MTW condition vs. convexity of injectivity domains

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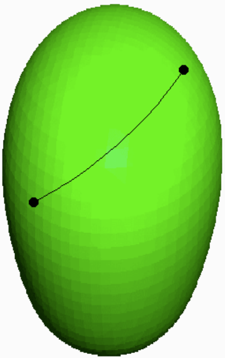
The framework

Let M be a **smooth connected compact manifold**.
For any $x, y \in M$, we define the geodesic distance between x and y, denoted by $d(x, y)$, as the minimum of the lengths of the curves (drawn on M) joining x to y.



Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_x M$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

where $\gamma_{x,v} : [0,1] \longrightarrow M$ is the unique geodesic starting at x with speed v.

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  where $\gamma_{x,v} : [0,1] \rightarrow M$ is the unique geodesic starting at x with speed v.

- We call **injectivity domain** at x, the set

$$\mathcal{I}(x) \subset T_x M$$

  of velocities v for which there exists $t > 1$ such that

  γ_{tv} is the unique minimizing geodesic
  between x and $\exp_x(tv)$.

Properties of injectivity domains

Proposition (Itoh-Tanaka '01)

For every $x \in M$, the set $\mathcal{I}(x)$ is a star-shaped (with respect to $0 \in T_xM$) bounded open set with Lipschitz boundary.

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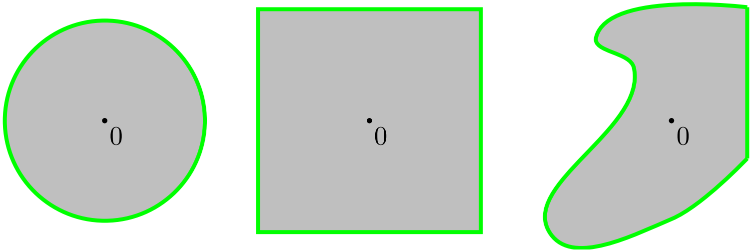

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- $\partial \mathcal{I}(x)$ more than Lipschitz ?
- Is the uniform convexity of injectivity domains stable under perturbation of the metric ?
- What kind of curvature-like condition implies the convexity of injectivity domains ?

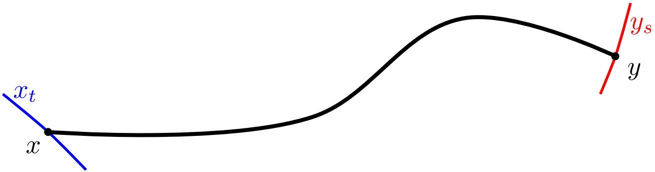
The Ma-Trudinger-Wang tensor

Definition

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0}\, d^2\Big(\exp_x(t\xi), \exp_x(v+s\eta)\Big),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.



$$y := \exp_x(v), \quad x_t := \exp_x(t\xi), \quad y_s := \exp_x(v+s\eta)$$

Two remarks

Remarks

- By an observation due to Loeper, one has

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = K_{\xi,\eta}$$

provided $\xi \perp \eta$ with $|\xi|_x = |\eta|_x = 1$.

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- We can extend $\mathfrak{S}$ up to the boundary of the nonfocal domain $\mathcal{NF}(x) \subset T_xM$ defined as the set of $v \in T_xM$ such that for any $t \in [0,1)$ the mapping

$$w \longmapsto \exp_x(w)$$

is nondegenerate at $w = tv$.

The Villani Conjecture

Definition

We say that (M, g) satisfies the MTW condition if the **MTW** tensor $\mathfrak{S} \succeq 0$, that is if for any $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq 0.$$

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Conjecture

If (M, g) satisfies the MTW condition, then all its injectivity domains are convex.

Examples

- On flat tori, we have $\mathfrak{S} \equiv 0$ and convexity of the $\mathcal{I}(x)$'s
- On $\mathbb{S}^2$ equipped with the unit round metric, we have

$$
\begin{aligned}
&\mathfrak{S}_{(x,v)}\left(\xi, \xi^\perp\right) \\
&\quad = 3\left[\frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4 \\
&\qquad + \frac{3}{2}\left[-\frac{6}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2 \\
&\qquad\qquad \geq 0,
\end{aligned}
$$

with
$x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := |v|, \xi = (\xi_1, \xi_2), \xi^\perp = (-\xi_2, \xi_1)$.

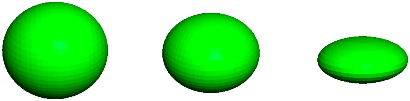
Back to examples..

- Ellipsoids of revolution (oblate case):

$$E_\mu : \quad x^2 + y^2 + \left(\frac{z}{\mu}\right)^2 = 1 \quad \mu \in (0, 1].$$

Theorem (Bonnard-Caillau-R '10)

The injectivity domains of an oblate ellipsoid of revolution are all convex if and only if and only if the ratio between the minor and the major axis is greater or equal to $1/\sqrt{3}\,(\simeq 0.58)$.



Why ?

Lemma

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle > 0.$$

Then F is quasiconvex.

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Proof of the lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$. Define $h : [0,1] \to \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t\in[0,1]} h(t) > \max\{h(0), h(1)\}.$$

There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.
Contradiction !! $\square$

Back to our problem

Given $v_0, v_1 \in \mathcal{I}(x)$ we set for every $t \in [0,1]$,

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with

$$F(v) = \frac{1}{2}|v|_x^2 - \frac{1}{2}d^2\big(x, \exp_x(v)\big) \qquad \forall v \in \mathcal{I}(x).$$

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Therefore

$$F \text{ quasiconvex } \Longrightarrow F(v_t) \leq 0 \Longrightarrow F(v_t) = 0 \qquad \forall t.$$

Thank you for your attention !!