Geometric control and dynamical systems

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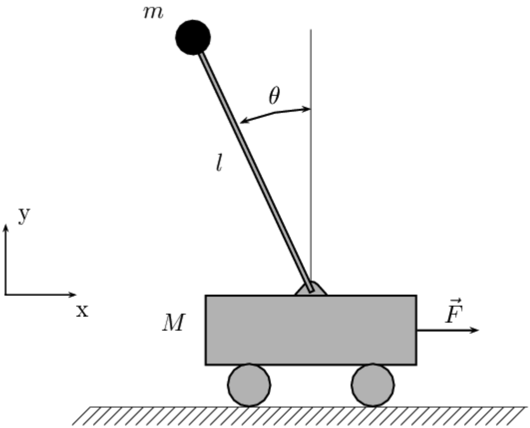
Control of an inverted pendulum



Control systems

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Proposition

Under classical assumptions on the datas, for every $x \in M$ and every measurable control $u : [0, T] \to U$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & a.e.\ t \in [0, T], \\ x(0) = x \end{cases}$$

admits a unique solution

$$x(\cdot) = x(\cdot; x, u) : [0, T] \longmapsto M.$$

Controllability issues

Given two points x_1, x_2 in the state space M and $T > 0$, can we find a control u such that the solution of

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x_1 \end{cases}$$

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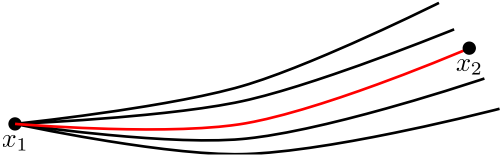
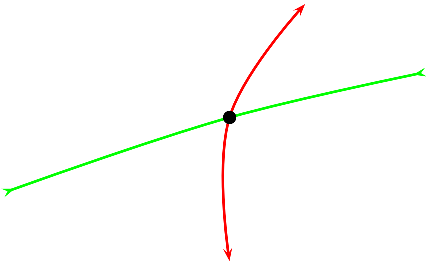


The Kupka-Smale Theorem for vector fields

Theorem (Kupka '63, Smale '63)

Let M be a smooth compact manifold . For C^k $(k \geq 1)$ vector fields on M, the following properties are generic:

1. *All closed orbits are hyperbolic.*
2. *Heteroclinic orbits are transversal, i.e. the intersections of stable and unstable manifolds of closed hyperbolic orbits are transversal.*



Mañé generic Hamiltonians

Let M be a smooth compact manifold and let T^*M be its cotangent bundle equipped with the canonical symplectic form.

Let $H : T^*M \to \mathbb{R}$ be an Hamiltonian of class at least C^2 and X_H be the associated Hamiltonian vector field which reads (in local coordinates)

$$X_H(x,p) = \left(\frac{\partial H}{\partial p}(x,p), -\frac{\partial H}{\partial x}(x,p) \right).$$

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Definition

Given an Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$, a property is called C^k Mañé generic if there is a residual set $\mathcal{G}$ in $C^k(M;\mathbb{R})$ such that the property holds for any $H + V$ with $V \in \mathcal{G}$.

Statement of the result

Theorem (Rifford-Ruggiero '10)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 2$. The following properties are C^k Mañé generic:

1. *Each closed orbit is either hyperbolic or no eigenvalue of the Poincaré transform of any closed orbit is a root of unity.*
2. *Heteroclinic orbits are trasnversal, i.e. the intersections of stable and unstable manifolds of closed hyperbolic orbits are transversal.*

Recall that H is Tonelli if it is superlinear and uniformly convex in the fibers.

The Poincaré map

Let $\bar{\theta} = (\bar{x}, \bar{p})$ be a periodic point for the Hamiltonian flow of positive period $T > 0$. Fix a local section Σ which is transversal to the flow at $\bar{\theta}$ and contained in the energy level of $\bar{\theta}$.

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Then consider the **Poincaré first return map**

$$\begin{array}{rccc} P : & \Sigma & \longrightarrow & \Sigma \\ & \theta & \longmapsto & \phi^H_{\tau(\theta)}(\theta), \end{array}$$

which is a local diffeomorphism and for which $\bar{\theta}$ is a fixed point.

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The Poincaré map is symplectic, i.e. it preserves the restriction of the symplectic form to $T_\theta \Sigma$.

The symplectic group

Let $\mathrm{Sp}(m)$ be the symplectic group in $M_{2m}(\mathbb{R})$ $(m = n - 1)$, that is the smooth submanifold of matrices $X \in M_{2m}(\mathbb{R})$ satisfying

$$X^* \mathbb{J} X = \mathbb{J} \quad \text{where } \mathbb{J} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

Choosing a convenient set of coordinates, the differential of the Poincaré map is the symplectic matrix $X(T)$ where $X : [0, T] \to \mathrm{Sp}(m)$ is solution to the Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) & \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where $A(t)$ has the form

$$A(t) = \begin{pmatrix} 0 & I_m \\ -K(t) & 0 \end{pmatrix} \qquad \forall t \in [0, T].$$

Perturbation of the Poincaré map

Let γ be the projection of the periodic orbit passing through $\bar{\theta}$, we are looking for a potential

$$V : M \longrightarrow \mathbb{R}$$

satisfying the following properties

$$V\big(\gamma(t)\big) = 0, \quad dV\big(\gamma(t)\big) = 0,$$

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$$d^2V\big(\gamma(t)\big) \quad \text{free.}$$

$$\Longrightarrow \quad d^2V\big(\gamma(t)\big) \quad \text{is the control.}$$

A controllability problem on $\mathrm{Sp}(m)$

The Poincaré map at time T associated with the new Hamiltonian

$$H + V$$

is given by $X_u(T)$ where $X_u : [0, T] \to \mathrm{Sp}(m)$ is solution to the control problem

$$\begin{cases} \dot{X}_u(t) = A(t)X_u(t) + \sum_{i\leq j=1}^m u_{ij}(t)\mathcal{E}(ij)X_u(t), \quad \forall t \in [0, T], \\ X(0) = I_{2m}, \end{cases}$$

where the $2m \times 2m$ matrices $\mathcal{E}(ij)$ are defined by

$$\mathcal{E}(ij) := \begin{pmatrix} 0 & 0 \\ E(ij) & 0 \end{pmatrix},$$

with
$$\begin{cases} (E(ii))_{k,l} := \delta_{ik}\delta_{il} \, \forall i = 1, \ldots, m, \\ (E(ij))_{k,l} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \, \forall i < j = 1, \ldots, m. \end{cases}$$

The End-Point mapping

We have to study the mapping

$$\begin{array}{rrcl} E \,:\, C^{\infty}\left([0,\tau];\mathbb{R}^{m(m+1)/2}\right) & \longrightarrow & \mathrm{Sp}(m) \\ u & \longmapsto & X_u(T). \end{array}$$

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If we can show that E is a submersion at $u=0$, we are done.

First-order controllability

Lemma

Assume that there is $\bar{t} \in [0, T]$ such that

$$dim\left(Span\Big\{[E(ij), K(\bar{t})] \mid i, j \in \{1, \ldots, m\}, i < j\Big\}\right) = \frac{m(m-1)}{2}.$$

Then we can reach a neighborhood of $X_0(T)$.

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Then we can reach a neighborhood of $X_0(T)$.

Lemma

The set of matrices $K \in \mathcal{S}(m)$ such that

$$dim\left(Span\Big\{[E(ij), K] \mid i,j \in \{1,\ldots,m\}, i<j\Big\}\right) = \frac{m(m-1)}{2}$$

is open and dense in $\mathcal{S}(m)$.

Thank you for your attention !!