Recent progress in sub-Riemannian geometry

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Outline

I. Introduction to sub-Riemannian geometry

II. A few open problems

III. A few partial results

Part I

I. Introduction to sub-Riemannian geometry

Sub-Riemannian structures

Let M be a smooth connected manifold of dimension n.

Definition

A sub-Riemannian structure of rank m in M is given by a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \leq n$ on M which is defined locally by

$$\Delta(x) = \mathrm{Span}\Big\{X^1(x), \ldots, X^m(x)\Big\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ is a family of m linearly independent smooth vector fields satisfying the **Hörmander condition**.
- g_x is a **scalar product** over $\Delta(x)$.

The Hörmander condition

We say that a family of smooth vector fields $X^1, \ldots, X^m$, satisfies the **Hörmander condition** if

$$\text{Lie}\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x,$$

where $\text{Lie}\{X^1, \ldots, X^m\}$ denotes the Lie algebra generated by $X^1, \ldots, X^m$, *i.e.* the smallest subspace of smooth vector fields that contains all the $X^1, \ldots, X^m$ and which is stable under Lie brackets.

Reminder

Given smooth vector fields X, Y in $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

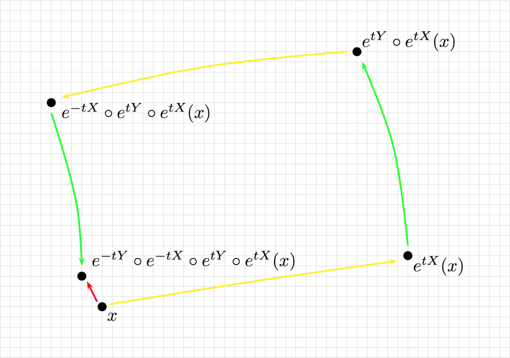
$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

Lie Bracket: Dynamic Viewpoint

Exercise

There holds

$$[X,Y](x) = \lim_{t\downarrow 0} \frac{\left(e^{-tY}\circ e^{-tX}\circ e^{tY}\circ e^{tX}\right)(x) - x}{t^2}.$$



The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any $\gamma \in W^{1,2}([0,1]; M)$ such that

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$

The following result is the cornerstone of the sub-Riemannian geometry. (Recall that M is assumed to be connected.)

Theorem (Chow-Rashevsky, 1938)

Let Δ be a totally nonholonomic distribution on M, then every pair of points can be joined by an horizontal path.

Since the distribution is equipped with a metric, we can measure the lengths of horizontal paths and consequently we can associate a metric with the sub-Riemannian structure.

Examples of sub-Riemannian structures

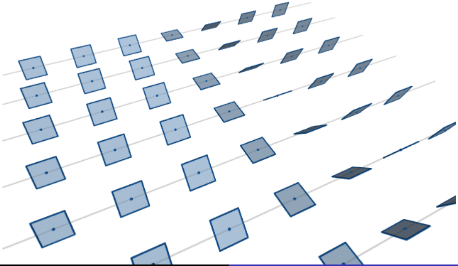
Example (Riemannian case)

Every Riemannian manifold (M, g) gives rise to a sub-Riemannian structure with $\Delta = TM$.

Example (Heisenberg)

In $\mathbb{R}^3$, $\Delta = Span\{X^1, X^2\}$ with

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z \quad et \quad g = dx^2 + dy^2.$$



Examples of sub-Riemannian structures

Example (Martinet)

In $\mathbb{R}^3$, $\Delta = Span\{X^1, X^2\}$ *with*

$$X^1 = \partial_x, \quad X^2 = \partial_y + x^2 \partial_z.$$

Since $[X^1, X^2] = 2x\partial_z$ *and* $[X^1, [X^1, X^2]] = 2\partial_z$, *only one bracket is sufficient to generate* $\mathbb{R}^3$ *if* $x \neq 0$, *however we needs two brackets if* $x = 0$.

Example (Rank 2 distribution in dimension 4)

In $\mathbb{R}^4$, $\Delta = Span\{X^1, X^2\}$ *with*

$$X^1 = \partial_x, \quad X^2 = \partial_y + x\partial_z + z\partial_w$$

satisfies $Vect\{X^1, X^2, [X^1, X^2], [[X^1, X^2], X^2]\} = \mathbb{R}^4$.

The sub-Riemannian distance

The **length** of an horizontal path γ is defined by

$$\text{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)} \, dt.$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is defined by

$$d_{SR}(x,y) := \inf\Big\{\text{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y\Big\}.$$

Proposition

The manifold M equipped with the distance d_{SR} is a metric space whose topology coincides with the one of M (as a manifold).

Sub-Riemannian geodesics

Definition

Given $x, y \in M$, we call **minimizing horizontal path** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y satisfying $d_{SR}(x,y) = \text{length}^g(\gamma)$.

The **energy** of the horizontal path $\gamma : [0,1] \to M$ is given by

$$\text{ener}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|^g_{\gamma(t)} \right)^2 dt.$$

Definition

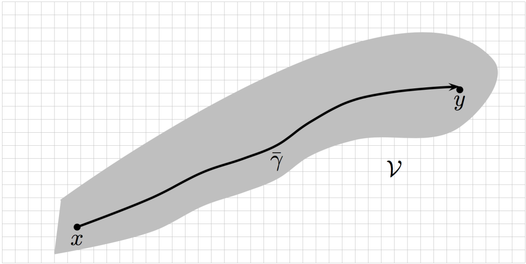
We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0,1] \to M$ joining x to y such that

$$d_{SR}(x,y)^2 = \text{ener}^g(\gamma).$$

Study of minimizing geodesics

Let $x, y \in M$ and $\bar{\gamma}$ be a **minimizing geodesic** between x and y be fixed. The SR structure admits an orthonormal parametrization along $\bar{\gamma}$, which means that there exists a neighborhood $\mathcal{V}$ of $\bar{\gamma}([0,1])$ and an orthonomal family of m vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \mathrm{Span}\left\{X^1(z), \ldots, X^m(z)\right\} \qquad \forall z \in \mathcal{V}.$$



Study of minimizing geodesics

There exists a control $\bar{u} \in L^2\big([0,1];\mathbb{R}^m\big)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\big(\bar{\gamma}(t)\big) \qquad \text{a.e. } t \in [0,1].$$

Moreover, any control $u \in \mathcal{U} \subset L^2\big([0,1];\mathbb{R}^m\big)$ (u sufficiently close to $\bar{u}$) gives rise to a trajectory γ_u solution of

$$\dot{\gamma}_u = \sum_{i=1}^{m} u^i\, X^i\big(\gamma_u\big) \quad \text{sur } [0,T], \quad \gamma_u(0) = x.$$

Furthermore, for every horizontal path $\gamma : [0,1] \to \mathcal{V}$ there exists a unique control $u \in L^2\big([0,1];\mathbb{R}^m\big)$ for which the above equation is satisfied.

Study of minimizing geodesics

Consider the **End-Point mapping**

$$E^{x,1} \; : \; L^2\big([0,1];\mathbb{R}^m\big) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and set $C(u) = \|u\|_{L^2}^2$, then $\bar{u}$ is a solution to the following **optimization problem with constraints**:

$$\bar{u} \text{ minimize } C(u) \text{ among all } u \in \mathcal{U} \text{ s.t. } E^{x,1}(u) = y.$$

(Since the family $X^1, \ldots, X^m$ is orthonormal, we have

$$\text{ener}^g(\gamma_u) = C(u) \qquad \forall u \in \mathcal{U}.)$$

Study of minimizing geodesics

Proposition (Lagrange Multipliers)

There exist $p \in T_y^*M \simeq (\mathbb{R}^n)^*$ *and* $\lambda_0 \in \{0,1\}$ *with* $(\lambda_0, p) \neq (0,0)$ *such that*

$$p \cdot d_{\bar{u}} E^{x,1} = \lambda_0 d_{\bar{u}} C.$$

As a matter of fact, the function given by

$$\Phi(u) := \big(C(u), E^{x,1}(u)\big)$$

cannot be a submersion at $\bar{u}$. Otherwise $D_{\bar{u}}\Phi$ would be surjective and so open at $\bar{u}$, which means that the image of Φ would contain some points of the form $(C(\bar{u}) - \delta, y)$ with $\delta > 0$ small.

$\rightsquigarrow$ Two cases may appear: $\lambda_0 = 1$ or $\lambda_0 = 0$.

Study of minimizing geodesics

First case : $\lambda_0 = 1$

This is the good case, the Riemannian-like case. The minimizing geodesic can be shown to be solution of a "geodesic equation". In fact, it is the projection of a **normal extremal**. It is smooth, there is a "geodesic flow"...

Second case : $\lambda_0 = 0$

In this case, we have

$$p \cdot D_{\bar{u}} E^{x,1} = 0 \text{ with } p \neq 0,$$

which means that $\bar{u}$ is **singular** as a critical point of the mapping $E^{x,1}$.

$\rightsquigarrow$ As shown by R. Montgomery, the case $\lambda_0 = 0$ cannot be ruled out.

Singular horizontal paths and Examples

Definition

An horizontal path is called **singular** if it is, through the correspondence $\gamma \leftrightarrow u$, a critical point of the End-Point mapping $E^{x,1} : L^2 \to M$.

Example 1: Riemannian case
Let $\Delta(x) = T_xM$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

Example 2: Heisenberg, fat distributions
In $\mathbb{R}^3$, Δ given by $X^1 = \partial_x, X^2 = \partial_y + x\partial_z$ does not admin nontrivial singular horizontal paths.

Examples

Example 3: Martinet-like distributions

In $\mathbb{R}^3$, let $\Delta = \text{Vect}\{X^1, X^2\}$ with X^1, X^2 of the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \left(1 + x_1\phi(x)\right)\partial_{x_2} + x_1^2\partial_{x_3},$$

where ϕ is a smooth function and let g be a metric over Δ.

Theorem (Montgomery)

There exists $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the singular horizontal path

$$\gamma(t) = (0, t, 0) \qquad \forall t \in [0, \epsilon],$$

is minimizing (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$. Moreover, if $\{X^1, X^2\}$ is orthonormal w.r.t. g and $\phi(0) \neq 0$, then γ is not the projection of a normal extremal ($\lambda_0 = 1$).

Summary

Given a sub-Riemannian structure (Δ, g) on M and a minimizing geodesic γ from x to y, two cases may happen:

- The geodesic γ is the projection of a normal extremal so it is smooth..
- The geodesic γ is a singular curve and could be non-smooth..

Questions:

When? How many? How?

Part II

II. A few open problems

A few open problems

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed.

How many?

$$\mathcal{S}^x_{\Delta,min^g} = \{\gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.}\}.$$

$$\mathcal{S}^x_{\Delta} = \{\gamma(1) | \gamma : [0,1] \to M, \gamma(0) = x, \gamma \text{ hor., sing.}\}.$$

Conjecture (Sard Conjectures)

The sets $\mathcal{S}^x_{\Delta,min^g}$ and $\mathcal{S}^x_{\Delta}$ have Lebesgue measure zero.

How?

Conjecture (Regularity Conjecture)

Minimizing geodesics are of class C^1 or smooth.

Part II

III. A few partial results

Characterization of singular curves

Let (Δ, g) be a SR structure on M and $x \in M$ be fixed. Set

$$\Delta^\perp := \Big\{ (x,p) \in T^*M \,|\, p \perp \Delta(x) \Big\} \subset T^*M$$

and (we assume here that Δ is generated by m vector fields $X^1, \ldots, X^m$) define

$$\vec{\Delta}(x,p) := \mathrm{Span} \Big\{ \vec{h}^1(x,p), \ldots, \vec{h}^m(x,p) \Big\} \quad \forall (x,p) \in T^*M,$$

where $h^i(x,p) = p \cdot X^i(x)$ and $\vec{h}^i$ is the associated Hamiltonian vector field in T^*M.

Proposition

An horizontal path $\gamma : [0,1] \to M$ is singular if and only if it is the projection of a path $\psi : [0,1] \to \Delta^\perp \setminus \{0\}$ which is horizontal w.r.t. $\vec{\Delta}$.

The case of Martinet surfaces

Let M be a smooth manifold of dimension 3 and Δ be a totally nonholonomic distribution of rank 2 on M. We define the **Martinet surface** by

$$\Sigma_\Delta = \{x \in M \,|\, \Delta(x) + [\Delta, \Delta](x) \neq T_x M\}$$

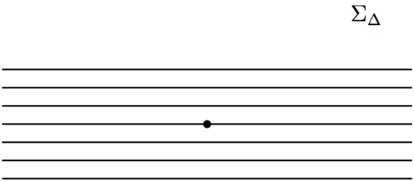
If Δ is generic, Σ_Δ is a surface in M. If Δ is analytic then Σ_Δ is analytic of dimension ≤ 2.

Proposition

The singular horizontal paths are the orbits of the trace of Δ on Σ_Δ.

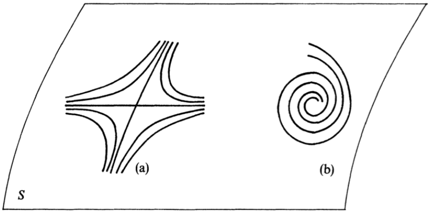
$\rightsquigarrow$ Let us fix x on Σ_Δ and see how its orbit look like.

Transverse case



The Sard Conjecture on Martinet surfaces

Generic tangent case
(Zelenko-Zhitomirskii, 1995)



The strong Sard Conjecture on Martinet surfaces

Let M be of dimension 3, Δ of rank 2 and g be fixed:

$$\mathcal{S}_{\Delta,g}^{x,L} = \{\gamma(1) | \gamma \in \mathcal{S}_{\Delta}^{x} \text{ and }, \text{length}^{g}(\gamma) \leq L\}.$$

Conjecture (Strong Sard Conjecture)

The set $\mathcal{S}_{\Delta}^{x,L}$ has finite $\mathcal{H}^1$-measure.

Theorem (Belotto-Figalli-Parusinski-R, 2018)

Assume that M and Δ are analytic and that g is smooth and complete. Then any singular horizontal curve is a semianalytic curve in M. Moreover, for every $x \in M$ and every $L \geq 0$, the set $\mathcal{S}_{\Delta,g}^{x,L}$ is a finite union of singular horizontal curves, so it is a semianalytic curve.

Proof

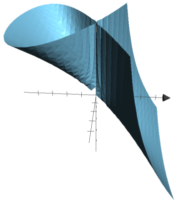
Ingredients of the proof

- Resolution of singularities.
- The vector field which generates the trace of $\tilde{\Delta}$ over $\tilde{\Sigma}$ (after resolution) has singularities of type saddle.
- A result of Speissegger (following Ilyashenko) on the regularity of Poincaré transitions mappings.

An example

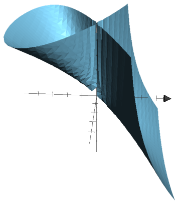
In $\mathbb{R}^3$,

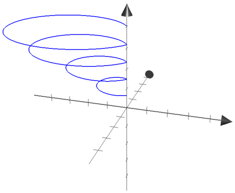
$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[\frac{y^3}{3} - x^2 y(x+z)\right] \partial_z.$$

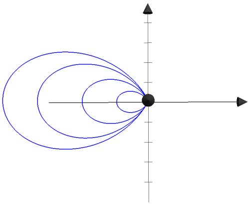


Martinet Surface: $\Sigma_\Delta = \left\{ y^2 - x^2(x+z) = 0 \right\}.$

An example







The Sard Conjecture on Martinet surfaces

As a consequence, thanks to a striking result by Hakavuori and Le Donne, we have:

Theorem (Belotto-Figalli-Parusinski-R, 2018)

Assume that M *and* Δ *are analytic and that* g *is smooth and complete and let* $\gamma : [0,1] \to M$ *be a singular minimizing geodesic. Then* γ *is of class* C^1 *on* $[0,1]$*. Furthermore,* $\gamma([0,1])$ *is semianalytic, and therefore it consists of finitely many points and finitely many analytic arcs.*

Thank you for your attention !!