On Riemannian manifolds satisfying the Transport Continuity Property

Ludovic Rifford

Université de Nice - Sophia Antipolis

(Joint work with A. Figalli and C. Villani)

I. Statement of the problem

Optimal transport on Riemannian manifolds

Let (M, g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$.
Denote by d_g the geodesic distance on M and define the quadratic cost $c : M \times M \longrightarrow [0, \infty)$ by

$$c(x, y) := \frac{1}{2} d_g(x, y)^2 \qquad \forall x, y \in M.$$

Given two Borelian probability measures μ_0, μ_1 on M, find a mesurable map $T : M \longrightarrow M$ satisfying

$$T_\sharp \mu_0 = \mu_1 \quad \text{(i.e. } \mu_1(B) = \mu_0\left(T^{-1}(B)\right), \forall B \text{ borelian } \subset M),$$

and minimizing

$$\int_M c(x, T(x)) d\mu_0(x).$$

The Brenier-McCann Theorem

Theorem (McCann '01)

Let μ_0, μ_1 be two probability measures on M. If μ_0 is absolutely continuous w.r.t. the Lebesgue measure, then there is a unique optimal transport map $T : M \to M$ satisfying $T_\sharp \mu_0 = \mu_1$ and minimizing

$$\int_M c(x, T(x)) d\mu_0(x).$$

It is characterized by the existence of a semiconvex function $\psi : M \to \mathbb{R}$ such that

$$T(x) = \exp_x (\nabla \psi(x)) \qquad \text{for } \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

The Transport Continuity Property

We say that (M, g) satisfies the **Transport Continuity Property (TCP)** if the following property is satisfied:
For any pair of probability measures μ_0, μ_1 associated with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \mathrm{vol}_g, \quad \mu_1 = \rho_1 \mathrm{vol}_g,$$

the optimal transport map between μ_0 and μ_1 is **continuous**.

II. Necessary conditions for **TCP**

Necessary conditions

Theorem (Villani '09, Figalli-R-Villani)

Assume that (M, g) *satisfies* **(TCP)** *then the following properties hold:*

- *all the injectivity domains are convex,*
- *the cost* c *is regular.*

Injectivity domains

Let $x \in M$ be fixed. We call exponential mapping from x, the mapping defined as

$$\begin{array}{rcl} \exp_x : T_xM & \longrightarrow & M \\ v & \longmapsto & \exp_x(v) := \gamma_v(1), \end{array}$$

where $\gamma_v : [0,1] \to M$ is the unique geodesic starting at x with speed $\dot{\gamma}_v(0) = v$. We call **injectivity domain** of x the set

$$\mathcal{I}(x) := \left\{ v \in T_xM \,\middle|\, \begin{array}{c} \gamma_v \text{ is the unique minimizing geodesic} \\ \text{between } x \text{ and } \exp_x(v) \end{array} \right\}.$$

It is a star-shaped (w.r.t. $0 \in T_xM$) domain with Lipschitz boundary.

Regular costs

The cost $c = d^2/2 : M \times M \longrightarrow \mathbb{R}$ is called **regular**, if for every $x \in M$ and every $v_0, v_1 \in \mathcal{I}(x)$, there holds

$$v_t := (1-t)v_0 + tv_1 \in \mathcal{I}(x) \qquad \forall t \in [0,1],$$

and

$$c\left(x', y_t\right) - c(x, y_t) \geq \min\Big(c\left(x', y_0\right) - c\left(x, y_0\right), \, c\left(x', y_1\right) - c\left(x, y_1\right)\Big),$$

for any $x' \in M$, where $y_t := \exp_x v_t$.

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0} c\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

Proposition (Villani '09, Figalli-R-Villani)

Assume that all the injectivity domains are convex. Then, the two following properties are equivalent:

- *the cost c is regular,*
- *the* **MTW** *tensor $\mathfrak{S}$ is ≥ 0, that is, for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \Longrightarrow \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

Necessary conditions

Theorem (Villani '09, Figalli-R-Villani)

Assume that (M, g) *satisfies* **(TCP)** *then the following properties hold:*

- *all the injectivity domains are convex,*
- *the cost* c *is regular,*
- *the* **MTW** *tensor* $\mathfrak{S}$ *is* ≥ 0.

Loeper noticed that for every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_xM$, there holds

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = \sigma_x(P),$$

where P is the plane generated by ξ and η. Consequently, any (M, g) satisfying **TCP** must have nonnegative sectional curvatures.

III. Sufficient conditions for **TCP**

Sufficient conditions

Theorem (Figalli-R-Villani)

Assume that (M, g) satisfies the following properties:

- *all the injectivity domains are strictly convex,*
- *the* **MTW** *tensor $\mathfrak{S}$ is > 0, that is, for every $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) > 0.$$

Then (M, g) satisfies **TCP**.

IV. Examples

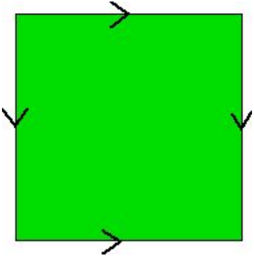
The flat torus

The **MTW** tensor of the flat torus $\left(\mathbb{T}^n, g^0\right)$ satisfies

$$\mathfrak{S}_{(x,v)} \equiv 0 \qquad \forall x \in \mathbb{T}^n, \forall v \in \mathcal{I}(x)$$

Theorem (Cordero-Erausquin '99)

The flat torus $\left(\mathbb{T}^n, g^0\right)$ *satisfies* **TCP**.



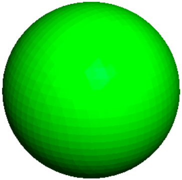
Round spheres

Loeper checked that the **MTW** tensor of the round sphere $(\mathbb{S}^n, g^0)$ satisfies for any $x \in \mathbb{S}^n$, $v \in \mathcal{I}(x)$ and $\xi, \eta \in T_x\mathbb{S}^n$,

$$\langle \xi, \eta \rangle_x = 0 \Longrightarrow \mathfrak{S}_{(x,v)}(\xi, \eta) \geq \|\xi\|_x^2 \|\eta\|_x^2.$$

Theorem (Loeper '06)

The round sphere $(\mathbb{S}^n, g^0)$ *satisfies* **TCP**.



Riemannian quotients

Let G be a discrete group of isometries of (M, g) acting freely and properly. Then there exists on the quotient manifold $N = M/G$ a unique Riemannian metric h such that the canonical projection $p : M \longrightarrow N$ is a Riemannian covering map.

Theorem (Delanoe-Ge '08)

If (M, g) satisfies **TCP**, *then $\left(N = M/G, h\right)$ satisfies* **TCP**.

Examples: $\left(\mathbb{RP}^n, g^0\right)$, the flat Klein bottle.

Riemannian submersions

We call **Riemannian submersion** from (M, g) to (N, h) any smooth submersion $p : M \longrightarrow N$ such that for every $x \in M$, the differential mapping $d_x p$ is an isometry between H_x and $T_{p(x)}N$, where $H_x \subset T_x M$ is the **horizontal subspace** defined as

$$H_x := \left\{ \left(d_x p\right)^{-1}(0) \right\}^{\perp}.$$

Theorem (Kim-McCann '08)

If (M, g) satisfies **MTW** > 0 *(resp. ≥ 0), then (N, h) satisfies* **MTW** > 0 *(resp. ≥ 0).*

Examples: complex projective spaces $\left(\mathbb{CP}^k, g^0\right)$ (dim $= 2k$), quaternionic projective spaces $\left(\mathbb{HP}^k, g^0\right)$ (dim $= 4k$).

Small deformations of $(\mathbb{S}^2, g^0)$

On $(\mathbb{S}^2, g^0)$, the **MTW** tensor is given by

$$
\begin{aligned}
&\mathfrak{S}_{(x,v)}(\xi, \xi^{\perp}) \\
&\quad = 3\left[\frac{1}{r^2} - \frac{\cos(r)}{r\sin(r)}\right]\xi_1^4 + 3\left[\frac{1}{\sin^2(r)} - \frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4 \\
&\qquad + \frac{3}{2}\left[-\frac{6}{r^2} + \frac{\cos(r)}{r\sin(r)} + \frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,
\end{aligned}
$$

with $x \in \mathbb{S}^2, v \in \mathcal{I}(x), r := \|v\|_x, \xi = (\xi_1, \xi_2), \xi^{\perp} = (-\xi_2, \xi_1)$.

Theorem (Figalli-R '09)

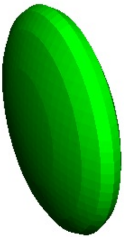
Any small deformation of the round sphere $(\mathbb{S}^2, g^0)$ in C^4 topology satisfies **TCP**.

Ellipsoids

The ellipsoid of revolution (E_ϵ) in $\mathbb{R}^3$ given by the equation

$$\frac{x^2}{\epsilon^2} + y^2 + z^2 = 1, \qquad \text{with } \epsilon = 0.29,$$

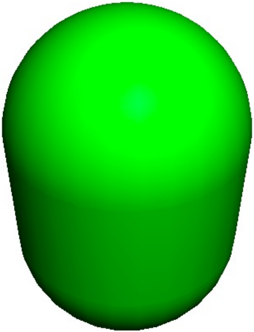
does not satisfies **MTW** ≥ 0.



In consequence, (E_ϵ) does not satisfy **TCP**.

Jump of curvature

The surface made with two half-balls joined by a cylinder has not a regular cost.



Then, it does not satisfy **TCP**.

Small deformations of $\left(\mathbb{S}^n, g^0\right)$

Theorem (Figalli-R-Villani'09)

Any small deformation of the round sphere $\left(\mathbb{S}^n, g^0\right)$ in C^4 topology satisfies **TCP**.

As a by-product, we obtain that the injectivity domains on small C^4 deformations of $\left(\mathbb{S}^n, g^0\right)$ are convex.

V. Conclusion and perspectives

Conclusion and perspectives

Theorem (Necessary conditions)

Assume that (M, g) *satisfies* **(TCP)** *then the following properties hold:*

- *all the injectivity domains are convex,*
- *the* **MTW** *tensor* $\mathfrak{S}$ *is* ≥ 0.

Theorem (Sufficient conditions)

Assume that (M, g) *satisfies the following properties:*

- *all the injectivity domains are strictly convex,*
- *the* **MTW** *tensor* $\mathfrak{S}$ *is* > 0.

Then (M, g) *satisfies* **TCP**.

There is a gap !

Thank you for your attention !