The Stabilization Problem: AGAS and SRS Feedbacks

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1 The Problem

Throughout this paper, M denotes a smooth manifold of dimension n. We are given a control system on M of the form,

$$\dot{x} = f(x,u) := \sum_{i=1}^{m} u_i f_i(x), \tag{1}$$

where $f_1, \cdots, f_m$ are smooth vector fields on M and where the control

$$u = (u_1, \cdots, u_m)$$

belongs to $\overline{B_m}$, the closed unit ball in $\mathbb{R}^m$. Throughout the paper, "smooth" means always "of class C^∞". Such a control system is said to be *Globally Asymptotically Controllable* at the point $O \in M$ (abbreviated GAC in the sequel) if the following two properties are satisfied:

1. Attractivity: For each $x \in M$ there exists a control $u(\cdot) : \mathbb{R}_{\geq 0} \to \overline{B_m}$ such that the corresponding trajectory $x(\cdot; x, u(\cdot))$ of (1) tends to O as t tends to infinity.
2. Lyapunov stability: For each neighborhood $\mathcal{V}$ of O, there exists some neighborhood $\mathcal{U}$ of O such that if $x \in \mathcal{U}$ then the control $u(\cdot)$ above can be chosen such that $x(t; x, u(\cdot)) \in \mathcal{V}, \forall t \geq 0$.

Example 1. The control system in the plane defined by

$$\begin{aligned} \dot{x} &= u(x^2 - y^2) \\ \dot{y} &= u(2xy), u \in [-1, 1], \end{aligned}$$

is GAC at the point $(0,0)$. In fact, as shown in Figure 1, for $(x,y) \neq (0,0)$ in the plane, the set $\{u(x^2 - y^2, 2xy) : u \in [-1,1]\}$ is a subinterval of the tangent space to the circle passing through (x,y) and $(0,0)$ with center on the y-axis. The GAC property becomes obvious.

Fig. 1.

Example 2. In $\mathbb{R}^3$, the nonholonomic integrator defined by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}$$

with $u_1^2 + u_2^2 \leq 1$, is a famous example of GAC control system (at any point of the space).

Example 3. More generally, if M is a connected manifold and if the vector fields $f_1, \cdots, f_m$ satisfy the Hörmander's bracket generating condition

$$\forall x \in M, \text{Lie}\ \{f_1, \cdots, f_m\}(x) = T_x M,$$

then a classical result of Chow says that every two points can be joined by a trajectory of the control system (1). Hence (1) is GAC at any point of the manifold.

Given a GAC control system of the form (1), the purpose of the stabilization problem is to study the possible existence of a feedback $k(\cdot) : M \mapsto \overline{B_m}$ which makes the closed-loop system

$$\dot{x} = f(x, k(x)) = \sum_{i=1}^{m} k_i(x) f_i(x), \tag{2}$$

globally asymptotically stable at the point O (abreviated GAS in the sequel); *i.e.* such that all the trajectories of the closed-loop system converge asymptotically to the point O, and in addition such that the local property of Lyapunov stability is satisfied.

Example 4. As shown in [19, pp. 561-562], the control system given in Example 1 above does not admit a continuous stabilizing feedback. One proof is by noticing that the circles defined in Figure 1 are invariant under the closed-loop system, and that on these circles the only way for a closed-loop system to go to O would be to have $k < 0$ or $k > 0$ along its trajectories. Since by Lyapunov stability, the closed-loop system should be negative (resp. positive) on the right side (resp. on the left side) of the origin, we conclude by connectedness of the circle that a continuous stabilizing feedback should have another equilibrium on each invariant circle!

As we shall see in the sequel, the absence of continuous stabilizing feedbacks for general GAC control systems motivated many authors to define new kinds of stabilizing feedbacks. The main contributions in that direction have been Sussmann [21], Artstein [3], Brockett [4], Sontag [18], Coron [7, 8], Clarke, Ledyaev, Sontag and Subbotin [6], and Ancona and Bressan [2].

2 The Kurzweil Theorem

Given a continuous vector field X on the manifold M which has an equilibrium at $O \in M$, the classical Lyapunov function method asserts that if there exists some function $V : M \to \mathbb{R}$ that satisfies the following properties:

(i) V is smooth on $M \setminus \{0\}$ and continuous at the origin,
(ii) $V \geq 0$ and $V(x) = 0 \Longleftrightarrow x = O$,
(iii) $\forall x \in M \setminus \{O\}, (L_g V)_x < 0$,

then the dynamical system given by

$$\dot{x} = X(x(t)) \tag{3}$$

is GAS at the point O. Such a function is called a *Lyapunov function* for the dynamical system (3). The Kurzweil Theorem establishes the converse result; it asserts that if the dynamical system is GAS (at O) on the manifold M, then it admits a Lyapunov function on M. This result has an important consequence for GAC control systems.

If the control system (1) admits a continuous stabilizing feedback $k : M \to \overline{B_m}$ then the dynamical system given by the closed-loop system (2) is GAS, and as a consequence it admits a Lyapunov function. Thus there exists some function $V : M \to \mathbb{R}$ which satisfies the following properties:

(i) V is smooth on $M \setminus \{0\}$ and continuous at the origin,
(ii) $V \geq 0$ and $V(x) = 0 \Longleftrightarrow x = O$,
(iii) $\forall x \in M \setminus \{O\}$, there exists $u \in \overline{B_m}$ such that $(L_{f(\cdot,u)}V)_x < 0$.

Such a function is called a *smooth control-Lyapunov function* for (1). To summarize, using the Kurzweil Theorem, we prove that if a control system has a continuous stabilizing feedback, then it admits a smooth control Lyapunov function.

Example 5. This result allows us to give another proof of the nonexistence of continuous stabilizing feedback for the control system given in Example 1.

If such a stabilizing feedback existed then the control system would admit a smooth control-Lyapunov. But such a function would attain its maximum over any compact circle of Figure 1, that is, there would be some point where the gradient of V and the tangent space to the circle are orthogonal, which contradicts (iii).

The converse of this result is true (for the type of control systems that we consider, that is, affine in the control and without drift). Artstein proved in [3] that if the control system admits a smooth control-Lyapunov function then there exists a continuous stabilizing feedback (and the feedback can indeed be taken to be smooth outside O). This gives the following characterization of control systems that admit continuous stabilizing feedbacks:

Theorem 1. *The control system (1) has a continuous stabilizing feedback on M if and only if it admits a smooth control-Lyapunov function on M.*

This classical result leads us to the consideration of two major obstructions to the existence of continuous stabilizing feedbacks. We consider these obstructions next.

3 Two obstructions

3.1 A local obstruction: The Brockett condition

Since the obstruction we are talking about is local, let us assume that we work in $\mathbb{R}^n$. If there exists a continuous vector field X defined in some neighborhood $\mathcal{U}$ of O which is locally GAS, then for all ϵ small enough,

$$\exists \delta > 0 \text{ such that } \delta B \subset X(\epsilon B).$$

This result, of topological nature, can indeed be seen as a consequence of Kurzweil Theorem. We refer the reader to the monograph of Sontag [19] for its proof based on Kurzweil and Brouwer theorems. This property enables us to deduce the Brockett's necessary condition.

If there exists a continuous feedback $k : \mathcal{U} \to \mathbb{R}^m$ ($k(O) = 0$) such that the closed-loop system (2) is locally GAS, then the result above applies to the dynamics $\dot{x} = f(x, k(x))$, and then we deduce that for all ϵ small enough,

$$\exists \delta > 0 \text{ such that } \delta B \subset f(\epsilon B, \overline{B_m}).$$

This necessary condition gives us an easy way to verify if a given GAC control system possesses continuous stabilizing feedbacks.

Example 6. The Brockett necessary condition is not satisfied for the nonholonomic integrator defined by (2). It is easy to see that the vector $(0, 0, \epsilon)$ is the set $f(x, \overline{B_2})$ for x in a neighborhood of the origin.

Example 7. More generally, if the tangent vectors $f_1(O), \cdots, f_m(O)$ are independent in $T_O M$ (with $m < n$), then the control system (1) does not satisfy the Brockett necessary condition.

3.2 A Global Obstruction

If (1) has a continuous stabilizing feedback, then it admits a smooth control-Lyapunov function. In fact, this function can be seen as a Morse function on M with a unique (possibly degenerate) critical point. Indeed, by a result of Milnor (see [10]), if a manifold M admits such a Morse function, then it is diffeomorphic to $\mathbb{R}^n$. Therefore, we proved that if a control system has a continuous stabilizing feedback on M, then the manifold M must be diffeomorphic to $\mathbb{R}^n$.

3.3 Consequences

The obstructions above make it impossible to prove the existence of continuous stabilizing feedbacks for general GAC control systems. Actually, they motivate the design of new kinds of stabilizing feedbacks and moreover, as we shall see, it turns out to be essential to consider nonsmooth control-Lyapunov functions.

In order to simplify the statements of the results, we assume from now that the manifold M is the Euclidean space $\mathbb{R}^n$. Of course, all the results remain on smooth manifolds.

4 Semiconcave Control-Lyapunov Functions

Let Ω be an open set in $\mathbb{R}^n$. A function $g : \Omega \to \mathbb{R}$ is said to be semiconcave on Ω provided it is continuous on Ω and for any $x_0 \in \Omega$ there are constants $\rho, C > 0$ such that

$$\frac{1}{2}\left(g(x) + g(y)\right) - g\left(\frac{x+y}{2}\right) \leq C\|x-y\|^2, \quad \forall x, y \in x_0 + \rho B_n. \tag{4}$$

Equivalently, this means that the function g can be written locally as the sum of a concave function and a smooth (quadratic) function:

$$g(x) = [g(x) - 4C\|x\|^2] + 4C\|x\|^2.$$

Observe that any function of class C^2 is semiconcave. Also, any semiconcave function is locally Lipschitz, since both concave functions and smooth functions have that property. Other classical types of semiconcave functions are given by the following examples:

Example 8. If S denotes some closed subset of $\mathbb{R}^n$, then the distance function to the set S is semiconcave on $\mathbb{R}^n \setminus S$.

Example 9. If $\phi_1, \cdots, \phi_p : \Omega \to \mathbb{R}$ is a finite family of smooth, then the function defined by $g(x) := \min\{\phi_1(x), \cdots, \phi_p(x)\}$ is semiconcave on Ω.

We next consider the notion of a *nonsmooth control-Lyapunov function.* A continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a control-Lyapunov function for the control system (1) provided it is a positive definite proper viscosity supersolution of the Hamilton-Jacobi equation

$$\max_{u \in \overline{B_m}} \{-\langle f(x,u), DV(x)\rangle\} - V(x) \geq 0. \tag{5}$$

In [13] we proved the following result:

Theorem 2. *If the control system (1) is GAC, then there exists a control-Lyapunov function $V : M \rightarrow \mathbb{R}$ which is continuous on M and semiconcave on $M \setminus \{O\}$.*

Notice that Sontag introduced in his seminal paper [18] a similar notion of nonsmooth control-Lyapunov function, and proved the equivalence of global asymptotic controllability and the existence of continuous control-Lyapunov function. The converse of this result is true. Furthermore, we notice that whenever a continuous function $V : M \rightarrow \mathbb{R}$ is semiconcave on $M \setminus \{O\}$, it is a control-Lyapunov function if and only if it is positive definite, proper and such that for every $x \in M \setminus \{0\}$ where V is differentiable, we have

$$\min_{u \in \overline{B_m}} \langle \nabla V(x), f(x,u)\rangle \leq -V(x).$$

(Note that since V is locally Lipschitz on $M \setminus \{O\}$, it is differentiable almost everywhere in M.) Let us show, in the following sections, the relevance of the semiconcave property to the construction of stabilizing feedbacks.

5 Stabilizing Feedbacks

Clarke *et al.* [6] proved that any GAC control system can be stabilized by means of a discontinuous stabilizing feedback. Later in [13], we showed that a semiconcave control-Lyapunov function generates this kind of stabilizing feedback in a very simple and natural way. (We refer the reader to the Clarke's article elsewhere in this volume for the presentation of the construction.) In the case of control systems that are affine in the control, we proved that in fact the method that we developed leads to the following result:

Theorem 3. *If the control system (1) is GAC, then there exists $\mathcal{D}$ an open dense set of full measure in $M \setminus \{O\}$ and a feedback $k : M \rightarrow \mathbb{R}^m$ such that*

(i) k is smooth on $\mathcal{D}$; and
(ii) the closed-loop system (2) is GAS in the sense of Carathéodory.

Recall that the closed-loop system (2) is said to be GAS in the sense of Carathéodory if for every $x \in \mathbb{R}^N$ the solutions (which are called Carathódory solutions) of

$$\dot{x}(t) = f(x(t), \alpha(x(t)) \text{ a.e. }, x(0) = x_0,$$

exist, converge to the origin as $t \to \infty$ and satisfy the property of Lyapunov stability. This result has been initially proven by Ancona and Bressan (see [2]) who do not make use of control-Lyapunov functions but paste together different stabilizing open-loops into what they call a patchy feedback. Although Theorem 3 provides an easy way to design stabilizing feedbacks (in the sense of Carathéodory), it does not explicitly detail the behavior of the closed-loop system near the singularities (*i.e.* the points where the feedback is not continuous). From now, our aim is to explain how further regularity of the control-Lyapunov function can improve the above result.

6 A Classification of Singularities in the Plane

In [14], we proved that the nature of the singularities of the stabilizing feedback given by Theorem 3 can be described very precisely, whenever we work on surfaces. In this case, we demonstrated that a part of the singular set of a semiconcave function (that is, the set of points where the function is not differentiable) can be "stratified" by Lipschitz submanifolds of dimension 1 and by isolated points. This permitted us to achieve the following result (and then to answer a conjecture of Bressan):

Theorem 4. *If M is a smooth manifold of dimension 2 and if the control system (1) is GAC, then there exists a Carathéodory stabilizing feedback $k : M \to \mathbb{R}^m$ with singularities as shown in Figure 2.*

Fig. 2. Different types of singularities on surfaces

Unfortunately, the "natural" stratification that appeared in dimension two cannot be achieved in higher dimensions. Hence, if we want to better understand the behavior of the closed-loop system near its singularities, we have to consider nonsmooth control-Lyapunov functions with a stronger property of semiconcavity.

7 Stratified Semiconcave Control-Lyapunov Functions

Let Ω be an open set of $\mathbb{R}^n$ and let $g : \Omega \to \mathbb{R}$ be a semiconcave function. By Rademacher's theorem, we know that g is differentiable almost everywhere in Ω. Let us denote by $\Sigma(g)$ the singular set of g, *i.e.* the set of points of Ω where g is not differentiable. We can also view $\Sigma(g)$ as the set of $x \in \Omega$ such that $\dim(\partial g(x)) \geq 1$; this point of view leads to a natural partition of the singular set. As a matter of fact, following more or less the seminal work of Alberti, Ambrosio and Cannarsa [1], $\Sigma(g)$ can be written as the disjoint union of n sets $\Sigma^k(g)$ (for $k \in \{1, \cdots, N\}$) defined by

$$\Sigma^k(g) := \{x \in \Omega : \dim(\partial g(x)) = k\}.$$

Alberti *et al.* proved that for any $k \in \{1, \cdots, N\}$, the set $\Sigma^k(u)$ is countably $\mathcal{H}^{n-k}$-rectifiable, *i.e.* it is contained (up to a $\mathcal{H}^{n-k}$-negligible set) in a countable union of C^1 hypersurfaces of dimension $N-k$. But each set $\Sigma^k(u)$ is certainly not an exact hypersurface (or submanifold) of Ω. That is why we introduce the concept of stratified semiconcave function.

The semiconcave function $g : \Omega \to \mathbb{R}$ is said to be stratified semiconcave (in Ω) if the following conditions are satisfied:

(1) The set $\Sigma(g)$ is a Whitney stratification[1] such that the strata of dimension $N-k$ are the connected components of $\Sigma^k(g)$;
(2) For every stratum S of $\Sigma(g)$, the set $\overline{S}$ is a smooth submanifold with boundary;
(3) For every stratum S of $\Sigma(g)$, the function g is smooth on $\overline{S}$; and
(4) For every $x \in \Sigma^k(g)$, the set $\partial g(x)$ is a compact convex set of dimension k with exactly $k+1$ extreme points $\zeta_1(x), \cdots, \zeta_{k+1}(x)$. In addition, the maps $\zeta_1(\cdot), \cdots, \zeta_{k+1}(\cdot)$ are smooth on $\Sigma^k(g)$ and for any stratum S of $\Sigma^k(g)$, they can be smoothly extended to $\overline{S}$.

Example 10. Let be given $(h_i)_{i \in I}$ a finite family of affine functions in $\mathbb{R}^N$. If $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$u := \min_{i \in I}\{h_i\},$$

then it is a stratified semiconcave function and moreover it satisfies:

(i) For every $k \in \{1, \cdots, N\}$, the set $\Sigma^k(u)$ is a finite disjoint union of open polyhedra of dimension $N-k$.
(ii) For every $k \in \{1, \cdots, N\}$, the multivalued map $x \mapsto \partial u(x) = \partial^P u(x)$ is constant on each connected component of $\Sigma^k(u)$.

[1] We refer the reader to our paper [15] for the precise definition of a Whitney stratification.

In [15], we showed that every semiconcave function can be approximated locally by a stratified semiconcave function, in such a way that the proximal subdifferentials of both functions are very close. Furthermore, by transversality arguments, we managed to paste together different approximations and to get the following:

Theorem 5. *If the control system (1) is GAC, then there exists a control-Lyapunov function which is stratified semiconcave on* $M \setminus \{O\}$.

Theorem 5 has its own importance, but in fact it appeared to be the key tool in the construction of AGAS feedbacks. We turn to this feedback construction next.

8 AGAS Feedbacks

A smooth dynamical system of the form $\dot{x} = X(x)$ is said to be *almost globally asymptotically stable* at the point O (abreviated AGAS) if the two following properties are satisfied:

1. Almost attractivity: For almost every $x \in M$, the solution of $\dot{x} = X(x)$ starting at x converges to O;
2. Lyapunov stability: For each neighborhood $\mathcal{V}$ of O, there exists some neighborhood $\mathcal{U}$ of O such that if $x \in \mathcal{U}$ then the solution of $\dot{x} = X(x)$ starting at x satisfies $x(t) \in \mathcal{V}, \forall t \geq 0$.

This particular property of stability has been introduced by Rantzer in [11]. The "very" regular control-Lyapunov function given by Theorem 5 permits us to prove the following result:

Theorem 6. *If the control system (1) is GAC, then there exists a continuous feedback* $k : M \to \mathbb{R}^m$ *which is smooth outside the origin, and such that the closed-loop system (2) is AGAS.*

Knowing this theorem, we can wonder if it is possible to detail the nature of the set (of measure zero) of points which are not stabilized by our smooth feedback. This set can be proved to be closed and repulsive in some cases.

9 SRS Feedbacks

The feedback $k : M \to \mathbb{R}^m$ is said to be a *smooth repulsive stabilizing* feedback (called SRS feedback) if the following properties are satisfied:

1. There exists a set $\mathcal{S} \subset M \setminus \{O\}$ which is closed in $M \setminus \{O\}$ and of measure zero;
2. The feedback k is smooth outside the set $\mathcal{S} \cup \{O\}$; and

3. The closed-loop system (2) is GAS in the sense of Carathéodory;
4. for all $t > 0$, the trajectories of the closed-loop system do not belong to the set $\mathcal{S}$.

The first result of existence of SRS feedbacks is a classical fact in sub-Riemannian geometry. We refer the reader to [9] and [20] for its proof and for the definition of fat distribution.

Theorem 7. *If the distribution defined by the control system (1) is fat, then the control system admits a SRS feedback.*

This theorem applies for instance in the case of the nonholonomic integrator. The second result that we want to highlight is concerned with control system in dimension three. In [17] we prove the following:

Theorem 8. *Assume that $n = 3$. If the control system (1) satisfies the Hörmander's condition at the origin,*

$$Lie\,\{f_1, \cdots, f_m\}(0) = \mathbb{R}^3,$$

then there exists a local SRS feedback.

The proof of this result is based on the classification of the singularities given in Section 6; it does not appear to extendable in greater dimension. Actually, we do not know if this result hold in dimension greater than three (or even globally).

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