Regularity of weak KAM solutions and Mañé's Conjecture

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Abstract

We provide a crash course in weak KAM theory and review recent results concerning the existence and uniqueness of weak KAM solutions and their link with the so-called Mañé conjecture.

1 Introduction

In the present paper, (M, g) will be a smooth connected compact Riemannian manifold without boundary of dimension $n \geq 2$, and $H : T^*M \to \mathbb{R}$ a C^k *Tonelli Hamiltonian* (with $k \geq 2$), that is, a Hamiltonian of class C^k satisfying the two following properties ($\|\cdot\|$ denotes the dual norm on T^*M):

(H1) *Superlinear growth:* For every $K \geq 0$, there is a finite constant $C^*(K)$ such that

$$H(x,p) \geq K\|p\|_x + C^*(K) \qquad \forall\, (x,p) \in T^*M.$$

(H2) *Uniform convexity:* For every $(x,p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

The Mañé critical value of H can be defined as follows.

Definition 1.1. *We call critical value of H, denoted by $c[H]$, the infimum of the values $c \in \mathbb{R}$ for which there exists a function $u : M \to \mathbb{R}$ of class C^1 satisfying*

$$H(x, du(x)) \leq c \qquad \forall\, x \in M.$$

Remark 1.2. *We can check easily that $c[H]$ satisfies the following inequalities*

$$\min_{x\in M} \{H(x,0)\} \leq c[H] \leq \max_{x\in M} \{H(x,0)\}.$$

The study of solutions of the critical Hamilton-Jacobi equation,

$$H\big(x, du(x)\big) = c[H] \qquad \forall x \in M, \tag{1.1}$$

is the core of Fathi's weak KAM theory developed in [9, 10, 11, 12, 13]. The aim of the present paper is to recall briefly the construction of Fathi's weak KAM solutions and to address uniqueness and regularity issues for the critical Hamilton-Jacobi equation.

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2 Critical subsolutions

A priori, the infimum in Definition 1.1 is not necessarily attained. For this reason, we introduce the notion of critical subsolutions. We recall that by Rademacher's theorem, Lipschitz functions are differentiable almost everywhere.

Definition 2.1. *A function $u : M \to \mathbb{R}$ is called a critical subsolution for H if it is Lipschitz and satisfies*

$$H\big(x, du(x)\big) \leq c[H] \qquad a.e.\ x \in M. \tag{2.1}$$

Let us denote by $\big(C^0(M;\mathbb{R}), \|\cdot\|_\infty\big)$ the Banach space of continuous functions on M equipped with the supremum norm.

Proposition 2.2. *The set $\mathcal{SS}[H]$ of critical subsolutions is a nonempty, compact and convex subset of $C^0(M;\mathbb{R})$.*

Proof. Pick a sequence of C^1 functions $\{u_k\}$ associated with a sequence of real numbers $\{c_k\}$ converging to $c[H]$ such that

$$H\big(x, du_k(x)\big) \leq c_k \qquad \forall x \in M,\, \forall k.$$

Thanks to the superlinear growth hypothesis (H1), the sequence $\{u_k\}$ is uniformly Lipschitz. Then by compactness, we may assume that it converges uniformly to some Lipschitz function $u : M \to \mathbb{R}$. The fact that u is a critical subsolution follows easily from the following lemma whose proof is left to the reader.

Lemma 2.3. *Let $\{u_k\}$ be a sequence of C^1 functions on M which converges uniformly to some Lipschitz function $u : M \to \mathbb{R}$. Assume that u is differentiable at $x \in M$. Then there is a sequence $x_k \to x$ such that $du_k(x_k) \to du(x)$.*

Thus we proved that $\mathcal{SS}[H]$ is nonempty. By (H1), any critical subsolution is Lipschitz on M (with universal Lipschitz constant). From the Arzela-Ascoli Theorem, the compactness of $\mathcal{SS}[H]$ follows easily. Finally, the convexity of $\mathcal{SS}[H]$ is straightforward from the convexity of H in the p variable (H2). □

The Lagrangian $L : TM \to \mathbb{R}$ associated with H by Legendre-Fenchel duality is defined by

$$L(x,v) := \max_{p \in T_x^*M} \Big\{ \langle p, v\rangle - H(x,p) \Big\}.$$

Thanks to (H1)-(H2), it can be shown (see [4, 13]) that L is a C^k Tonelli Lagrangian, that is it is C^k and satisfies the two following properties ($\|\cdot\|$ denotes the norm on TM):

(L1) *Superlinear growth:* For every $K \geq 0$, there is a finite constant $C(K)$ such that

$$L(x,v) \geq K\|v\|_x + C(K) \qquad \forall\,(x,v) \in TM.$$

(L2) *Uniform convexity:* For every $(x,v) \in TM$, $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite.

Note that the Fenchel inequality

$$\langle p, v\rangle \leq L(x,v) + H(x,p) \tag{2.2}$$

holds for any $x \in M$ and $v \in T_xM, p \in T_x^*M$ with equality if and only if (in local coordinates)

$$v = \frac{\partial H}{\partial p}(x,p) \Leftrightarrow p = \frac{\partial L}{\partial v}(x,v). \tag{2.3}$$

The Legendre-Fenchel duality allows us to characterize the critical subsolutions in a variational way.

Proposition 2.4. *A function $u : M \to \mathbb{R}$ is a critical subsolution (for H) if and only if*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds + c[H]\,(b-a), \tag{2.4}$$

for any Lipschitz curve $\gamma : [a,b] \to M$.

Proof. Let $u : M \to \mathbb{R}$ be a critical subsolution. If u is of class C^1, then we can write for any Lipschitz curve $\gamma : [a,b] \to M$ (remember (2.2)),

$$\begin{aligned} u\big(\gamma(b)\big) - u\big(\gamma(a)\big) &= \int_a^b \langle du\big(\gamma(s)\big), \dot\gamma(s)\rangle\, ds \\ &\leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds + \int_a^b H\big(\gamma(s), du(\gamma(s))\big)\, ds \\ &\leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, dt + c[H]\,(b-a). \end{aligned}$$

If u is not C^1, then by convolution we can approximate it by a smooth function $u^\epsilon : M \to \mathbb{R}$ satisfying

$$H\big(x, du(x)\big) \leq c[H] + \epsilon \qquad \forall x \in M.$$

Then we can apply the previous argument to u^ϵ and let ϵ tend to 0.

Assume now that a function $u : M \to \mathbb{R}$ satisfies (2.4) for any Lipschitz curve $\gamma : [a,b] \to M$. Let $x, y \in M$ be fixed. Denote by d the Riemannian distance associated with g on $M \times M$ and let $\alpha : [0, d(x,y)] \to M$ be a unit speed geodesic joining x to y in time $d(x,y)$. By assumption on u, one has

$$u(y) - u(x) \leq \int_0^{d(x,y)} L\big(\alpha(s), \dot\alpha(s)\big)\, ds + c[H] d(x,y).$$

Setting

$$S(1) := \max\Big\{ L(z,w) \mid (z,w) \in TM, \|w\|_z \leq 1 \Big\},$$

we infer that

$$u(y) - u(x) \leq (S(1) + c[H])\, d(x,y) \qquad \forall x, y \in M,$$

which shows that u is Lipschitz on M. Let $x \in M$ be such that u is differentiable at x and let $(x(\cdot), p(\cdot)) : [0, \epsilon] \to T^*M$ be a local solution of the Hamiltonian system associated with H, which reads in local coordinates,

$$\begin{cases} \dot x(t) &= \frac{\partial H}{\partial p}\big(x(t), p(t)\big) \\ \dot p(t) &= -\frac{\partial H}{\partial x}\big(x(t), p(t)\big), \end{cases}$$

such that $(x(0), p(0)) = (x, du(x))$. By assumption, we have for every $t \in (0, \epsilon]$,

$$\frac{u\big(x(t)\big) - u\big(x(0)\big)}{t} \leq \frac{1}{t} \int_0^t L\big(x(s), \dot x(s)\big)\, ds + c[H].$$

Since u is differentiable at $x = x(0)$ and $\dot x(0)$ satisfies (remember (2.3))

$$H\big(x, du(x)\big) = \langle du(x), \dot x(0)\rangle - L\big(x, \dot x(0)\big),$$

letting t tends to 0 yields $H(x, du(x)) \leq c[H]$. $\square$

3 The Fathi weak KAM Theorem

Following [13], the Lax-Oleinik semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ associated with L is defined as

$$\{\mathcal{T}_t\}_{t\geq 0} : C^0(M;\mathbb{R}) \longrightarrow C^0(M;\mathbb{R})$$

where for every $t \geq 0$ and for any $u \in C^0(M;\mathbb{R})$, $\mathcal{T}_t u = \mathcal{T}_t(u)$ is given by

$$\mathcal{T}_t u(x) := \inf \left\{ u\big(\gamma(-t)\big) + \int_{-t}^{0} L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds \right\} \qquad \forall x \in M, \tag{3.1}$$

where the infimum is taken among the Lipschitz curves $\gamma : [-t, 0] \to M$ such that $\gamma(0) = x$. If for every $t > 0$, we define the function $h_t : M \times M \to \mathbb{R}$ by

$$h_t(x, y) := \inf_{\gamma} \left\{ \int_0^t L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds \right\} \qquad \forall x, y \in M, \tag{3.2}$$

where the infimum is taken on the set of Lipschitz curves $\gamma : [0, t] \to M$ which satisfy $\gamma(0) = x$ and $\gamma(t) = y$, then $\mathcal{T}_t u$ can also be written as (for $t > 0$)

$$\mathcal{T}_t u(x) := \inf_{z\in M} \left\{ u(z) + h_t(z, x) \right\} \qquad \forall x \in M. \tag{3.3}$$

In Appendix A, we recall that, under the present hypotheses, the infimum in the definition of h_t is always attained and that each function h_t is indeed Lipschitz on $M \times M$. This shows that $\mathcal{T}_t$ is well-defined for all $t \geq 0$ and that the infimum in (3.1) is always attained. By the way, we note that by Proposition 2.2, if $u : M \to \mathbb{R}$ is a critical subsolution then there holds for any $t > 0$,

$$u(x) - u(z) \leq h_t(z, x) + c[H]\, t \qquad \forall x, z \in M. \tag{3.4}$$

In fact, $\{\mathcal{T}_t\}_{t\geq 0}$ enjoys the properties of a continuous nonexpansive semigroup.

Proposition 3.1. *The following properties hold:*

(i) $\mathcal{T}_0 = Id$ *and* $\mathcal{T}_{t+t'} = \mathcal{T}_t \circ \mathcal{T}_{t'}$, *for any* $t, t' \geq 0$.

(ii) For every $t \geq 0$, $\|\mathcal{T}_t u - \mathcal{T}_t v\|_\infty \leq \|u - v\|_\infty$ *for any* $u, v \in C^0(M;\mathbb{R})$.

(iii) For every $u \in C^0(M;\mathbb{R})$, *the mapping* $t \in [0, \infty) \mapsto \mathcal{T}_t u$ *is continuous.*

(iv) The set $\mathcal{SS}[H]$ *is invariant with respect to* $\{\mathcal{T}_t\}$.

Proof. Let $t, t' > 0$ and $u \in C^0(M;\mathbb{R})$ be fixed. We have for every $x \in M$,

$$\begin{aligned}
\mathcal{T}_{t+t'} u(x) = \inf_{y\in M} \left\{ u(y) + h_{t+t'}(y, x) \right\} &= \inf_{y\in M} \left\{ u(y) + \inf_{z\in M} \{ h_{t'}(y, z) + h_t(z, x) \} \right\} \\
&= \inf_{y,z\in M} \left\{ u(y) + h_{t'}(y, z) + h_t(z, x) \right\} \\
&= \inf_{z\in M} \left\{ \inf_{y\in M} \{ u(y) + h_{t'}(y, z) \} + h_t(z, x) \right\} \\
&= \inf_{z\in M} \left\{ \mathcal{T}_{t'} u(z) + h_t(z, x) \right\} = \mathcal{T}_t \left[\mathcal{T}_{t'} u \right](x).
\end{aligned}$$

This proves (i). The proof of (ii) is easy. Let $u, v \in C^0(M;\mathbb{R})$, $x \in M$ and $t > 0$ be fixed. There is $z \in M$ such that $\mathcal{T}_t u(x) = u(z) + h_t(z, x)$. Moreover we have necessarily $\mathcal{T}_t v(x) \leq v(z) + h_t(z, x)$. We deduce that

$$\mathcal{T}_t v(x) - \mathcal{T}_t u(x) \leq v(z) - u(z) \leq \|v - u\|_\infty.$$

Exchanging the roles of u and v yields the result. To prove (iii) we first assume that $u : M \to \mathbb{R}$ is K-Lipschitz on M. By (L1), there is $C(K) \in \mathbb{R}$ such that

$$L(x,v) \geq K\|v\|_x + C(K) \qquad \forall\,(x,v) \in TM.$$

Then for every $t > 0$ and any Lipschitz curve $\gamma : [-t, 0] \to M$, we have

$$\int_{-t}^{0} L\big(\gamma(s),\dot\gamma(s)\big)\,ds \geq K \int_{-t}^{0} \big\|\dot\gamma(s)\big\|_{\gamma(s)}\,ds + C(K)t \geq Kd\big(\gamma(-t),\gamma(0)\big) + C(K)t,$$

which implies

$$u\big(\gamma(-t)\big) + \int_{-t}^{0} L\big(\gamma(s),\dot\gamma(s)\big)\,ds \geq u\big(\gamma(0)\big) + C(K)t.$$

We deduce that $\mathcal{T}_t u \geq u + C(K)t$. On the other hand, we have easily for any $t > 0$,

$$\mathcal{T}_t u(x) \leq u(x) + h_t(x,x) \leq tL(x,0) \qquad \forall x \in M.$$

We infer that $\mathcal{T}_t u \to u$ as $t \to 0$. Note that for any $t, t' \geq 0$, by (i)-(ii) above, one has

$$\|\mathcal{T}_{t'} u - \mathcal{T}_t u\|_\infty \leq \big\|\mathcal{T}_{|t'-t|} u - u\big\|_\infty .$$

This shows that $t \in [0,\infty) \mapsto \mathcal{T}_t u$ is continuous. If u is merely continuous, then for every $\epsilon > 0$, there is a Lipschitz function $v : M \to \mathbb{R}$ such that $\|v - u\| \leq \epsilon$. By the above argument together with (ii), we deduce easily that $\|\mathcal{T}_t u - u\|_\infty \leq 3\epsilon$ for $t \geq 0$ small enough. We deduce easily that $t \in [0,\infty) \mapsto \mathcal{T}_t u$ is continuous.

It remains to prove (iv). We need to show that given u in $\mathcal{SS}[H]$, $\mathcal{T}_t u$ satisfies the characterization given in Proposition 2.4 for any $t > 0$. Let $t > 0$ be fixed, we observe that it is sufficient to prove (2.4) for any Lipschitz curve $\gamma : [a,b] \to M$ with $b - a < t/2$. Let γ be such a curve. There is $z \in M$ such that

$$\mathcal{T}_t u\big(\gamma(a)\big) = u(z) + h_t\big(z,\gamma(a)\big).$$

Then we have for any $y \in M$,

$$\begin{aligned}
\mathcal{T}_t u\big(\gamma(b)\big) - \mathcal{T}_t u\big(\gamma(a)\big) \;&\leq\; u(y) + h_t\big(y,\gamma(b)\big) - u(z) - h_t\big(z,\gamma(a)\big)\\
&\leq\; u(y) - u(z) + h_{t-(b-a)}\big(y,\gamma(a)\big) + h_{b-a}\big(\gamma(a),\gamma(b)\big) - h_t\big(z,\gamma(a)\big).
\end{aligned}$$

Let $\alpha : [0,t] \to M$ be a Lipschitz curve with $\alpha(0) = z, \alpha(t) = \gamma(a)$ and $h_t(z,\gamma(a)) = \int_0^t L\big(\alpha(s),\dot\alpha(s)\big)\,ds$. Applying the previous inequality with $y = \alpha(b-a)$ and using that

$$u(y) - u(z) \leq h_{b-a}(z,y) \quad \text{ and } \quad h_t\big(z,\gamma(a)\big) = h_{b-a}(z,y) + h_{t-(b-a)}\big(y,\gamma(a)\big),$$

yields the result. □

As shown by Fathi [13], the existence of weak KAM solutions can be obtained as a consequence of a fixed point theorem for continuous semigroups acting on compact convex sets (see Theorem B.1).

Theorem 3.2. *There exists a critical subsolution $u : M \to \mathbb{R}$ such that*

$$\mathcal{T}_t u = u - c[H]\,t \qquad \forall t \geq 0. \tag{3.5}$$

Such a function is called a weak KAM solution or a critical solution (for H).

Proof. Let E be the normed space obtained as the quotient of $C^0(M;\mathbb{R})$ by constant functions equipped with the norm

$$\|[u]\| = \inf_{a\in\mathbb{R}} \{u + a\} \qquad \forall [u] \in E.$$

Since $\mathcal{T}_t(u + a) = \mathcal{T}_t u + a$ for any $a \in \mathbb{R}$, the maps $\mathcal{T}_t$ pass to the quotient. Then the family $\{\mathcal{T}_t\}_{t\geq 0}$ defines a continuous nonexpansive semigroup which preserves the compact set

$$\mathcal{K} := \Big\{ [u] \,|\, u \in \mathcal{SS}(H) \Big\}.$$

Thus by Theorem B.1, there is a family $\{c_t\}_{t\geq 0}$ of real numbers and $u \in \mathcal{SS}(H)$ such that

$$\mathcal{T}_t u = u + c_t \qquad \forall t \geq 0.$$

By Proposition 3.1 (i) and (iii), the mapping $t \mapsto c_t$ satisfies the semigroup property and is continuous. Then there is $c \in \mathbb{R}$ such that $c_t = tc$ for all $t \geq 0$. We need to show that $c = -c[H]$. On the one hand, since u is a critical subsolution, remembering (3.4) we check easily that

$$\mathcal{T}_t u \geq u - c[H]\, t \qquad \forall t \geq 0.$$

Which shows that $c \geq -c[H]$. On the other hand, $\mathcal{T}_t u \geq u + c_t (\forall t \geq 0)$ implies that for any Lipschitz curve $\gamma : [a, b] \to M$, we have

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq h_{b-a}\big(\gamma(a), \gamma(b)\big) - ct \leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds - ct.$$

Proceeding as in the proof of Proposition 2.4, this shows that $H(x, du(x)) \leq -c$ almost everywhere. Regularizing u by convolution, we deduce that for every $\epsilon > 0$ there is a smooth function $u^\epsilon : M \to \mathbb{R}$ such that $H(x, du^\epsilon(x)) \leq -c + \epsilon$ for all $x \in M$. By definition 1.1, this shows that $-c \geq c[H] \Leftrightarrow c \leq -c[H]$. □

Remark 3.3. *The existence of weak KAM solutions can also be shown by proving that for any subsolution $u \in \mathcal{SS}(H)$, the function $\mathcal{T}_t u + c[H]\, t$ converges uniformly as t tends to ∞ to a weak KAM solution, see [13].*

Remark 3.4. *It can be shown easily that $u : M \to \mathbb{R}$ is a weak KAM solution (for H) if and only if it is a viscosity of the Hamilton-Jacobi equation (1.1). We refer the reader to [13, 24] for further details on weak KAM solutions from the viscosity viewpoint.*

Remark 3.5. *Roughly speaking, in the classical KAM theory, weak KAM solutions are smooth and the graphs of its differentials are indeed invariant tori, see [3].*

The following result provides several characterizations of weak KAM solutions.

Proposition 3.6. *Let $u \in C^0(M;\mathbb{R})$, the following properties are equivalent:*

(i) *u is a weak KAM solution.*

(ii) *$\mathcal{T}_t u = u - c[H]\, t$, for all $t \geq 0$.*

(iii) *$u \in \mathcal{SS}(H)$ and for every $x \in M$, there exists a Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ such that*

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\,(b - a) \qquad \forall a < b \leq 0. \tag{3.6}$$

(iv) *$u \in \mathcal{SS}(H)$ and for every smooth function $\phi : M \to \mathbb{R}$ with $\phi \leq u$ and all $x \in M$,*

$$\phi(x) = u(x) \quad \Longrightarrow \quad H\big(x, d\phi(x)\big) \geq c[H]. \tag{3.7}$$

Moreover, the curve appearing in (iii) is necessary of class C^2 and solution to the Euler-Lagrange equation.

Remark 3.7. *Actually, assertion (iv) states that u is a viscosity solution of (1.1). It is also equivalent to the following:*

(iv)' For every smooth function $\phi : M \to \mathbb{R}$ with $\phi \leq u$ and all $x \in M$, $\phi(x) = u(x) \quad \Rightarrow H\big(x, d\phi(x)\big) = c[H]$.

Proof. (i) $\Rightarrow$ *(ii)* is obvious. Let us show that (ii) $\Rightarrow$ (iii). Let $u \in C^0(M;\mathbb{R})$ satisfying (3.5) be fixed. Given a Lipschitz curve $\gamma : [a,b] \to M$ we have

$$u\big(\gamma(b)\big) - c[H]\,(b-a) = \mathcal{T}_{b-a}u\big(\gamma(b)\big) \leq u\big(\gamma(a)\big) + \int_a^b L\big(\gamma(s),\dot\gamma(s)\big)\,ds,$$

which by Proposition 2.4 shows that u is a critical subsolution. Let $x \in M$ be fixed, let us construct γ_x. Since the infimum in the definition of $h_t(x,y)$ is always attained, for every positive integrer k, there is a curve $\gamma_k : [-k,0] \to M$ with $\gamma(0) = x$ such that

$$\mathcal{T}_k u(x) = u\big(\gamma_k(-k)\big) + h_k\big(\gamma_k(-k),x\big) = u\big(\gamma_k(-k)\big) + \int_{-k}^0 L\big(\gamma_k(s),\dot\gamma_k(s)\big)\,ds.$$

Since $\mathcal{T}_k u(x) = u(x) - c[H]\,k$ and $u \in \mathcal{SS}(H)$, we deduce that

$$u\big(\gamma_k(b)\big) - u\big(\gamma_k(a)\big) = \int_a^b L\big(\gamma_k(s),\dot\gamma_k(s)\big)\,ds + c[H]\,(b-a) \qquad \forall a < b \in [-k,0].$$

By Theorem A.1, the curves γ_k are uniformly Lipschitz. Then we conclude easily by the Arzela-Ascoli Theorem. (iii) $\Rightarrow$ (i) is easy. Let us show that (iii) $\Leftrightarrow$ (iv). Let $\phi : M \to \mathbb{R}$ be a smooth function such that $\phi \leq u$ and let $x \in M$ with $u(x) = \phi(x)$ be fixed. By (iii), there is a Lipschitz curve $\gamma_x : (-\infty,0] \to M$ with $\gamma(0) = x$ satisfying (3.6). Then

$$\frac{\phi(x) - \phi\big(\gamma_x(-t)\big)}{t} \geq \frac{1}{t}\int_{-t}^0 L\big(\gamma_x(s),\dot\gamma_x(s)\big)\,ds + c[H] \qquad \forall t > 0.$$

Taking the limit as $t \to 0$ and using Legendre-Fenchel duality, yields $H(x, d\phi(x)) \geq c[H]$. This shows that (iii) $\Rightarrow$ (iv). Assume now that (iv) holds and prove (iii) for some $x \in M$. It is sufficient to show how to construct a curve γ_x satisfying (3.6) on a small interval $[-\epsilon,0]$. Taking a local chart around x if necessary, we may assume that we work in $\mathbb{R}^n$. Let $\alpha \in (0,1]$ and $N \in \mathbb{N}^*$ be fixed and $u_\alpha : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$u_\alpha(x') := \inf_{y\in\mathbb{R}^n}\left\{u(y) + \frac{1}{2\alpha^2}|y-x'|^2\right\} \qquad \forall x' \in \mathbb{R}^n.$$

Note that since u is Lipschitz, for any $\alpha \in (0,1]$ small enough, for every x' close to x the infimum in the above formula is attained for some y close to x'. Let $y_\alpha(x) \in \mathbb{R}^n$ be such that

$$u_\alpha(x) = u\big(y_\alpha(x)\big) + \frac{1}{2\alpha^2}\big|y_\alpha(x) - x\big|^2$$

and set

$$p_\alpha(x) := \frac{x - y_\alpha(x)}{\alpha^2}.$$

Note that by construction

$$u(y) \geq u\big(y_\alpha(x)\big) + \langle p_\alpha(x), y - y_\alpha(x)\rangle - \frac{1}{2\alpha^2}\big|y - y_\alpha(x)\big|^2 \qquad \forall y \in \mathbb{R}^n.$$

This means that the right hand side in the above inequality is a smoooth support function from below for u at $y_\alpha(x)$. Then by (iv), we can pick $v_\alpha(x) \in \mathbb{R}^n$ such that

$$\langle p_\alpha(x), v_\alpha(x)\rangle - L\big(y_\alpha(x), v_\alpha(x)\big) = H\big(y_\alpha(x), p_\alpha(x)\big) \geq c[H].$$

Then setting $x_1 := x - \frac{1}{N} v_\alpha(x)$ implies

$$\begin{aligned} u_\alpha(x_1) - u_\alpha(x) &\leq u\big(y_\alpha(x)\big) + \frac{1}{2\alpha^2}\big|y_\alpha(x) - x_1\big|^2 - u\big(y_\alpha(x)\big) - \frac{1}{2\alpha^2}\big|y_\alpha(x) - x\big|^2 \\ &\leq \langle p_\alpha(x), x_1 - x\rangle + \frac{1}{2\alpha^2}\big|x_1 - x\big|^2 \\ &\leq -\frac{1}{N} L\big(y_\alpha(x), v_\alpha(x)\big) - \frac{1}{N} c[H] + \frac{1}{2\alpha^2 N^2}\big|v_\alpha(x)\big|^2, \end{aligned}$$

which can be written as

$$u_\alpha(x_1) \leq u_\alpha(x) - \frac{1}{N} L\big(y_\alpha(x), v_\alpha(x)\big) - \frac{1}{N} c[H] + \frac{1}{2\alpha^2 N^2}\big|v_\alpha(x)\big|^2.$$

Repeating this construction yields finite sequences

$$\big\{x_0 = x, x_1, \ldots, x_N\big\}, \big\{y_\alpha(x_0), \ldots, y_\alpha(x_N)\big\}, \big\{p_\alpha(x_0), \ldots, p_\alpha(x_N)\big\}, \big\{v_\alpha(x_0), \ldots, v_\alpha(x_N)\big\}$$

such that for all $i = 1, \ldots, N$,

$$u_\alpha(x_i) \leq u_\alpha(x) - \sum_{j=0}^{i-1} \left[\frac{1}{N} L\big(y_\alpha(x_j), v_\alpha(x_j)\big)\right] - \frac{i}{N} c[H] + \sum_{j=0}^{i-1} \left[\frac{1}{2\alpha^2 N^2}\big|v_\alpha(x_i)\big|^2\right]. \tag{3.8}$$

Since u is Lipschitz, the $p_\alpha(x_i), v_\alpha(x_i)$'s are bounded and the paths $x_{\delta,N}(\cdot) : [-1, 0] \to \mathbb{R}^n$ defined by

$$x_{\delta,N}(t) = x_i - (t - i/N) v_\alpha(x_i) \qquad \forall t \in (-i/N, -(i-1)/N], \forall i = 1, \ldots, N,$$

are uniformly Lipschitz and satisfies $x_{\delta,N}(-i/N) = x_i$ for any $i = 0, \ldots, N$. Then by (3.8), letting $\delta \downarrow 0$ and $N \uparrow \infty$, we infer that there is a Lipschitz curve $x(\cdot) : [-1, 0] \to \mathbb{R}^n$ with $x(0) = x$ and satisfying

$$u\big(x(-t)\big) \leq u(x) - \int_{-t}^0 L\big(x(s), \dot{x}(s)\big)\, ds - c[H]\, t \qquad \forall t \in [0, 1].$$

We conclude easily by the fact that $u \in \mathcal{SS}(H)$. □

Remark 3.8. *The proof of (ii) ⇒ (iii) shows that if some continuous function $v : M \to \mathbb{R}$ satisfies*

$$\mathcal{T}_t v = v - c\, t \qquad \forall t \geq 0,$$

for some constant $c \in \mathbb{R}$, then $c = c[H]$ and v is a weak KAM solution. As a matter of fact, by the above argument, v is Lipschitz and satisfies $H(x, dv(x)) \leq c$ for almost every $x \in M$. Which means that $c \geq c[H]$. Moreover, for every $x \in M$, there is a curve $\alpha_x : (-\infty, 0] \to M$ with $\alpha_x(0) = x$ such that

$$v\big(\alpha_x(b)\big) - v\big(\alpha_x(a)\big) = \int_a^b L\big(\alpha_x(s), \dot{\alpha}_x(s)\big)\, ds + c\,(b - a) \qquad \forall a < b \leq 0.$$

Taking a critical subsolution u yields for any $t > 0$,

$$u(x) - u\big(\alpha_x(-t)\big) \leq \int_{-t}^0 L\big(\alpha_x(s), \dot{\alpha}_x(s)\big)\, ds + c[H]\, t = v(x) - v\big(\alpha_x(-t)\big) + (c[H] - c)\, t.$$

Letting t tend to ∞ gives the result.

Example 3.9 (Mechanical Lagrangians)**.** *Consider a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of the form kinetic energy plus potential,*

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x) \qquad \forall (x,p) \in T^*M,$$

where $V : M \to \mathbb{R}$ is a function of class C^2. We check easily that

$$c[H] = \max_{x\in M} \{V(x)\}.$$

As a matter of fact, one has $H(x,0) \leq \max V$ which yields $c[H] \leq 0$. In addition, if $u : M \to \mathbb{R}$ is a C^1 function then for every $x \in M$ with $V(x) = \max V$, one has $H(x,du(x)) \geq \max V$, which shows that $c[H] \geq \max V$. We can also observe that any constant function is a critical subsolution for H.

Example 3.10 (Mañé's Lagrangians)**.** *Let X be a vector field of class C^k (with $k \geq 2$) on M. The Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated with X is the C^k Tonelli Lagrangian defined as*

$$L(x,v) = \frac{1}{2}\|v - X(x)\|_x^2 \qquad \forall (x,v) \in TM.$$

The Hamiltonian H_X associated to L_X by Legendre-Fenchel duality is given by

$$H_X(x,p) = \frac{1}{2}\|p\|_x^2 + \langle p, X(x)\rangle \qquad \forall (x,p) \in T^*M.$$

By Remark 1.2, $c[H] = 0$. In fact, constant functions are weak KAM solution.

4 The Peierls barrier

The Peierls barrier $h : M \times M \to \mathbb{R}$ is defined by

$$h(x,y) := \liminf_{t\to+\infty} \Big\{ h_t(x,y) + c[H]t \Big\} \qquad \forall x,y \in M.$$

The following result is crucial.

Proposition 4.1. *Let $u : M \to \mathbb{R}$ be a weak KAM solution. For every $x \in M$ and every curve $\gamma_x : (-\infty,0] \to M$ with $\gamma_x(0) = x$ satisfying (3.6), any α-limit point z of γ_x, that is any*

$$z \in \bigcap_{t<0} \overline{\gamma_x\big((-\infty,t]\big)}$$

satisfies $h(z,z) = 0$.

Proof. Let z be an α-limit point of γ_x. There is a increasing sequence of times $\{t_k\}$ tending to ∞ such that $z = \lim_{k\to\infty} z_k$ with $z_k := \gamma_x(-t_k)$. Since $h(z,z) \geq 0$ by (3.4), we need to construct a sequence of Lipschitz curves $\gamma_k : [0,T_k] \to M$ such that

$$\lim_{k\to\infty} \int_0^{T_k} L\big(\gamma_k(s), \dot\gamma_k(s)\big)\, ds + c[H]\, T_k = 0.$$

For that we simply concatenate the restriction of γ_x to $[-t_k, 0]$ with some unit speed geodesic joining z to z_k. The continuity of u together with (3.6) yields the result. $\square$

We can check easily that h satisfies for any $x, y, z \in M$ and any $t > 0$,

$$h(x,z) \leq h(x,y) + h_t(y,z) + c[H]\,t \quad \text{and} \quad h(x,z) \leq h_t(x,y) + c[H]\,t + h(y,z). \tag{4.1}$$

Then from the above proposition we deduce that $h(x,y)$ is well-defined for all $x, y \in M$ and satisfies the triangle inequality

$$h(x,z) \leq h(x,y) + h(y,z) \qquad \forall x, y, z \in M. \tag{4.2}$$

Remembering (3.4), we also notice that for every $u \in \mathcal{SS}(H)$, we have

$$u(y) - u(x) \leq h(x,y) \qquad \forall x, y \in M, \tag{4.3}$$

which implies that

$$h(x,x),\, h(x,y) + h(y,x) \geq 0 \qquad \forall x, y \in M. \tag{4.4}$$

The Peierls barrier can indeed be used to construct weak KAM solutions.

Proposition 4.2. *For every $x \in M$, the "pointed" Peierls barrier $h_x : M \to \mathbb{R}$ defined by*

$$h_x(y) := h(x,y) \qquad \forall y \in M.$$

is a weak KAM solution.

Proof. Let $\gamma : [a,b] \to M$ be a Lipschitz curve (with $b - a > 0$), by (4.1) we have

$$\begin{aligned} h\big(x,\gamma(b)\big) &\leq h\big(x,\gamma(a)\big) + h_{b-a}\big(\gamma(a),\gamma(b)\big) + c[H]\,(b-a) \\ &\leq h\big(x,\gamma(a)\big) + \int_a^b L\big(\gamma(s),\dot\gamma(s)\big)\,ds + c[H]\,(b-a), \end{aligned}$$

which shows by Proposition 2.4 that h_x is a critical subsolution.

Lemma 4.3. *For every $y \in M$, there exists a Lipschitz curve $\gamma_y : (-\infty, 0] \to M$ with $\gamma_y(0) = y$ satisfying*

$$h_x\big(\gamma_y(b)\big) - h_x\big(\gamma_y(a)\big) = \int_a^b L\big(\gamma_y(s),\dot\gamma_y(s)\big)\,ds + c[H](b-a) \qquad \forall a < b \leq 0. \tag{4.5}$$

Moreover, the curve γ_y is of class C^2 and solution to the Euler-Lagrange equation.

Proof of Lemma 4.5. Let $y \in M$ be fixed. By definition of $h(x,y)$, there is a sequence of times $\{t_k\}$ tending to ∞ and a sequence of curves $\gamma_k : [0, t_k] \to M$ with $\gamma_k(0) = x, \gamma_k(t_k) = y$ such that

$$h(x,y) = \lim_{k\to\infty} h_{t_k}(x,y) + c[H]\,t_k = \lim_{k\to\infty} \int_0^{t_k} L\big(\gamma_k(s),\dot\gamma_k(s)\big)\,ds + c[H]\,t_k.$$

Foe each $t > 0$, one has for k large enough(with $t_k > t$)

$$\begin{aligned} h_{t_k}(x,y) &= h_{t_k-t}\big(x,\gamma_k(t_k-t)\big) + h_t\big(\gamma_k(t_k-t),y\big) \\ &= h_{t_k-t}\big(x,\gamma_k(t_k-t)\big) + \int_{t_k-t}^{t_k} L\big(\gamma_k(s),\dot\gamma_k(s)\big)\,ds. \end{aligned}$$

By Theorem A.1, the curves γ_k are uniformly Lipschitz. Then by the Arzela-Ascoli Theorem, taking a subsequence if necessary, we may assume that there is a Lipschitz curve $\gamma_y : (-\infty, 0] \to M$ with $\gamma_y(0) = y$ such that

$$h(x,y) \geq h\big(x,\gamma_y(-t)\big) + \int_{-t}^0 L\big(\gamma_y(s),\dot\gamma_y(s)\big)\,ds + c[H]t \qquad \forall t \geq 0.$$

We conclude easily by the fact that $h_x \in \mathcal{SS}(H)$. ☐

Proposition 3.6 concludes the proof. ☐

5 The projected Aubry set and the Aubry set

Definition 5.1. *We call projected Aubry set the nonempty compact subset of M defined by*

$$\mathcal{A}(H) := \Big\{ x \in M \,|\, h(x,x) = 0 \Big\}.$$

The fact that $\mathcal{A}(H)$ is nonempty follows from Proposition 4.1 while the compactness is a consequence of the Lipschitz regularity of h (which is itself a consequence of Proposition A.3). Proposition 4.1 shows indeed that $\mathcal{A}(H)$ plays the role of a boundary at infinity. As shown below, the points of the projected Aubry set are the only points where all critical subsolutions are differentiable.

Proposition 5.2. *The following properties hold:*

(i) For every $x \in \mathcal{A}(H)$, there is a C^2 curve $\gamma_x : \mathbb{R} \to \mathcal{A}(H)$ with $\gamma_x(0) = x$ which is solution to the Euler-Lagrange equation and such that

$$h\big(x, \gamma_x(t)\big) = \int_0^t L\big(\gamma_x(s), \dot\gamma(s)\big)\, ds + c[H]\, t = -h\big(\gamma_x(t), x\big) \qquad \forall t \in \mathbb{R}. \tag{5.1}$$

(ii) For every $x \in \mathcal{A}(H)$, there is $P(x) \in T_x^*M$ with $H(x, P(x)) = c[H]$ such that any critical subsolution u is differentiable at x and satisfies $du(x) = P(x) = \frac{\partial L}{\partial v}(x, \dot\gamma_x(0))$.

(iii) For every $x \in \mathcal{A}(H)$ and every $u \in \mathcal{SS}(H)$, we have

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\,(b-a) \qquad \forall a < b \in \mathbb{R}. \tag{5.2}$$

Moreover, any Lipschitz curve $\gamma : [a,b] \to M$ with $0 \in [a,b]$, $\gamma(0) = x$ and $u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + c[H]\,(b-a)$ satisfies $\gamma(t) = \gamma_x(t)$, for any $t \in [a,b]$.

(iv) For every $x \notin \mathcal{A}(H)$, there is a critical subsolution u which is smooth in an open neighborhood $\mathcal{V}_x$ of x and such that $H(x, du(x)) < c[H]$ for any $x' \in \mathcal{V}_x$.

Proof. Assertion (i) follows by the same arguments as the ones given in the proof of Lemma 4.5. Let us prove (ii) and fix a critical subsolution u. Then we have for any $t \geq 0$

$$u\big(\gamma_x(t)\big) - u(x) \leq \int_0^t L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\, t$$

$$\text{and} \quad u(x) - u\big(\gamma_x(t)\big) \leq h\big(\gamma_x(t), x\big).$$

By (5.1) (summing both inequalities), we infer that

$$u\big(\gamma_x(t)\big) - u(x) = \int_0^t L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\, t \qquad \forall t \geq 0, \tag{5.3}$$

Repeating the same argument for negative times, we get

$$u(x) - u\big(\gamma_x(-t)\big) = \int_{-t}^0 L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + c[H]\, t \qquad \forall t \geq 0. \tag{5.4}$$

Let us now show that we can put a C^2 support function for the graph of u at x from above and from below. Taking a local chart in a neighborhood of x, we may assume that there is $\epsilon > 0$

such that the restriction of γ to the interval $[-\epsilon, \epsilon]$ is valued in $\mathbb{R}^n$. For every x' close to x, we define a C^2 curve $\alpha_{x'} : [0, \epsilon] \to M$ steering $\gamma_x(-\epsilon)$ to x' by

$$\alpha_{x'}(s) = \gamma_x(-\epsilon + s) + \frac{s}{\epsilon}\,(x' - x) \qquad \forall s \in [0, \epsilon].$$

By (5.4) and (2.4), we deduce that

$$u(x') \le u(x) + \int_0^\epsilon L\big(\alpha_{x'}(s), \dot{\alpha}_{x'}(s)\big)\, ds - \int_{-\epsilon}^0 L\big(\gamma_x(s), \dot{\gamma}_x(s)\big)\, ds.$$

Which shows that we can put from above a function of class C^2 on the graph of u at x. Now for every x' close to x, we can define a C^2 curve $\beta_{x'} : [0, \epsilon] \to M$ steering x' to $\gamma_x(\epsilon)$ by

$$\beta_{x'}(s) = \frac{\epsilon - s}{\epsilon}\,(x' - x) + \gamma_x(s) \qquad \forall s \in [0, \epsilon].$$

By (5.3) and (2.4), we deduce that we can put from below a function of class C^2 on the graph of u at x. Thus u is differentiable at x. The restriction of γ_x to the interval $[0, \epsilon]$ minimizes the quantity

$$u\big(\gamma(0)\big) + \int_0^\epsilon L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds$$

among the Lipschitz curves $\gamma : [0, \epsilon] \to M$ such that $\gamma(\epsilon) = \gamma_x(\epsilon)$. Since u is differentiable at $x = \gamma_x(0)$, by Proposition A.4, we deduce that

$$du(x) = \frac{\partial L}{\partial v}\big(x, \dot{\gamma}_x(0)\big) =: P(x).$$

This concludes the proof of assertion (ii). Assertion (iii) is an easy consequence of (5.3)-(5.4) and Cauchy-Lipschitz's theorem. Let us now prove (iv). Fix $x \notin \mathcal{A}(H)$ and define the "pointed" Mañé potential $v : M \to \mathbb{R}$ by

$$v(y) := \inf_{t>0} \Big\{ h_t(x, y) + c[H]\, t \Big\} \qquad \forall y \in M.$$

The function v is well-defined ($v \le h_x$ and one has (3.4)) and satisfies

$$v(z) - v(y) \le h_t(y, z) + c[H]\, t \qquad \forall t > 0.$$

Then proceeding as in the proof of Proposition 4.2, we deduce that $v \in \mathcal{SS}(H)$. Let $y \neq x$ be fixed. Either there is $\bar{t}$ such that $v(y) = h_{\bar{t}}(x, y) + c[H]\, \bar{t}$ or we have $v(y) = h(x, y)$. In both cases, proceeding as in the proof of Lemma 4.5, we can show that there is $\epsilon_y > 0$ and a Lipschitz curve $\gamma_y : (-\epsilon_y, 0] \to M$ with $\gamma_y(0) = y$ such that

$$v\big(\gamma_y(b)\big) - v\big(\gamma_y(a)\big) = \int_a^b L\big(\gamma_y(s), \dot{\gamma}_y(s)\big)\, ds + c[H](b - a) \qquad \forall a < b \in \big(-\epsilon_y, 0\big].$$

By the argument given in the proof of Proposition 3.6 (iii) $\Rightarrow$ (iv), we deduce that for every smooth function $\phi : M \to \mathbb{R}$ with $\phi \le v$ and all $y \in M \setminus \{x\}$,

$$\phi(y) = v(y) \quad \Longrightarrow \quad H\big(y, d\phi(y)\big) \ge c[H].$$

We claim that the above property cannot be satisfied for $y = x$. If not, from Proposition 3.6 this means that v is a weak KAM solution. Then there is a Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma(0) = x$ satisfying (3.6). We check easily that $v(x) = 0$. Then we get

$$-v\big(\gamma_x(-t)\big) = \int_{-t}^0 L\big(\gamma_x(s), \dot{\gamma}_x(s)\big)\, ds + c[H]\, t \qquad \forall t \ge 0.$$

Fix $t > 0$ and $\epsilon > 0$ small. By definition of $v(\gamma_x(-t))$ there is $t_\epsilon > 0$ and a Lipschitz curve $\gamma_\epsilon : [0, t_\epsilon] \to M$ with $\gamma_\epsilon(0) = x, \gamma_\epsilon(t_\epsilon) = \gamma_x(-t)$ such that

$$\int_0^{t_\epsilon} L\big(\gamma_\epsilon(s), \dot{\gamma}_\epsilon(s)\big)\, ds + c[H]\, t_\epsilon \leq v\big(\gamma_x(-t)\big) + \epsilon.$$

Then the concatenation of γ_ϵ with γ_x restricted to $[-t, 0]$ yields

$$h_{t_\epsilon + t}(x, x) + c[H]\ (t_\epsilon + t) \leq \epsilon.$$

Thus letting t tend to ∞ and ϵ tend to zero implies $h(x, x) = 0$, which gives a contradiction. So we deduce the existence of a smooth function $\phi : M \to \mathbb{R}$ with $\phi \leq v$ and such that

$$\phi(x) = v(x) \quad \text{ and } \quad H\big(x, d\phi(x)\big) < c[H].$$

Changing ϕ is necessary we may assume that there are $\epsilon > 0$ and an open neighborhood Ω of x such that the following properties are satisfied:

$$\begin{cases} \phi(y) \geq v(y) - \epsilon\, \forall y \in \Omega, \\ \phi(y) = v(y) - \epsilon\, \forall y \in \partial\Omega \\ \phi(y) < v(y) - \epsilon\, \forall y \in M \setminus \Omega \\ H(y, d\phi(y)) < c[H]\, \forall y \in \Omega. \end{cases}$$

Then we define $u : M \to \mathbb{R}$ as

$$u(y) := \max\Big\{ v(y), \phi(y) + \epsilon \Big\} \qquad \forall y \in M.$$

By construction, u satisfies the properties of assertion (iv). □

Definition 5.3. *We call Aubry set the subset of T^*M defined by*

$$\tilde{\mathcal{A}}(H) := \{(x, P(x)) \in T^*M \,|\, x \in \mathcal{A}(H)\}\,.$$

The following result is due to Mather [20, 21].

Theorem 5.4. *The set $\tilde{\mathcal{A}}(H)$ is a nonempty compact subset of T^*M which is invariant under the Hamiltonian flow. Moreover it is a Lipschitz graph over $\mathcal{A}(H)$.*

Proof. Let $u \in \mathcal{SS}(H)$ be fixed. By Proposition 5.2 (ii), u is differentiable at each point x of $\mathcal{A}(H)$ and satisfies $du(x) = P(x)$. In fact, in the proof we saw that for each $x \in \mathcal{A}(H)$, we can put a C^2 support function for the graph of u at x from above and from below. The Lipschitz regularity of the mapping $x \in \mathcal{A}(H) \mapsto P(x)$ is a easy consequence of the following lemma taken from [13] and whose proof is given in Section C.

Lemma 5.5. *Let $\mathcal{B}$ be a open unit ball in $\mathbb{R}^n$, $f : B \to \mathbb{R}$ be a continuous function, $K > 0$ and $E \subset \mathcal{B}$ be such that for every $x \in E$, there is $p_x \in (\mathbb{R}^n)^*$ veryfing*

$$|f(y) - f(x) - \langle p_x, y - x\rangle| \leq K|y - x|^2 \qquad \forall y \in \mathcal{B}.$$

Then u is differentiable on E, $du(x) = p_x\, \forall x \in E$, and the mapping

$$x \in E \text{ with } |x| \leq \frac{1}{3} \longmapsto du(x)$$

is $6K$-Lipschitz.

The remaining part follows from Proposition 5.2 (i)-(ii) and Remark A.2

□

Example 5.6 (Mechanical Lagrangians)**.** *Consider an mechanical Hamiltonian $H : T^*M \to \mathbb{R}$ of the form given in Example 3.9. The Aubry set consists in a set of equilibria:*

$$\tilde{\mathcal{A}}(H) = \Big\{(x,0) \,|\, V(x) = \max V\Big\}.$$

Example 5.7 (Mañé's Lagrangians)**.** *Let X be a vector field of class C^k (with $k \geq 2$) on M and $H_X : T^*M \to \mathbb{R}$ be the Hamiltonian associated to the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ given in Example 3.10. it can be shown that the projected Aubry set of H_X contains the set of recurrent points of the flow of X. The Aubry set is given by*

$$\tilde{\mathcal{A}}(H) = \Big\{(x,0) \,|\, x \in \mathcal{A}(H)\Big\}.$$

Its orbits are orbits of X lifted in T^*M, that is have the form $(x(\cdot), p(\cdot)))$ with $\dot{x}(\cdot) = X(x(\cdot))$ and $p(\cdot) = 0$.

Let $u \in \mathcal{SS}(H)$ be fixed. By Proposition 5.2, the set of critical subsolutions which coincide with u on $\mathcal{A}(H)$ is convex, compact, and is invariant with respect to the critical Lax-Oleinik semigroup. Then the same proof as for Theorem 3.2 gives.

Proposition 5.8. *For every $u \in \mathcal{SS}(H)$, there exists a weak KAM solution v such that $v = u$ on $\mathcal{A}(H)$.*

We mention that there is a comparison theory for weak KAM solutions. Fathi proves in [13] that if two weak KAM solutions can be compared on the projected Aubry set, then they can be compared globally.

6 The uniqueness issue

Of course if a given function u is a weak KAM solution for H, then for every constant $a \in \mathbb{R}$ the function $u + a$ is weak KAM solution. We shall say that (1.1) has a unique solution if for every pair u, v of weak KAM solutions , the function $u - v$ is constant.

Theorem 6.1. *Let $H : T^*M \to \mathbb{R}$ be a C^k Tonelli Hamiltonian with $k \geq 2$ such that (1.1) has a unique solution. Then $\mathcal{A}(H)$ is connected.*

Proof. We argue by contradiction and assume that $\mathcal{A}(H)$ is not connected. From Proposition 5.8, it is sufficient to construct two critical subsolutions $u, v : M \to$ such that the restriction of $v - u$ to $\mathcal{A}(H)$ is not constant. Since $\mathcal{A}(H)$ is not connected, there are two disjoint compact sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}(H)$ such that $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}(H)$. Let $\Omega_1, \Omega_2 \subset M$ be two disjoint open sets containing respectively $\mathcal{A}_1$ and $\mathcal{A}_2$, by Proposition 5.2 (iv), for every $x \in M \setminus (\Omega_1 \cup \Omega_2)$ there is a critical subsolution $u_x : M \to \mathbb{R}$ which is smooth in an open neighborhood $\mathcal{V}_x$ and such that $H(x, du_x(x')) < c[H]$ for any $x' \in \mathcal{V}_x$. Note that taking $\mathcal{V}_x$ smaller if necessary we may assume that $\mathcal{V}_x \cap \mathcal{A}(H) = \emptyset$. By compactness, there are $x_1, \ldots, x_N$ in $M \setminus (\Omega_1 \cup \Omega_2)$ such that

$$M \setminus (\Omega_1 \cup \Omega_2) \subset \bigcup_{i=1,\ldots,N} \mathcal{V}_{x_i}.$$

By (H2), the function $u : M \to \mathbb{R}$ defined by

$$u(x) := \frac{u_{x_1}(x) + \cdots + u_{x_N}(x)}{N} \qquad \forall x \in M,$$

is a critical subsolution. In addition, by construction there is $\epsilon > 0$ such that

$$H\big(x, du(x)\big) \leq c[H] - \epsilon \qquad \text{a.e. } x \in M \setminus (\Omega_1 \cup \Omega_2).$$

Then by regularizing u in a neighborhood of $M \setminus (\Omega_1 \cup \Omega_2)$, we can assume that u is a critical subsolution which is smooth in an open neighborhood $\mathcal{V}$ of $M \setminus (\Omega_1 \cup \Omega_2)$ and satisfies

$$H\big(x, du(x)\big) \leq c[H] - \epsilon \qquad \forall x \in M \setminus (\Omega_1 \cup \Omega_2).$$

By Proposition 5.8, there is a critical solution $\tilde{u}$ which coincides with u on $\mathcal{A}(H)$. For every $x \in M \setminus (\Omega_1 \cup \Omega_2)$ and every γ_x given by Proposition 3.6 (iii), we have

$$\tilde{u}(x) - \tilde{u}\big(\gamma_x(-t)\big) = \int_{-t}^{0} L\big(\gamma_x(s), \dot{\gamma}_x(s)\big)\, ds + c[H]\, t \qquad \forall t \geq 0$$

and

$$u(x) - u\big(\gamma_x(-t)\big) \leq \int_{-t}^{0} L\big(\gamma_x(s), \dot{\gamma}_x(s)\big)\, ds + (c[H] - \epsilon)\, t \quad \forall t \geq 0 \text{ such that } \gamma_x\big([-t, 0]\big) \subset \mathcal{V}.$$

Since $u \in \mathcal{SS}(H)$ and $(u - \tilde{u})\big(\gamma_x(-t)\big)$ tends to zero as t tends to ∞ (by Proposition 4.1), we infer that $u < \tilde{u}$ on $M \setminus (\Omega_1 \cup \Omega_2)$. Let $\alpha > 0$ be such that $u + \alpha \leq \tilde{u}$ on $M \setminus (\Omega_1 \cup \Omega_2)$, the function $v : M \to \mathbb{R}$ defined by

$$v(x) = \begin{cases} \tilde{u}(x) \text{ if } x \in \Omega_1, \\ \max\{\tilde{u}(x), u(x) + \alpha/2\} \text{ otherwise}, \end{cases}$$

is a critical subsolution such that the the restriction of $v - u$ to $\mathcal{A}(H)$ is not constant. □

Remark 6.2. *In the above proof, we constructed a critical subsolution which is strict on a compact subset of $M \setminus \mathcal{A}(H)$. In fact, as shown by Bernard [2] (thanks to a seminal result by Fathi and Siconolfi [15]), there is a critical subsolution u of class $C^{1,1}$ such that $H(x, du(x)) < c[H]$ on $M \setminus \mathcal{A}(H)$. In particular, the infimum in the definition of $c[H]$ is attained for a function of class $C^{1,1}$. We refer the reader to [24] fur further details.*

Thanks to a Sard-type result proven in [14], the converse result holds in low dimension.

Theorem 6.3. *Let $H : T^*M \to \mathbb{R}$ be a C^k Tonelli Hamiltonian with $k \geq 2$ and $k \geq 4$ if dim $M \geq 3$. Assume that $\mathcal{A}(H)$ is connected. Then (1.1) has a unique solution.*

Proof. Let $u, v : M \to \mathbb{R}$ be two weak KAM solutions. As shown in [14], under the present assumptions, the function $(u - v)_{|\mathcal{A}(H)}$ satisfies Sard's Theorem, that is $(u - v)(\mathcal{A}(H))$ has zero Lebesgue measure. Since $\mathcal{A}(H)$ is connected, its image has to be an interval. Consequently, it is a singleton. Furthermore, thanks to a comparison theorem by Fathi [13, Theorem 8.5.5], if two weak KAM solutions coincide on $\mathcal{A}(H)$ then they coincide on all M. □

Example 6.4 (Mechanical Lagrangians)**.** *In [22] Mather provides examples of potentials $V : M \to \mathbb{R}$ of class C^k such that the projected Aubry set of the Hamiltonian given in Example 3.9 is connected but without uniqueness. No smooth or analytic counterexamples to Theorem 6.3 are known.*

Example 6.5 (Mañé's Lagrangians)**.** *In [14], we show that (at least in low dimension) the uniqueness property for Mañé Lagrangians is related to chain-recurrent properties of the flow of X.*

Following Mañé [19], given a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$) and a potential $V : M \to \mathbb{R}$ of class C^k (with $k \geq 2$), we define the Hamiltonian $H_V : T^*M \to \mathbb{R}$ by

$$H_V(x, p) := H(x, p) + V(x) \qquad \forall\, (x, p) \in T^*M.$$

Denote by $C^k(M)$ the set of C^k potentials on M equipped with the C^k topology. Generically on the potential, we have uniqueness. The following result is due to Mañé [19].

Theorem 6.6. *Let $H : T^*M \to \mathbb{R}$ be a C^k Tonelli Hamiltonian (with $k \geq 2$). There is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ in $C^k(M)$ such that for every $V \in \mathcal{G}$, the critical Hamilton-Jacobi associated with H_V has a unique solution.*

7 Regularity of weak KAM solutions

The following is proven in [23]. We refer to the reader to [4, 23, 24] for the definition of semiconcave functions.

Theorem 7.1. *Let $u : M \to \mathbb{R}$ be a weak KAM solution. Then u is semiconcave on M and $C^{1,1}_{loc}$ on an open dense subset $\mathcal{O}$ of M.*

Since weak KAM solutions u are Lipschitz, the limiting differential of u at x defined by

$$du^*(x) = \Big\{ \lim du(x_k) \,|\, x_k \to x, u \text{ diff at } x_k \Big\},$$

is always a nonempty compact subset of $T^*_x M$. In [23], we show that there is a one-to-one correspondence between the set of limiting differentials at x and the set of curves $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ and such that

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s), \dot{\gamma}_x(s)\big)\, ds + c[H]\,(b-a) \qquad \forall a < b \leq 0.$$

In fact (by the same arguments as in the proof of Proposition 5.2 (i)), u can be shown to be C^1 at every $\gamma_x(-t)$ with $t > 0$. Since $\gamma_x(-t)$ tends to the projected Aubry set as t tends to ∞ (by Proposition 4.1), regularity properties for weak KAM solutions in a neighborhood of $\mathcal{A}(H)$ imply more regularity for u globally (in the spirit of classical results for Dirichlet-type problems [5, 18, 23]). Some properties on the behavior of the Hamiltonian flow in a neighborhood of $\tilde{\mathcal{A}}(H)$ may also bring regularity properties. This is the purpose of the following result by Bernard [1].

Theorem 7.2. *Let H be an Hamiltonian whose the Aubry set is an hyperbolic periodic orbit. Then there is a unique weak KAM solution. Moreover it is C^k in a neighborhood of $\mathcal{A}(H)$.*

We refer the reader to [1] for the proof which is based on the fact (thanks to Proposition 4.1) every limiting subdifferential has to be in the unstable manifold of the periodic orbit.

8 The Mañé conjecture

The Mañé conjecture in C^k topology (with $k \geq 2$) can be stated as follows.

Conjecture 8.1 (Mañé's Conjecture)**.** *For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set $\tilde{\mathcal{A}}(H_V)$ of the Hamiltonian H_V is either an equilibrium point or a periodic orbit.*

A natural way to attack the Mañé Conjecture in any dimension would be to prove first a density result, then a stability result. Namely, given an Hamiltonian of class C^k satisfying (H1) and (H2), first one could show that the set of potentials $V \in C^k(M)$ such that $\tilde{A}(H_V)$ is either a hyperbolic equilibrium point or a hyperbolic periodic orbit is dense, and then prove that the latter property is open in C^k topology. The stability part is indeed contained in results obtained by Contreras and Iturriaga in [8], so we can consider that the Mañé Conjecture reduces to the density part.

Conjecture 8.2 (Mañé's density Conjecture)**.** *For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of H_V is either an equilibrium point or a periodic orbit.*

In a series of papers in collaboration with Figalli [16, 17], we made progress toward a proof of the Mañé Conjecture in C^2 topology. Our approach is based on a combination of techniques coming from finite dimensional control theory and Hamilton-Jacobi theory, together with some of the ideas which were used to prove C^1-closing lemmas for dynamical systems. The following result is a weak form of some of the results that we obtained in [16, 17].

Theorem 8.3. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$, and fix $\epsilon > 0$. Assume that there is a critical subsolution which is of class C^{k+1}. Then there exists a potential $V : M \to \mathbb{R}$ of class C^{k-1}, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit.*

This result together with stability results by Contreras and Iturriaga [8] shows that we can more or less consider that the Mañé Conjecture for Hamiltonians of class at least C^4 is equivalent to the:

Conjecture 8.4 (Mañé regularity Conjecture)**.** *For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k, with $k \geq 4$, there is a set $\mathcal{D} \subset C^4(M)$ which is dense in $C^2(M)$ (with respect to the C^2 topology) such that the following holds: For every $V \in \mathcal{D}$, the Hamiltonian H_V admits a critical subsolution of class C^5.*

A Reminders in calculus of variations

Let $L : TM \to \mathbb{R}$ be a C^k Tonelli Lagrangian (with $k \geq 2$), that is a Lagrangian of class C^k satisfying the two following properties:

(L1) *Superlinear growth:* For every $K \geq 0$, there is a finite constant $C(K)$ such that

$$L(x, v) \geq K\|v\|_x + C(K) \qquad \forall\, (x, v) \in TM.$$

(L2) *Uniform convexity:* For every $(x, v) \in TM$, the second derivative along the fibers $\frac{\partial^2 L}{\partial v^2}(x, v)$ is positive definite.

The purpose of the present section is to recall basic facts on minimizing problems associated with Tonelli Lagrangians. Given $t > 0$ we study the minimal action problem in time t, that is we study the function $h_t : M \times M \to \mathbb{R}$ defined by

$$h_t(x, y) := \inf_{\gamma} \left\{ \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds \right\} \qquad \forall x, y \in M, \tag{A.1}$$

where the infimum is taken on the set of Lipschitz curves $\gamma : [0, t] \to M$ which satisfy $\gamma(0) = x$ and $\gamma(t) = y$.

Theorem A.1. *For any $x, y \in M$ and $t > 0$, there exists a Lipschitz curve $\gamma : [0, t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$ such that*

$$h_t(x, y) = \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds. \tag{A.2}$$

The curve γ is indeed of class C^2 and satisfies the Euler-Lagrange equation

$$\frac{d}{ds}\left[\frac{\partial L}{\partial v}\left(\gamma(s), \dot\gamma(s)\right)\right] = \frac{\partial L}{\partial x}\left(\gamma(s), \dot\gamma(s)\right) \qquad \forall s \in [0, t]. \tag{A.3}$$

Moreover, there is a continuous increasing function $\theta : (0, \infty) \to (0, \infty)$ depending only on L such that

$$\big\|\dot\gamma(s)\big\|_{\gamma(s)} \leq \theta(1/t) \qquad \forall s \in [0, t]. \tag{A.4}$$

Remark A.2. *If $H : T^*M \to \mathbb{R}$ denotes the C^k Tonelli Hamiltonian associated with L by Legendre duality, then for any curve $\gamma : [0,t] \to M$ satisfying the Euler-Lagrange equation (A.3), the curve $\big(x(\cdot), p(\cdot)\big) : [0,t] \to T^*M$ defined by*

$$\big(x(s), p(s)\big) := \left(\gamma(s), \frac{\partial L}{\partial v}\big(\gamma(s), \dot\gamma(s)\big)\right) \qquad \forall s \in [0,t],$$

is a trajectory of the Hamiltonian system associated with H.

Proof. Let $x, y \in M$ and $t > 0$ be fixed. Set

$$\tilde h_t(x,y) := \inf\left\{ \int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\, ds \,|\, \gamma \in AC([0,t],M), \gamma(0) = x, \gamma(t) = y \right\},$$

where $AC([0,t],M)$ signifies the class of absolutely continuous functions mapping $[0,t]$ to M. Tonelli's celebrated theorem asserts that under the present hypotheses, the infimum in the above formula is attained, namely there is an absolutely continuous curve $\gamma : [0,t] \to M$ with $\gamma(0) = x$ and $\gamma(t) = y$ such that

$$\tilde h_t(x,y) = \int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\, ds.$$

Since the Lagrangian is autonomous (it does not depend on time), it can be shown that γ is indeed Lipschitz, see [6]. In particular, this shows that $\tilde h_t(x,y) = h_t(x,y)$. Then the duBois-Reymond Theorem asserts that γ satisfies an integral Euler equation. From the latter equation and the uniform convexity of L in the fibers, it can be shown that γ has indeed the same regularity as L. We refer the reader to the textbook [7] for the proofs of these facts. In conclusion, we infer that the infimum in (A.1) is attained by a curve γ which is at least C^2 and satisfies the Euler-Lagrange equation (A.3). It remains to show the existence of a continuous increasing function satisfying (A.4).

Denote by d the Riemannian distance associated with g on $M \times M$. Let $\alpha : [0, d(x,y)] \to M$ be a unit speed geodesic joining x to y in time $d(x,y)$. There holds

$$h_t(x,y) \leq \int_0^t L\left(\alpha(sd(x,y)/t), (d(x,y)/t)\dot\alpha(sd(x,y)/t)\right) ds.$$

Set for any $R > 0$,

$$S(R) := \max\Big\{ L(z,w) \,|\, (z,w) \in TM, \|w\|_z \leq R \Big\}.$$

Then one has

$$h_t(x,y) \leq tS\big(d(x,y)/t\big) \leq tS\big(\mathrm{diam}(M)/t\big). \tag{A.5}$$

Let $C(1) \in \mathbb{R}$ be the constant given by (L1), that is such that $L(x,v) \geq \|v\|_x + C(1)$ for any $(x,v) \in TM$. From (A.2) and (A.5), we infer that there is $\bar s \in [0,t]$ such that

$$\big\|\dot\gamma(\bar s)\big\|_{\gamma(\bar s} \leq S\big(\mathrm{diam}(M)/t\big) - C(1). \tag{A.6}$$

By Remark A.2, the curve $\big(x(\cdot), p(\cdot)\big) : [0,t] \to T^*M$ defined by

$$\big(x(s), p(s)\big) := \left(\gamma(s), \frac{\partial L}{\partial v}\big(\gamma(s), \dot\gamma(s)\big)\right) \qquad \forall s \in [0,t],$$

is a trajectory of the Hamiltonian system associated with H. Then since H is constant along $\big(x(\cdot), p(\cdot)\big)$, by (A.6) we infer that for all $s \in [0,t]$,

$$H\big(x(s), p(s)\big) = H\big(x(\bar s), p(\bar s)\big) \leq S^*\left(S_1\big(S\big(\mathrm{diam}(M)/t\big) - C(1)\big)\right),$$

where for any $R > 0$, $S^*(R)$ and $S_1(R)$ are given by

$$S^*(R) := \max\Big\{H(z,p) \,|\, (z,p) \in T^*M, \|p\|_z \le R\Big\},$$

and

$$S_1(R) := \max\left\{\left\|\frac{\partial L}{\partial v}(z,w)\right\|_z \,|\, (z,w) \in TM, \|w\|_z \le R\right\}.$$

By superlinear growth of H, there is $C^*(1) \in \mathbb{R}$ such that $H(x,p) \ge \|p\|_x + C^*(1)$ for any $(x,p) \in T^*M$. Then we get

$$\|p(s)\|_{\gamma(s)} \le S^*\left(S_1\big(S\big(\mathrm{diam}(M)/t\big) - C(1)\big)\right) - C^*(1) \qquad \forall s \in [0,t].$$

Since $\dot\gamma(s) = \frac{\partial H}{\partial p}\big(x(s), p(s)\big)$ for all $s \in [0,t]$, setting

$$S_1^*(R) := \max\left\{\left\|\frac{\partial H}{\partial p}(z,p)\right\|_z \,|\, (z,p) \in T^*M, \|p\|_z \le R\right\}$$

yields

$$\|\dot\gamma(s)\|_{\gamma(s)} \le S_1^*\left(S^*\left(S_1\big(S\big(\mathrm{diam}(M)/t\big) - C(1)\big)\right) - C^*(1)\right) \qquad \forall s \in [0,t].$$

We leave the reader to construct the continuous increasing function θ satisfying (A.4). $\square$

Proposition A.3. *For every $\bar t > 0$, there is $K_t > 0$ such that the functions h_t are K_t-Lipschitz on $M \times M$ for all $t \ge \bar t$.*

Proof. This is a consequence of (A.4). Let $x, y \in M$ and $t \ge \bar t$ be fixed. By Theorem A.1, there is a C^2 curve $\gamma : [0,t] \to M$ with $\gamma(0) = x, \gamma(t) = y$ satisfying (A.2)-(A.4). Taking a local chart in a neighborhood of y, we may assume that there is a universal $\epsilon > 0$ such that the restriction of γ to the interval $[t-\epsilon, t]$ is valued in $\mathbb{R}^n$. For every y' close to y, define $\alpha_{y'} : [0,t] \to M$ by

$$\alpha_{y'}(s) = \gamma(s) + \frac{\psi(t-s)}{\epsilon}(y' - y) \qquad \forall s \in [0,t],$$

where $\psi : \mathbb{R} \to [0,1]$ is a smooth function such that $\psi(s) = 0$ for $s \ge \epsilon$ and $\psi(0) = 1$. The curve $\alpha_{y'}$ is a C^2 curve steering x to y' in time t. Hence

$$\begin{aligned} h_t(x,y') &\le \int_0^t L\big(\alpha_{y'}(s), \dot\alpha_{y'}(s)\big)\, ds \\ &\le h_t(x,y) + \int_0^t L\big(\alpha_{y'}(s), \dot\alpha_{y'}(s)\big)\, ds - \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds. \end{aligned}$$

The function

$$y' \quad \longmapsto \quad \int_0^t L\big(\alpha_{y'}(s), \dot\alpha_{y'}(s)\big)\, ds - \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds$$

is of class C^2, it vanishes at $y' = y$ and by (A.4) its derivative at $y' = y$ is bounded by a constant depending only on $\bar t$. This proves the result. $\square$

Finally we recall the following result which follows easily from Theorem A.1.

Proposition A.4. *Let $\varphi : M \to \mathbb{R}$ be a Lipschitz function and $y \in M$ and $t > 0$ be fixed. Let$\gamma : [0,t] \to M$ be a Lipschitz curve with $\gamma(t) = y$ which minimizes the quantity*

$$\varphi\big(\alpha(0)\big) + \int_0^t L\big(\alpha(s), \dot\alpha(s)\big)\, ds$$

among all Lipschitz curves $\alpha : [0,t] \to M$ with $\alpha(t) = y$. Assume that φ is differentiable at $\gamma(0)$. Then γ is of class C^2, satisfies the Euler-Lagrange equation, and verifies the transversality condition

$$d\varphi\big(\gamma(0)\big) = \frac{\partial L}{\partial v}\big(\gamma(0), \dot\gamma(0)\big).$$

B A fixed point theorem for nonexpansive semigroups

The proof of the following result is sketched in [13]. For sake of completeness we provide its proof.

Theorem B.1. *Let $\mathcal{K}$ be a nonempty compact convex set in a normed space $(E, \|\cdot\|_E)$. Let $\{\varphi_t\}_{t\geq 0}$ be a family of mappings $\varphi_t : \mathcal{K} \to \mathcal{K}$ satisfying the following properties:*

(i) $\forall t, t' \geq 0, \varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$ *(semigroup property).*

(ii) $\forall t \geq 0$, $\|\varphi_t(e) - \varphi_t(e')\|_E \leq \|e - e'\|_E$ *(nonexpansivity).*

(iii) For every $e \in E$, the mapping $t \in [0, \infty) \mapsto \varphi_t(e)$ is continuous.

Then there is $e \in E$ such that $\varphi_t(e) = e$ for all $t \geq 0$.

Proof. First we show that each φ_t has a fixed point. Let $t > 0$ be fixed, fix $\bar{e} \in \mathcal{K}$ and define for every positive integer k the function $\varphi_t^k : \mathcal{K} \to \mathcal{K}$ by

$$\varphi_t^k(e) = \frac{1}{k}\,\varphi_t(\bar{e}) + \left(1 - \frac{1}{k}\right)\varphi_t^k(e) \qquad \forall e \in \mathcal{K}.$$

Note that φ_t^k is well-defined by convexity of $\mathcal{K}$. By (ii), φ_t^k is a $(1 - 1/k)$ contraction. Then by Banach's fixed point theorem, each φ_t^k has a fixed point e_t^k. By compactness, taking a subsequence if necessary, $\{e_t^k\}$ tends as k tends to ∞ to a fixed point of φ_t. Denote by $\mathrm{Fix}(\varphi_t)$ the set of $e \in \mathcal{K}$ such that $\varphi_t(e) = e$. The set $\mathrm{Fix}(\varphi_t)$ is a nonempty compact subset of $\mathcal{K}$. Let us show that there is a nonexpansive retraction from $\mathcal{K}$ to $\mathrm{Fix}(\varphi_t)$. Denote by $\tilde{\mathcal{K}}_t$ the set of mapping $r : \mathcal{K} \to \mathcal{K}$ which are nonexpansive and such that $r(e) = e$ for any $e \in \mathrm{Fix}(\varphi_t)$. Note that φ_t belongs to $\tilde{\mathcal{K}}_t$. Since $\mathcal{K}$ is convex, the set $\tilde{\mathcal{K}}_t$ is a compact convex subset of the set of continuous mapping from $\mathcal{K}$ to $\mathcal{K}$ equipped with the supremum norm. Define the mapping $G_t : \tilde{\mathcal{K}}_t \to \tilde{\mathcal{K}}_t$ by

$$G_t(r) = \varphi_t \circ r \qquad \forall r \in \tilde{\mathcal{K}}_t.$$

Using (ii) we check easily that G_t is a nonexpansive mapping. Thus by the above argument, there is $r_t \in \tilde{\mathcal{K}}_t$ such that $G_t(r_t) = \varphi_t \circ r_t = r_t$. This means that $r_t : \mathcal{K} \to \mathcal{K}$ is nonexpansive and satisfies

$$r_t(e) \in \mathrm{Fix}(\varphi_t) \quad \forall e \in \mathcal{K} \quad \text{ and } \quad r_t(e) = e \quad \forall e \in \mathrm{Fix}(\varphi_t).$$

Let us now show that for any $t, t' > 0$, the mapping $\varphi_t, \varphi_{t'}$ share a common fixed point. By (ii) and the nonexpansiveness of r_t, the mapping

$$e \in \mathcal{K} \quad \longmapsto \quad \bigl(r_t \circ \varphi_{t'}\bigr)(e) \in \mathcal{K}$$

is nonexpansive. Again, by the above argument there is $e \in \mathcal{K}$ such that

$$\bigl(r_t \circ \varphi_{t'}\bigr)(e) = r_t\bigl(\varphi_{t'}(e)\bigr) = e.$$

Since e belongs to the image of r_t, it belongs to $\mathrm{Fix}(\varphi_t)$. By (i), $\varphi_t \circ \varphi_{t'} = \varphi_{t+t'} = \varphi_{t'} \circ \varphi_t$. Thus $\varphi_{t'}(e)$ also belongs to $\mathrm{Fix}(\varphi_t)$. Then we get

$$e = r_t\bigl(\varphi_{t'}(e)\bigr) = \varphi_{t'}(e),$$

which shows that e is a common fixed point of φ_t and $\varphi_{t'}$. Denote by $\mathrm{Fix}(\varphi_t, \varphi_{t'})$ the set of $e \in \mathcal{K}$ such that $\varphi_t(e) = \varphi_{t'}(e) = e$. Considering the set of mappings $r \in \tilde{\mathcal{K}}_t$ satisfying $r(e) = e$ for any $e \in \mathrm{Fix}(\varphi_t, \varphi_{t'})$ and the mapping $r \mapsto r_t \circ \varphi_{t'} \circ r$ and arguing as above, we show easily the existence of a nonexpansive contraction from $\mathcal{K}$ to $\mathrm{Fix}(\varphi_t, \varphi_{t'})$. Then repeating the previous arguments, we show that for every finite set $t_1, \ldots, t_N$ of positive times, the mappings

$\varphi_{t_1}, \ldots, \varphi_{t_N}$ have a common fixed point. Let us now consider a countable family of positive times $\{t_l\}_{l\in\mathbb{N}}$ which is dense in $[0,\infty)$. For every $l \in \mathbb{N}$, there is $e_l \in \mathcal{K}$ such that $\varphi_{t_i}(e_l) = e_l$ for every $i \in \{0,\ldots,l\}$. By compactness of $\mathcal{K}$, taking a subsequence if necessary we may assume that e_l converges to some $e \in E$ as l tends to ∞. It remains to show that $\varphi_t(e) = e$ for all $t \geq 0$. Let $t > 0$ be fixed, there is a sequence of positive times $\{t_{l_k}\}$ converging to t as k tends to ∞. For every k and every integer $l > t_{l_k}$, we have

$$\begin{aligned}
\|\varphi_t(e) - e\|_E &\leq \|\varphi_t(e) - e_l\|_E + \|e_l - e\|_E \\
&\leq \left\|\varphi_t(e) - \varphi_{t_{l_k}}(e_l)\right\|_E + \|e_l - e\|_E \\
&\leq \left\|\varphi_t(e) - \varphi_{t_{l_k}}(e)\right\|_E + \left\|\varphi_{t_{l_k}}(e) - \varphi_{t_{l_k}}(e_l)\right\|_E + \|e_l - e\|_E \\
&\leq \left\|\varphi_t(e) - \varphi_{t_{l_k}}(e)\right\|_E + 2\,\|e_l - e\|_E\,.
\end{aligned}$$

We conclude easily by (iii). □

C Proof of Lemma 5.5

The differentiability of f on E is easy. Let $x, x' \in E$ with $|x - x'| \leq 1/3$ and $p_x \neq p_{x'}$ be fixed, set $h := \frac{|x-x'|}{|p_x - p_{x'}|}(p_{x'} - p_x)$. Then both $x + h$ and $x' + h$ belong to $\mathcal{B}$ and we have

$$f(x+h) - f(x) - \langle p_x, h\rangle \leq \left|f(x+h) - f(x) - \langle p_x, h\rangle\right| \leq K|h|^2 = K|x-x'|^2,$$

$$f(x) - f(x') - \langle p_{x'}, x - x'\rangle \leq \left|f(x) - f(x') - \langle p_{x'}, x - x'\rangle\right| \leq K|x-x'|^2,$$

and

$$\begin{aligned}
f(x') - f(x+h) + \langle p_{x'}, x + h - x'\rangle &\leq \left|f(x+h) - f(x') - \langle p_{x'}, x + h - x'\rangle\right| \\
&\leq K|x+h-x'|^2 \leq K\big(|x-x'| + |h|\big)^2 = 4K|x-x'|^2.
\end{aligned}$$

Summing the above inequalities yields

$$\left|p_{x'} - p_x\right| |x' - x| = \langle p_{x'} - p_x, h\rangle \leq 6K|x-x'|^2.$$

References

[1] P. Bernard. Smooth critical sub-solutions of the Hamilton-Jacobi equation. *Math. Res. Lett.*, 14(3):503–511, 2007.

[2] P. Bernard. Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equation on compact manifolds. *Ann. Sci. École Norm. Sup.*, 40(3):445–452, 2007.

[3] J.-B. Bost. Tores invariants des systèmes dynamiques hamiltoniens. In Séminaire Bourbaki, Vol. 1984/85 *Astérisque*, 133-134:113–157, 1986.

[4] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[5] M. Castelpietra and L. Rifford. Regularity properties of the distance function to conjugate and cut loci for viscosity solutions of Hamilton-Jacobi equations and applications in Riemannian geometry. *ESAIM Control Optim. Calc. Var.*, 16 (3):695–718, 2010.

[6] F. Clarke. A Lipschitz regularity theorem. *Ergodic Theory Dynam. Systems*, 27(6):1713–1718, 2007.

[7] F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control.* To appear.

[8] G. Contreras and R. Iturriaga. Convex Hamiltonians without conjugate points. *Ergodic Theory Dynam. Systems*, 19(4):901–952, 1999.

[9] A. Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(9):1043–1046, 1997.

[10] A. Fathi. Solutions KAM faible conjuguées et barrières de Peierls. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(6):649–652, 1997.

[11] A. Fathi. Orbites hétéroclones et ensemble de Peierl. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(10):1213–1216, 1998.

[12] A. Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(3):267–270, 1998.

[13] A. Fathi. *Weak KAM Theorem and Lagrangian Dynamics.* Cambridge University Press, to appear.

[14] A. Fathi, A. Figalli and L. Rifford. On the Hausdorff dimension of the Mather quotient. *Comm. Pure Appl. Math.*, 62(4):445–500, 2009.

[15] A. Fathi and A. Siconolfi. Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. *Invent. math.*, 1155:363–388, 2004.

[16] A. Figalli and L. Rifford. Closing Aubry sets I. Preprint, 2010.

[17] A. Figalli and L. Rifford. Closing Aubry sets II. Preprint, 2010.

[18] Y. Li and L. Nirenberg. The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations. *Comm. Pure Appl. Math.*, 58(1):85–146, 2005.

[19] R. Mañé. Generic properties and problems of minimizing measures of Lagrangian systems, *Nonlinearity*, 9(2):273–310, 1996.

[20] J. N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.*, 207:169–207, 1991.

[21] J. N. Mather. Variational construction of connecting orbits. *Ann. Inst. Fourier*, 43:1349–1386, 1993.

[22] J. N. Mather. Examples of Aubry sets. *Ergod. Th. Dynam. Sys.*, 24:1667–1723, 2004.

[23] L. Rifford. On viscosity solutions of certain Hamilton-Jacobi equations: Regularity results and generalized Sard's Theorems. *Comm. Partial Differential Equations*, 33(3):517–559, 2008.

[24] J.-M. Roquejoffre. Propriétés qualitatives des solutions des équations de Hamilton-Jacobi (d'après A. Fathi, A. Siconolfi, P. Bernard). In Séminaire Bourbaki, Vol. 2006/2007 *Astérisque*, 317:269–293, 2008.