# Nearly round spheres look convex

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(Joint work with A. Figalli and C. Villani)

# The statement

**Theorem (Figalli, R, Villani '09)**

*Let $(M,g)$ be a $C^4$ perturbation of the round sphere $\mathbb{S}^n$. Then all injectivity domains of $M$ are uniformly convex.*

# Injectivity domains

Let $(M, g)$ be a smooth compact Riemannian manifold and $x \in M$ be fixed. We call exponential mapping from $x$, the mapping defined as

$$\begin{array}{rcl} \exp_x : T_xM & \longrightarrow & M \\ v & \longmapsto & \exp_x(v) := \gamma_v(1), \end{array}$$

where $\gamma_v : [0, 1] \to M$ is the unique geodesic starting at $x$ with speed $\dot{\gamma}_v(0) = v$.

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$$\mathcal{I}(x) := \left\{ v \in T_xM \;\middle|\; \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minimizing} \\ \text{geodesic between } x \text{ and } \exp_x(tv) \end{array} \right\}.$$

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It is a bounded open set which is star-shaped w.r.t. $0 \in T_xM$.

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$$t_{cut}(x;v) := \inf\Big\{ t > 0 \,|\, tv \notin \mathcal{I}(x) \Big\}.$$

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The boundary $\mathrm{TCL}(x)$ of $\mathcal{I}(x)$ may be seen as the graph of the function $t_{cut}(x;\cdot)$:

$$\mathrm{TCL}(x) = \Big\{ t_{cut}(x;v)v \,|\, v \in U_xM \Big\}.$$

# Sketch of proof

It is enough to show that there is $K > 0$ such for every $v \in U_x M$, there is a neighborhood $\mathcal{V}$ of $v$ in $U_x M$ together with a Lipschitz function $\tau : \mathcal{V} \to \mathbb{R}$ such that

$$t_{cut}(x; v) = \tau(v), \quad t_{cut}(x; \cdot) \leq \tau, \quad \mathrm{Lip}(\tau) \leq K.$$

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Let $v \in U_x M$ be fixed, set $t_v := t_{cut}(x; v)$.
Three cases appear:

- Case 1: $t_v v$ is a cut speed.
- Case 2: $t_v v$ is not a cut speed (we call it purely focal).
- Case 3: $t_v v$ is a cut speed not far from being purely focal.

# Case I : $t_v v$ cut speed

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By semiconcavity of $z \mapsto d_g(x,z)$, there is a $C^2$ function $g : \mathcal{O} \to \mathbb{R}$ such that (we set $y := \exp(t_v v)$)

$$d_g(x,y) = g(y), d_g(x,\cdot) < g(\cdot) \text{ on } \mathcal{O} \setminus \{y\}, \nabla^g g(y) = \dot{\gamma}_{v'}(t_v).$$

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Set for any $t > 0, w \in U_x M$ close to $t_v, v$,

$$\Psi(t,w) := g(\exp_x(tw)) - t.$$

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The function $\Psi$ is $C^2$ and there holds

$$\Psi(t_v, v) = 0, \quad \frac{\partial \Psi}{\partial t}(t_v, v) = \langle \nabla^g g(y), \dot{\gamma}_v(t_v) \rangle_y - 1 \neq 0$$

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By the Implicit Function Theorem, there is a $C^2$ function $\tau(\cdot)$ such that $\tau(v) = t_v$ and $\Psi(\tau(\cdot), \cdot) = 0$. By construction, $t_{cut}(x; \cdot) \leq \tau$ and $\mathrm{Lip}(\tau)$ is controlled by $\left|\left|\frac{\partial \Psi}{\partial w}\right|\right| / \left|\frac{\partial \Psi}{\partial t}\right|$.

# Case II : $t_v v$ focal speed

We call **distance function to the conjugate locus**, the function $t_{conj}(x;\cdot) : U_xM \longrightarrow (0,\infty)$ defined by

$$t_{conj}(x;w) := \inf\Big\{ t>0 \,|\, \exp_x \text{ is not a submersion at } tw \Big\}.$$

We have

$$t_{cut}(x;\cdot) \leq t_{conj}(x;\cdot).$$

If $t_v v$ is purely focal, then $t_{cut}(x;\cdot) = t_{conj}(x;\cdot)$.

**Theorem (Castelpietra, R '08)**

*The function $t_{conj}(x;\cdot)$ is locally semiconcave on its domain.*

# Case III : $t_v v$ almost focal speed

We can indeed control the Lipschitz constant of $\tau$ when $v$ approaches the set of (purely) focal speeds.

# Perturbation result for the focal domain

Given $x \in M$, we call **nonfocal domain** of $x$ the set

$$\mathrm{NF}(x) := \{ tv \,|\, v \in U_x M, t < t_{conj}(x; v) \}.$$

# Perturbation result for the focal domain

Given $x \in M$, we call **nonfocal domain** of $x$ the set

$$\mathrm{NF}(x) := \{tv \mid v \in U_x M, t < t_{conj}(x; v)\}.$$

**Theorem**

*If $(M, g)$ is $C^4$ close to the round sphere, then all nonfocal domains are uniformly convex.*

Define for every $x \in M$, $h_x : NF(x) \longrightarrow \mathbb{R}$ by

$$h_x(v) = |v|_x^2 - d_g\left(x, \exp_x(v)\right)^2 \qquad \forall v \in NF(x).$$

**Lemma**

- $v \in \overline{\mathcal{I}(x)} \Longrightarrow h_x(v) = 0.$
- $h_x(v) \leq 0 \Longrightarrow v \in \overline{\mathcal{I}(x)}.$

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As a consequence,

$$\overline{\mathcal{I}(x)} = \left\{ v \in \mathrm{NF}(x) \,|\, h_x(v) \leq 0 \right\}.$$

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From now on, the strategy is to show that all the $h_x$ are quasiconvex.

# An easy lemma

**Lemma**

*Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$, the following property holds*

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle > 0.$$

*Then $F$ is quasiconvex.*

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$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

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If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) > \max\{h(0), h(1)\}.$$

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{and} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

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Since $\tau$ is a local maximum, we get a contradiction. □

# The extended Ma-Trudinger-Wang tensor

There holds $\langle \nabla_v h_x, w \rangle = \langle \xi, q - \overline{q} \rangle_x$ and

$$\langle \nabla_v^2 h_x \, w, w \rangle = \frac{2}{3} \int_0^1 (1-s) \, \overline{\mathfrak{S}}_{(y,(1-s)\overline{q}+sq)}(\xi, q - \overline{q}) \, ds$$

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where the **MTW** tensor $\overline{\mathfrak{S}}$ is defined as

$$\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = -\frac{3}{2} \frac{d^2}{ds^2}\Big|_{s=0} \frac{d^2}{dt^2}\Big|_{t=0} d_g^2 \left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$, and by

$$\overline{\mathfrak{S}}_{(x,v)}(\xi,\eta) = -\frac{3}{2} \frac{d^2}{ds^2}\Big|_{s=0} \frac{d^2}{dt^2}\Big|_{t=0} \overline{d_g}^2 \left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathrm{NF}(x)$, and $\xi, \eta \in T_xM$, where $\overline{d_g}$ denotes a extended distance on a neighborhood of $(x, \exp_x(v))$.

On $\left(\mathbb{S}^2, g^0\right)$, the **MTW** tensor is given by

$$\begin{aligned}
&\mathfrak{S}_{(x,v)}\left(\xi, \xi^{\perp}\right) \\
&\quad = 3\left[\frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4 \\
&\qquad + \frac{3}{2}\left[-\frac{6}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,
\end{aligned}$$

with $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := |v|_x, \xi = (\xi_1, \xi_2), \xi^{\perp} = (-\xi_2, \xi_1)$.

### Theorem (Figalli, R, Villani '09)

*If $(M,g)$ is a $C^4$ perturbation of the round sphere, then it satisfies an extended uniform Ma–Trudinger–Wang condition of the form*

$$\forall x \in M,\ \forall v \in \mathrm{NF}(x) \setminus \{0\},$$
$$\mathfrak{S}_{(x,v)}(\xi,\eta) \geq \kappa\big(|\xi|_x^2 + |\Lambda^{-1}\xi|_x^2\big)|\eta|_x^2 - c < \xi,\eta >_x^2,$$

*where $\kappa, c$ are positive constants, and $\Lambda^{-1}$ is a symmetric nonnegative matrix.*

In conclusion,

$$\langle \nabla_v h_x, w \rangle = 0 \quad \Longrightarrow \quad \langle \nabla_v^2 h_x\, w, w \rangle > 0.$$

Which gives the quasiconvexity of the $h_x$'s.

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Which gives the quasiconvexity of the $h_x$'s.

Be careful with the regularity oh $h_x$ !!!

Thank you for your attention !

# Sufficient conditions for **TCP**

**Theorem (Figalli, R, Villani '10)**

*Assume that $(M, g)$ satisfies the following properties:*

- *all the injectivity domains are strictly convex,*
- *the* **MTW** *tensor $\mathfrak{S}$ is $\succ 0$, that is, for every $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_x M \setminus \{0\}$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) > 0.$$

*Then $(M, g)$ satisfies* **TCP**.

# Necessary conditions for **TCP**

**Theorem (Figalli, R, Villani '10)**

*Assume that* $(M, g)$ *satisfies* **(TCP)** *then the following properties hold:*

- *all the injectivity domains are convex,*
- *the cost* $c$ *is regular,*
- *the* **MTW** *tensor* $\mathfrak{S}$ *is* $\succeq 0$.