Regularity of solutions to Hamilton-Jacobi equations for Tonelli Hamiltonians

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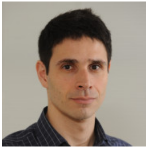
In honour of Francis and Richard

Other students and friends



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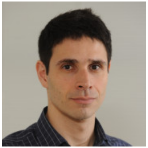

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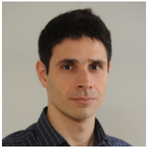

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Sébastien Cambier and Nicolas Deniau

Setting

Let M be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^2 satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall (x,p) \in T^*M.$$

(H2) **Uniform convexity:**
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For sake of simplicity, we may assume that $M = \mathbb{T}^n$, that is that $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H1)-(H2) and is periodic with respect to the x variable.

A first result of regularity

We are concerned with the regularity properties of **viscosity solutions** of the **Hamilton-Jacobi equation**

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Theorem (Rifford 2007)

Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^2 satisfying (H1)-(H2) and $u : M \to \mathbb{R}$ be a viscosity solution of (HJ). Then the function u is semiconcave on M. Moreover, the singular set of u is nowhere dense in M and u is $C^{1,1}_{loc}$ on the open dense set $M \setminus \overline{\Sigma(u)}$.

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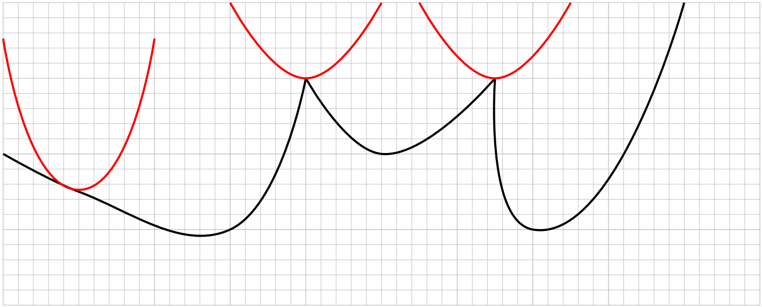
Reminder:

$$\Sigma(u) = \left\{ x \in M \,|\, u \text{ not diff. at } x \right\}$$

Semiconcave functions



A function $u : M \to \mathbb{R}$ is called **semiconcave** if it can be written locally (in charts) as

$$u = g + h,$$

the sum of a smooth function g and a concave function h with a universal upper bound on the C^2-norm of g.

Graph of a semiconcave function

Characterization of viscosity solutions

Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated with H by Legendre-Fenchel duality, that is

$$L(x,v) := \max_{p \in T_x^* M} \left\{ p \cdot v - H(x,p) \right\} \qquad \forall (x,v) \in TM.$$

Proposition

The function $u : M \to \mathbb{R}$ is a viscosity solution of (HJ) iff:

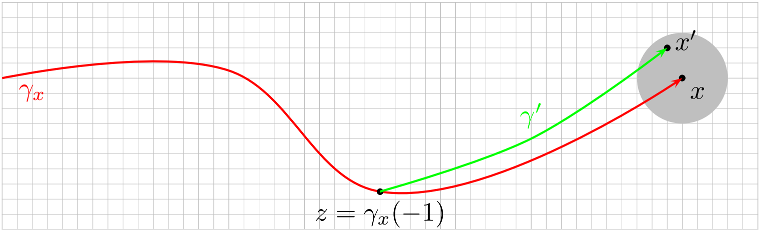
(i) *For every Lipschitz curve $\gamma : [a,b] \to M$, we have*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\, ds.$$

(ii) *$\forall x \in M$, there is a curve $\gamma_x : (-\infty, 0] \to M$ such that*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) = \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\, ds \quad \forall a < b < 0.$$

Semiconcavity of critical solutions



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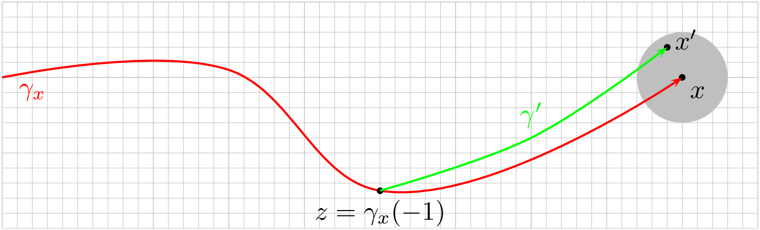

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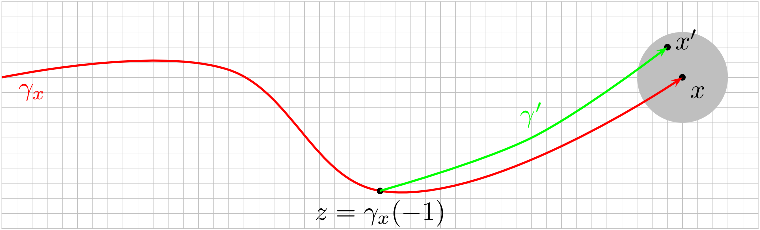

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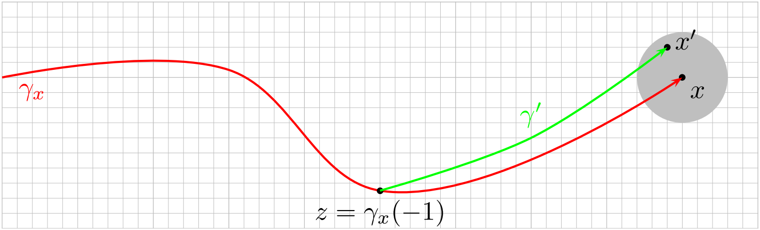

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Thus

$$u(x') \leq u(x) + \int_{-1}^{0} L\big(\gamma'(t), \dot{\gamma}'(t)\big) - L\big(\gamma_x(t), \dot{\gamma}_x(t)\big)\, dt$$

Rest of the proof

We can repeat the previous argument to show that for every $x \in M$, every semi-calibrated curve $\gamma_x : (-\infty, 0] \to M$ and every $t > 0$, the graph of u at $\gamma_x(-t)$ admits a support function of class C^2 from below.

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for every $x \in M$, there is a one-to-one correspondence between the limiting differential of u at x,

$$d^*u(x) := \{\lim du(x_k) \,|\, x_k \to x, u \text{ diff at } x_k\},$$

and the set of semi-calibrated curves.

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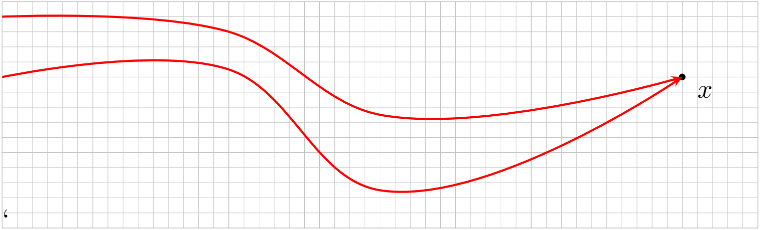

A beautiful result by Arnaud

Theorem (Arnaud '08)

Let $M = \mathbb{T}^2$ *and* $H : T^*M \to \mathbb{R}$ *be an Hamiltonian of class* C^2 *satisfying (H1)-(H2). Let* $u : M \to \mathbb{R}$ *be a solution of (HJ) of class* C^1 *without singularities. Then* u *is* $C^{1,1}$

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We call **singularity** any equilibrium of the characteristic flow of u, that is any $x \in M$ such that

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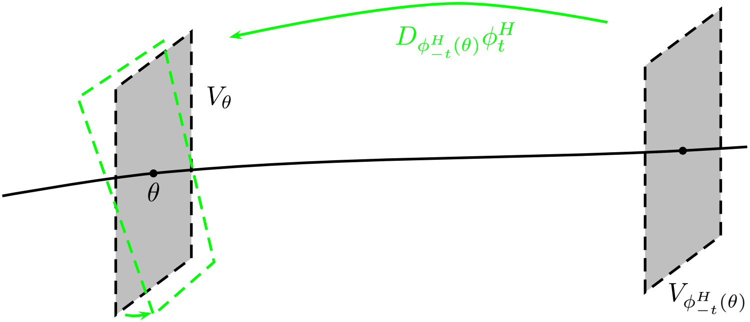
Remark

Note that $u\, C^1 \Rightarrow u\, C^{1,1}$ and the graph of $x \mapsto d_x u$ is a Lipschitz Lagrangian submanifold of T^*M which is invariant by the Hamiltonian flow ϕ_t^H.

Green bundles I



For every $\theta = (x, d_x u) \in T^*M$ and every $t \in \mathbb{R}$, we define the Lagrangian subspace $G_\theta^t \subset T_\theta T^*M$ by ($V_\theta \simeq \{0\} \times \mathbb{R}^2$)

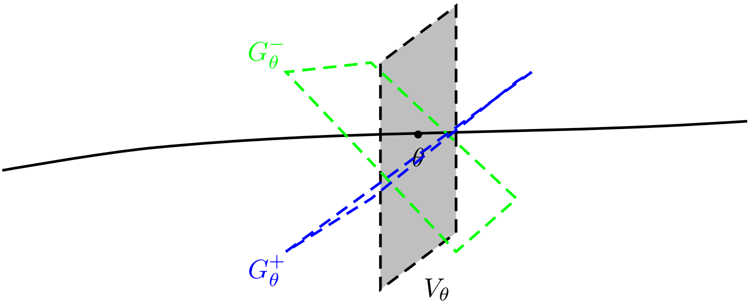
$$G_\theta^t := \left(\phi_t^H\right)_* \left(V_{\phi^H_{-t}(\theta)}\right).$$

Green bundles II

Definition

For every $\theta = (x, d_x u)$, we define the positive and negative Green bundles at θ as

$$G_\theta^+ := \lim_{t \to +\infty} G_\theta^t \quad \text{and} \quad G_\theta^- := \lim_{t \to -\infty} G_\theta^t$$



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- The function $x \in M \mapsto G_{(x,d_x u)}^+$ is upper-semicontinuous.
- The function $x \in M \mapsto G_{(x,d_x u)}^-$ is lower-semicontinuous.
- So, if $G_{(x,d_x u)}^+ = G_{(x,d_x u)}^-$ for some x then both functions are continuous at x.
- For every $x \in M$, we have

$$G_{(x,d_x u)}^- \preceq \mathcal{H}ess_C u(x) \preceq G_{(x,d_x u)}^+,$$

where $\mathcal{H}ess_C u(x)$ denotes the Clarke generalized Hessian of u at x.

Thank you for your attention !!