{R}^n \setminus \{0\}$ *the following property holds :*

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle > 0.$$

Then F *is quasiconvex.*

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$.

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If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

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Since τ is a local maximum, one has $\dot{h}(\tau) = 0$. Contradiction !! □

Exercise 1

The following lemma is FALSE !!

False Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n$, the following property holds:

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F \, w, w \rangle \geq 0.$$

Then F is quasiconvex.

Exercise 2

However the following result holds true.

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that there is a constant $C > 0$ such that

$$\langle \nabla_v^2 F\, w, w \rangle \geq -C \left| \langle \nabla_v F, w \rangle \right| |w| \qquad \forall v \in U, \forall w \in \mathbb{R}^n,$$

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Back to our problem

Recall that $F(v) = F_{x,x'}(v) = c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$

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Hence

$$\frac{\partial F}{\partial v}(v) \cdot h = \frac{\partial c}{\partial y}\big(x, \exp_x(v)\big) \cdot \frac{\partial \exp_x}{\partial v}(v) \cdot h \\ - \frac{\partial c}{\partial y}\big(x', \exp_x(v)\big) \cdot \frac{\partial \exp_x}{\partial v}(v) \cdot h,$$

Back to our problem

Recall that $F(v) = F_{x,x'}(v) = c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$
Hence

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$$\begin{aligned}\frac{\partial^2 F}{\partial v^2}(v)\cdot(h,h) &= \frac{\partial^2 c}{\partial y^2}\big(x, \exp_x(v)\big)\cdot\left(\frac{\partial \exp_x}{\partial v}(v)\cdot h, \frac{\partial \exp_x}{\partial v}(v)\cdot h\right)\\ &\quad - \frac{\partial^2 c}{\partial y^2}\big(x', \exp_x(v)\big)\cdot\left(\frac{\partial \exp_x}{\partial v}(v)\cdot h, \frac{\partial \exp_x}{\partial v}(v)\cdot h\right)\\ &\quad + \left(\frac{\partial c}{\partial y}\big(x, \exp_x(v)\big) - \frac{\partial c}{\partial y}\big(x', \exp_x(v)\big)\right)\cdot \frac{\partial^2 \exp_x}{\partial v^2}(v)\cdot(h,h)\end{aligned}$$

Set $y := \exp_x(v)$, $q := \frac{\partial c}{\partial y}(x, y)$, $q' := \frac{\partial c}{\partial y}(x', y)$. There holds

$$\begin{aligned} \frac{\partial^2 F}{\partial v^2}(v) \cdot (h, h) = & \frac{\partial^2 c}{\partial y^2}\left(\exp_y(-q), y\right) \cdot \left(\tilde{h}, \tilde{h}\right) \\ & - \frac{\partial^2 c}{\partial y^2}\left(\exp_y(-q'), y\right) \cdot \left(\tilde{h}, \tilde{h}\right) \\ & + (q - q') \cdot \frac{\partial^2 \exp_x}{\partial v^2}(v) \cdot (h, h) \end{aligned}$$

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But $\frac{\partial c}{\partial x}(x,\exp_x(v)) = -v$, then

$$\frac{\partial^2 \exp_x}{\partial v^2}(v)\cdot(h,h) = -\left(\frac{\partial^2 c}{\partial x\partial y}(x,y)\right)^{-1}\frac{\partial^3 c}{\partial x\partial y^2}(x,y)\cdot\left(\tilde{h},\tilde{h}\right)$$

Setting

$$\Phi(p) = \Phi_{x,y,h}(p) = -\frac{\partial^2 c}{\partial y^2}\left(\exp_y(-p), y\right) \cdot \left(\tilde{h}, \tilde{h}\right),$$

we get

$$\begin{aligned} & \frac{\partial^2 F}{\partial v^2}(v) \cdot (h, h) \\ = & -\Phi(q) + \Phi(q') - D\Phi(q) \cdot (q' - q) \end{aligned}$$

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The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0} c\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

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Proposition (Villani '09, Figalli-R-Villani '10)

Let M be a surface all of whose injectivity domains are convex. Then the following properties are equivalent:

- *The cost $c = \frac{1}{2}d^2$ is regular.*
- *The* **MTW** *tensor is $\succeq 0$, that is for any $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$,*

$$\langle \xi,\eta\rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

Caracterization of **TCP** on surfaces

Theorem (Figalli-R-Villani '10)

Let M be a surface in $\mathbb{R}^n$. It satisfies **TCP** *if and only if the two following properties holds:*

- *all the injectivity domains are convex,*
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Loeper notices that for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_x M$, there holds

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = \sigma_x,$$

where σ_x denote the gaussian curvature of M at x. As a consequence,

$$\textbf{TCP} \Longrightarrow \sigma \geq 0.$$

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Therefore, if $M \subset \mathbb{R}^3$ satisfies **TCP**, then it is **convex**.

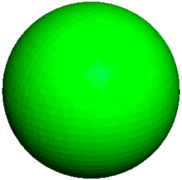
Spheres

Loeper checked that the **MTW** tensor of the round sphere $\mathbb{S}^2$ satisfies for any $x \in \mathbb{S}^2, v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x\mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$

Theorem (Loeper '06)

The round sphere $\mathbb{S}^2$ satisifes **TCP**.



Small deformations of $\mathbb{S}^2$

On $\mathbb{S}^2$, the **MTW** tensor is given by

$$\begin{aligned}
&\mathfrak{S}_{(x,v)}\left(\xi,\xi^{\perp}\right)\\
&\quad = 3\left[\frac{1}{r^2}-\frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)}-\frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4\\
&\qquad + \frac{3}{2}\left[-\frac{6}{r^2}+\frac{\cos(r)}{r\sin(r)}+\frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,
\end{aligned}$$

with $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := |v|, \xi = (\xi_1, \xi_2), \xi^{\perp} = (-\xi_2, \xi_1)$.

Theorem (Figalli-R '09)

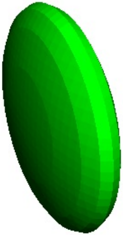
Any small deformation of $\mathbb{S}^2$ in C^4 topology satisfies **TCP**.

Ellipsoids

The ellipsoid of revolution $(E_\epsilon) \subset \mathbb{R}^3$

$$\frac{x^2}{\epsilon^2} + y^2 + z^2 = 1, \qquad \text{with } \epsilon = 0.29,$$

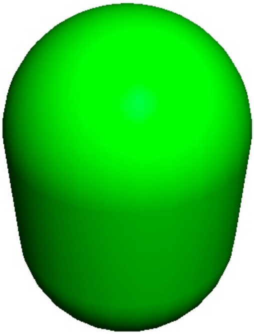
does not satisfy **MTW** $\succeq 0$.



Consequently, (E_ϵ) cannot satisfy **TCP**.

Curvature jumps

The surface made by two hemispheres glued at the ends of a cylinder with same radius does not have a regular cost.



Then, it does not satisfy **TCP**.

Perspectives

Conjectures

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MTW $\succeq 0 \Longrightarrow$ *convexity of all injectivity domains.*

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Conjecture

A satisfies **TCP** *if and only if* $\mathfrak{S} \succeq 0$.

Greater dimensions

Let (M, g) be a smooth connected compact Riemannian manifold of dimension $n \geq 2$.

Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies **(TCP)**. *Then*

- *all its injectivity domains are convex,*
- *the* **MTW** *tensor is $\succeq 0$.*

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Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies the two following properties:

- *all its injectivity domains are strictly convex,*
- *the* **MTW** *tensor is $\succ 0$,*

Then, it satisfies **TCP**.

Thank you for your attention !!