Generic Aubry sets on surfaces

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Setting

Let M be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $\mathbb{H} : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall (x,p) \in T^*M.$$

(H2) **Uniform convexity:**
For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

Critical value of H

Definition

We call **critical value** of H the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M;\mathbb{R})} \left\{ \max_{x \in M} \left\{ H\big(x, du(x)\big) \right\} \right\}.$$

In other terms, $c[H]$ is the infimum of numbers $c \in \mathbb{R}$ such that there is a C^1 function $u : M \to \mathbb{R}$ satisfying

$$H\big(x, du(x)\big) \leq c \qquad \forall x \in M.$$

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Note that

$$\min_{x \in M} \{H(x,0)\} \leq c[H] \leq \max_{x \in M} \{H(x,0)\}.$$

Critical subsolutions

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Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated with H by Legendre-Fenchel duality, that is

$$L(x, v) := \max_{p \in T_x^* M} \left\{ p \cdot v - H(x, p) \right\} \qquad \forall (x, v) \in TM.$$

Proposition

A Lipschitz function $u : M \to \mathbb{R}$ is a critical subsolution if and only if for every Lipschitz curve $\gamma : [a, b] \to M$,

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot{\gamma}(t))\, ds + c[H]\,(b - a).$$

The weak KAM Theorem

Definition

Given $u : M \to \mathbb{R}$ and $t \geq 0$, $\mathcal{T}_t u : M \to \mathbb{R}$ is defined by

$$\mathcal{T}_t u(x) \quad := \quad \min_{y \in M} \left\{ u(y) + A_t(y,x) \right\},$$

$$\text{with} \qquad A_t(z,z') := \inf \left\{ \int_0^t L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds + c[H]\, t \right\},$$

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Theorem (Fathi, 1997)

There is a critical subsolution $u : M \to \mathbb{R}$ such that

$$\mathcal{T}_t u = u \qquad \forall t \geq 0.$$

It is called a **critical** *or a* **weak KAM solution** *of H.*

More on critical solutions

Given a critical solution $u : M \to \mathbb{R}$, for every $x \in M$, there is a curve

$$\gamma : (-\infty, 0] \to M \quad \text{with} \quad \gamma(0) = x$$

such that, for any $a < b \leq 0$,

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) = \int_a^b L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds + c(b-a).$$

Therefore, any restriction of γ minimizes the action between its end-points. Then, it satisfies the Euler-Lagrange equations.

Projected Aubry set and Aubry set

Definition and Proposition

- The **projected Aubry set** of H defined as

$$\mathcal{A}(H) = \{x \in M \,|\, A_t(x,x) = 0\}.$$

is compact and nonempty.

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- For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at x does not depend on u.
- The **Aubry set** of H defined by

$$\tilde{\mathcal{A}}(H) := \big\{\big(x, du(x)\big) \,|\, x \in \mathcal{A}(H), u \text{ crit. subsol.}\big\} \subset T^*M$$

  is compact, invariant by ϕ_t^H, and is a Lipschitz graph over $\mathcal{A}(H)$.

Questions

- Uniqueness (up to constants) of critical solutions ?
- Regularity of critical solutions ?
- Structure of the Aubry sets ?
- Size of the (quotiented) Aubry set ?
- Dynamics of the Aubry set ?

The Mañé Conjecture

Conjecture (Mañé, 96)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ *of class* C^k *(with* $k \geq 2$*), there is a residual subset (i.e., a countable intersection of open and dense subsets)* $\mathcal{G}$ *of* $C^k(M)$ *such that, for every* $V \in \mathcal{G}$*, the Aubry set of the Hamiltonian* $H_V := H + V$ *is either an equilibrium point or a periodic orbit.*

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Strategy of proof:

- Density result.
- Stability result.

Partial results

Theorem (Figalli-LR, 2011)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2. If there is a critical subsolution sufficiently regular on a neighborhood of $\mathcal{A}(H)$, then for every $\epsilon > 0$, there exists $V \in C^2(M)$,with $\|V\|_{C^2} < \epsilon$ such that the Aubry set of $H + V$ is a hyperbolic periodic orbit.

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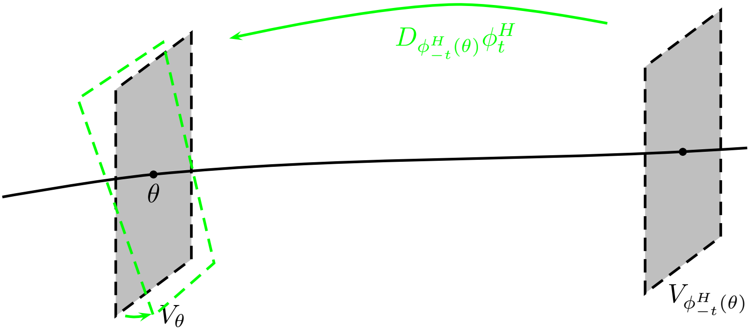
Theorem (Contreras-Figalli-LR, 2013)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2, and assume that $\dim M = 2$. Then there is an open dense set of potentials $\mathcal{V} \subset C^2(M)$ such that, for every $V \in \mathcal{V}$, the Aubry set of $H + V$ is hyperbolic in its energy level.

Key ingredients of the proof

- Green bundles
- Nonsmooth analysis
- Techniques from closing lemmas
- Geometric control theory
- Geometric measure theory

Green bundles I



For every $\theta \in T^*M$ and every $t \in \mathbb{R}$, we define the Lagrangian subspace $G_\theta^t \subset T_\theta T^*M$ by

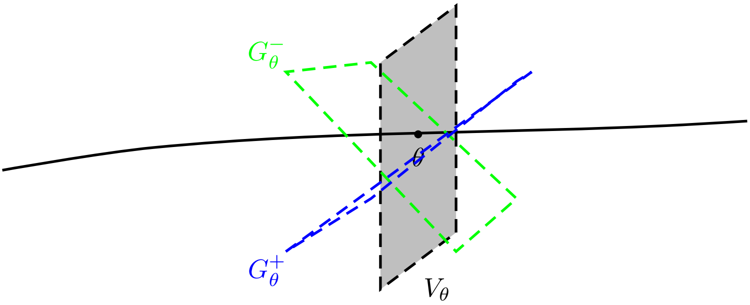
$$G_\theta^t := \left(\phi_t^H\right)_* \left(V_{\phi^H_{-t}(\theta)}\right).$$

Green bundles II

Definition

For every $\theta \in \tilde{\mathcal{A}}(H)$, we define the positive and negative Green bundles at θ as

$$G_\theta^+ := \lim_{t \to +\infty} G_\theta^t \quad \text{and} \quad G_\theta^- := \lim_{t \to -\infty} G_\theta^t$$



A dichotomy

Two cases may appear:

- For every $\theta \in \tilde{\mathcal{A}}(H)$ the Green bundles G_θ^- and G_θ^+ are transverse
- There is $\bar{\theta} \in \tilde{\mathcal{A}}(H)$ such that $G_{\bar{\theta}}^- = G_{\bar{\theta}}^+$

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- There is $\bar{\theta} \in \tilde{\mathcal{A}}(H)$ such that $G_{\bar{\theta}}^- = G_{\bar{\theta}}^+$ $\rightsquigarrow$ further regularity for critical solutions

Further regularity (after Arnaud)

Definition

Let $S \subset \mathbb{R}^k$ be a compact set which has the origin as a cluster point. The **paratingent cone** to S at 0 is the cone defined as

$$C_0(S) := \left\{ \lambda \lim_{i \to \infty} \frac{x_i - y_i}{|x_i - y_i|} \mid \lambda \in \mathbb{R}, x_i \neq y_i \xrightarrow{S} 0 \right\}.$$

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For every $\theta \in \tilde{\mathcal{A}}(H)$, there holds

$$G_\theta^- \preceq C_\theta \left(\tilde{\mathcal{A}}(H) \right) \preceq G_\theta^+.$$

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As a consequence, if $G_\theta^- = G_\theta^+$ for some $\theta \in \tilde{\mathcal{A}}(H)$, then $\tilde{\mathcal{A}}(H)$ is locally contained in the graph of a Lipschitz 1-form which is C^1 at θ.

Closing the Aubry set

Under this additional regularity, given $\epsilon > 0$, we are able to

- a C^2 potential $V : M \to \mathbb{R}$ with $\|V\|_{C^2} < \epsilon$,
- a periodic orbit $\gamma : [0, T] \to M$ $(\gamma(0) = \gamma(T))$,
- a Lipschitz function $v : M \to \mathbb{R}$,

in such a way that the following properties are satisfied:

- $H\big(x, dv(x)\big) + V(x) \leq 0$ for a.e. $x \in M$,
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This shows that the Aubry set of $H + V$ contains a periodic orbit.

Thank you for your attention !!