Chapter 1
Stabilized spectral element approximation of the Saint Venant system using the entropy viscosity technique

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Abstract We consider the Saint Venant system (shallow water equations), i.e. an approximation of the incompressible Euler equations widely used to describe river flows, flooding phenomena or erosion problems. We focus on problems involving dry-wet transitions and propose a solution technique using the Spectral Element Method (SEM) stabilized with a variant of the Entropy Viscosity Method (EVM) that is adapted to treat dry zones.

1.1 Introduction

Because high-order methods are known to produce spurious oscillations in shocks, solving non-linear hyperbolic systems of conservation equations with high accuracy is a challenging task. Assuming that an entropy does exist for the considered physical problem, the Entropy Viscosity Method (EVM) offers an elegant way to stabilize various numerical discretizations, including the standard Finite Element Method or Spectral Element Method (SEM) and even Fourier expansions [3]. The basic idea consists of introducing in the governing equations a nonlinear viscous term based on the residual of the Partial Differential Equation (PDE) that governs the evolution of the entropy and to bound from above this term by a first order viscosity.

We consider in the present paper the Saint Venant system, i.e. a simplified form of the incompressible Euler equations well adapted to describe free surface flows like rivers or flooding phenomena. We especially focus on problems involving dry-wet transitions, e.g. the classical dram break problem.

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This class of problems is generally addressed in the finite volume literature by using Godunov-type methods, i.e. Riemann solvers together with flux or slope limiters, see e.g. [6] for a review. We introduce a new ingredient in the EVM that enables the method to handle the dry-wet transition problem satisfactorily. The numerical discretization is based on the SEM in space and on a standard forth order Runge-Kutta (RK4) scheme in time. Although all the numerical simulations shown in the paper are one-dimensional, the method is a priori multi-dimensional. Finally, the proposed approach can be used to treat problems in gas dynamics with vacuum.

The paper is organized as follows. We introduce the Saint Venant system and recall its basic properties in Section 2 . The SEM approximation and the EVM stabilization are described and discussed in Section 3. Some examples of applications, all of them involving dry-wet transitions, are presented in Section 4.

1.2 The Saint Venant system

The Saint-Venant system (shallow water equations) is an approximation of the incompressible Euler equations assuming that the pressure is hydrostatic and the free surface perturbations are small compared to the water height. The one-dimensional version of this system is

$$\partial_t h + \partial_x (hu) = 0 \tag{1.1}$$

$$\partial_t (hu) + \partial_x (hu^2 + gh^2/2) + gh\partial_x z = 0, \tag{1.2}$$

where $h(x,t)$ is the water height, $u(x,t)$ the horizontal velocity, g the gravity acceleration, $z(x)$ the topography, for which it is assumed that $\partial_x z \ll 1$. The independent variables are time $t \in (0, t_F)$ and space $x \in D = (x_{\text{inf}}, x_{\text{sup}})$. These PDEs are obtained by integrating the mass and momentum conservation equations in the Euler system over the vertical direction. This nonlinear two equations system has the following properties:

- The system is hyperbolic, which means that discontinuities may develop;
- Assuming that the inlet flow-rate equals the outlet flow-rate, the total mass is preserved: $d_t \int_D h\, dx = 0$;
- The height h is nonnegative: $\forall x, t,\ h(x,t) \geq 0$;
- Rest solutions are stable: $u = 0, \quad h(x,t) + z(x) = \textit{constant}$;
- There exists a convex entropy (actually the energy E) such that:

$$\partial_t E + \partial_x ((E + gh^2/2)u) \leq 0, \quad E = hu^2/2 + gh^2/2 + ghz. \tag{1.3}$$

1.3 Stabilized SEM approximation

The EVM-stabilization is obtained via the introduction of nonlinear viscous terms in the governing equations. The *entropy viscosity* is computed from the residual of the entropy inequality and bounded from above by a first order viscosity. In case of a scalar conservation law, with δx for the grid size, we generally set, see [3] for details:

$$\nu = \mathcal{S}(\min(\nu_{max}, \nu_E)) \quad \text{where} \tag{1.4}$$

$$\nu_{max} = \alpha \max_{loc} |\boldsymbol{f}'(u)| \delta x \tag{1.5}$$

$$\nu_E = \beta \delta x^2 |r_E| / \Delta E \tag{1.6}$$

where r_E is the residual of the entropy inequality; $\boldsymbol{f}(u)$, $\boldsymbol{f}'(u)$ are the flux and derivative of the flux; α and β are user defined parameters; ΔE is a scaling parameter equal to the amplitude of variations of the entropy. The local maximum is generally based on the computational cell. $\mathcal{S}$ is a smoothing operator required by the fact that at the discrete level the residual r_E is oscillatory. For hyperbolic systems $\boldsymbol{f}'(u)$ is the Jacobian matrix of $\boldsymbol{f}$, and $|\boldsymbol{f}'|$ is defined to be the absolute value of the largest eigenvalue of $\boldsymbol{f}'(u)$.

Discretization of the Saint Venant system: Set $q = hu$ and, for any t, let $h_N(x,t)$ (resp. $q_N(x,t)$) to be the continuous piecewise polynomial approximation of degree N of $h(x,t)$ (resp. $q(x,t)$) built on a discretization of $D = (x_{\inf}, x_{\sup})$; i.e. we use the standard SEM for the space approximation, see e.g. [5] . Then we propose the following EVM-stabilized weak formulation of the Saint Venant system:

$$\int_D (\partial_t h_N + \partial_x q_N) v_N = -\int_D \nu \partial_x h_N \, \partial_x v_N \tag{1.7}$$

$$\int_D (\partial_t q_N + \partial_x (q_N^2/h_N + g h_N^2/2) + g h_N z_x) w_N = -\int_D \nu \partial_x q_N \, \partial_x w_N, \tag{1.8}$$

where v_N, w_N are test functions spanning the approximation space and ν is the entropy viscosity, still to be defined. As usual, the viscous (stabilization) terms have been integrated by parts. Note that (i) viscous stabilization is added to the mass equation and that (ii) the stabilization is done on q instead of u in the momentum equation. This differs from the physically and mathematically well justified viscous form of the Saint-Venant system, which makes only use of $\partial_x(h\nu\partial_x u)$ in the momentum equation [2]. In [4], where the Euler system is addressed, it is however outlined that the physical stabilization may not be the best suited one for numerical purposes.

Time is approximated using an explicit RK4 scheme.

Entropy viscosity for the Saint-Venant system: First we define the viscosity ν_E associated to the residual of the entropy equation. Using the expression (1.3) leads to a viscosity ν_E that depends on z, i.e. on the choice of the coordinate system. To avoid this arbitrariness, we take into account the mass conservation equation in (1.3) to derive an expression that only depends on $\partial_x z$ and governs the evolution of an entropy $\tilde{E}$ which satisfies:

$$\partial_t \tilde{E} + \partial_x((\tilde{E} + gh^2/2)u) + ghu\partial_x z \leq 0, \quad \tilde{E} = hu^2/2 + gh^2/2 \tag{1.9}$$

The evaluation of the entropy viscosity is done at each time step before entering the RK explicit time scheme. This is done at time t_n by using a Backward Difference Formula (e.g. BDF2) for the approximation of $\partial_t \tilde{E}_N$; more precisely, denoting by $\Delta\tilde{E}_N/\Delta t$ the approximation of $\partial_t \tilde{E}_N$, we compute

$$r_E = \Delta\tilde{E}_N/\Delta t + \partial_x((\tilde{E}_N + gh_N^2/2)q_N/h_N) + gq_N\partial_x z \tag{1.10}$$

with $\tilde{E}_N = q_N^2/(2h_N) + gh_N^2/2$, and we set

$$\nu_E = \beta |r_E|/\Delta E_N \delta x^2\,, \quad \Delta E_N = \max_D E_N - \min_D E_N \tag{1.11}$$

where the grid size δx is that of the Gauss-Lobatto-Legendre (GLL) mesh.

The first order viscosity $\nu_{\max}$ for the Saint Venant system must be based on a wave speed that should be larger than $\lambda_\pm = u \pm \sqrt{gh}$. We set

$$\nu_{\max} = \alpha \max_D(|q_N/h_N| + \sqrt{gh_N})\delta x \tag{1.12}$$

where again δx is the GLL grid-size.

The viscosity is then defined by $\nu = \min(\nu_{\max}, \nu_E)$. This viscosity is additionally smoothed by using a two-step procedure:

- first locally (in each element), e.g. $(\nu_{i-1} + 2\nu_i + \nu_{i+1})/4 \to \nu_i$
- then globally, by projection onto the space of the C^0 piecewise polynomial of degree N. Note that this is easy to do, since the SEM mass matrix is diagonal.

We now finally recall how to adjust the values of the EVM control parameters:

- First, one solves the problem with the viscosity $\nu_{\max}$ and adjust α to obtain a smooth solution.
- Second, one solves with the entropy viscosity ν and adjust β.

Properties of the approximation: The following properties are expected from the SEM/EVM approximation:

- Mass conservation: Setting $v_N = 1$ in the equation for h_N yields

$$\int_D (\partial_t h_N + \partial_x q_N)\, dx = \int_D \partial_t h_N\, dx + 0 = d_t \int_D h_N\, dx = 0 \tag{1.13}$$

if $q_N(x_{\text{sup}}) - q_N(x_{\text{inf}}) = 0$, which means that the total mass is preserved.

- Conservation of energy for smooth solutions. There is no guaranty here, because the equation for the energy involves non-linear terms that are approximated by quadratures.
- Positivity of h. Here again, one may expect difficulties as soon as $N > 1$, i.e. when the space approximation is not simply piecewise linear. For problems in which we are interested in, i.e. involving dry-wet transitions, numerical difficulties systematically occur when using the standard form of the EVM. To overcome these difficulties, we suggest to use the first order viscosity as soon as the fluid height becomes small. We thus supplement the EVM with the following step:

$$\nu = \nu_{\max} \quad \text{if} \quad h_N < h_{\text{thres}} \tag{1.14}$$

where the threshold height h_{thres} is small, i.e. typically 10^{-3} of the mean fluid height. Moreover, we have not based $\nu_{\max}$ on a local but on a global maximum of the wave speed, see eq. (1.12).

1.4 Examples of applications

The following test-cases have been considered: (i) Lake at rest with an emerged bump. The surface water should remain flat. This is what one usually expects of a *well balanced scheme*. (ii) Oscillations in a parabolic cup. The solution to this problem being smooth, the energy should remain constant over time. (iii) Dam break on a dry domain. The main problem here is to get the right velocity at the front of the water wave. (iv) Dam break on a sinusoidal topography. This problem combines different aspects previously mentioned. It should be remarked that all these test-cases have dry-wet transitions. The first three test cases have analytical solutions, see e.g. [1].

Lake at rest with an emerged bump: In this test the free surface should remain flat and the velocity must be zero at all times. Fig. 1.1 shows the EVM solution as well as the entropy viscosity. As desired, the viscosity is maximal in the dry part of the bump. The result is satisfactory, even if one observes (on an animation) some traveling waves with very low amplitude.

Oscillations in a cup: The topography is a parabolic bowl. The fluid level, $h + z$, at the initial time is defined by an inclined line. Since the solution to the problem is smooth there is no dissipation and the fluid oscillates indefinitely. Fig. 1.2 compares the exact solution with the computed one at the final time, $t_F = 50$. The entropy viscosity is also shown.

It is interesting for this problem to verify how well the energy is conserved. Fig. 1.3 shows the time evolution of the total energy for both the EVM and

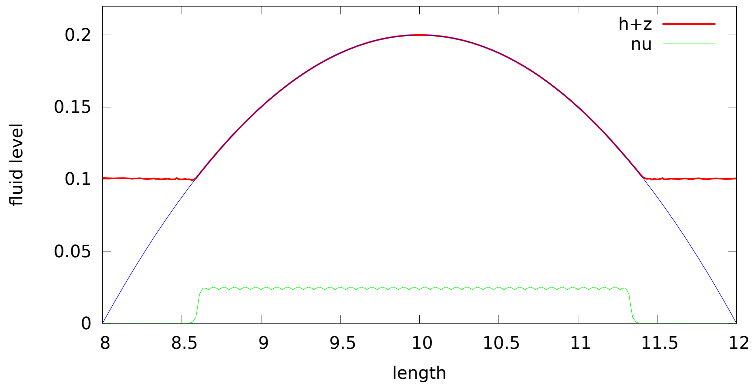

Fig. 1.1 Bump problem: $D = (8, 12)$, $t_F = 400$, 60 elements, $N = 4$, $\alpha = 1$, $\beta = 10$, $h_{\mathrm{thres}} = 10^{-4}$. EVM solution and entropy viscosity at time $t_F = 400$.

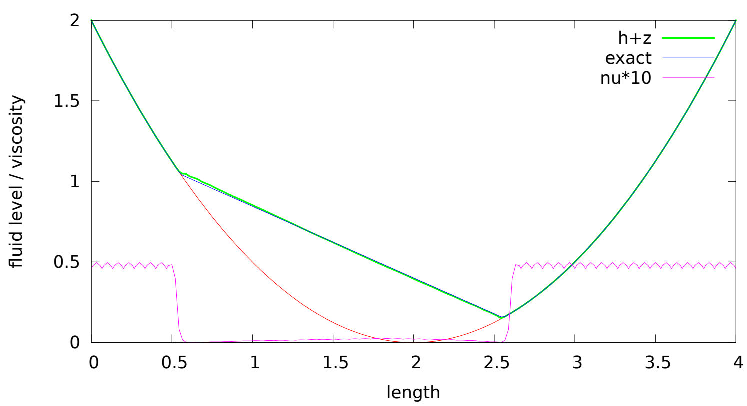

Fig. 1.2 Cup problem: $D = (0, 4)$, $t_F = 50$, 60 elements, $N = 4$, $\alpha = 1$, $\beta = 10$, $h_{\mathrm{thres}} = 10^{-3}$. EVM and exact solutions, entropy viscosity at time $t_F = 50$.

the first order viscosity solutions. One observes some oscillations, especially for the first order viscosity solution, and there is a slight increase in energy for the EVM solution. The result is however satisfactory since the oscillatory motion is well maintained, i.e. there is no significant artificial dissipation of the energy.

Dam break: The dam break on dry domain is a classical test-case. It is especially of interest to verify whether the velocity of the leading wave is correct. Fig. 1.4 shows that the results from the EVM are satisfactory, even

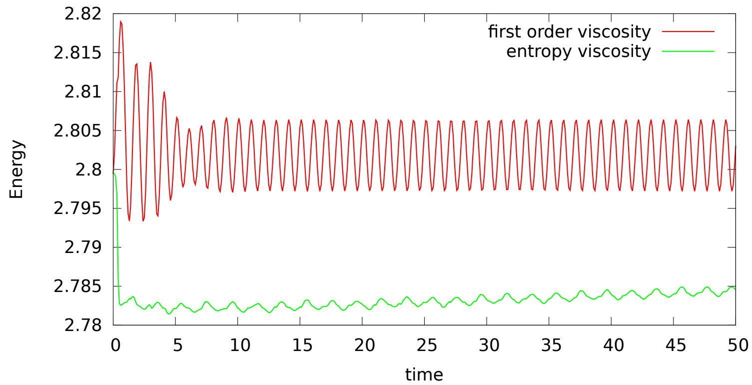

Fig. 1.3 Cup problem: Time-variations of the total energy for the solutions obtained with the entropy viscosity and with the first order viscosity

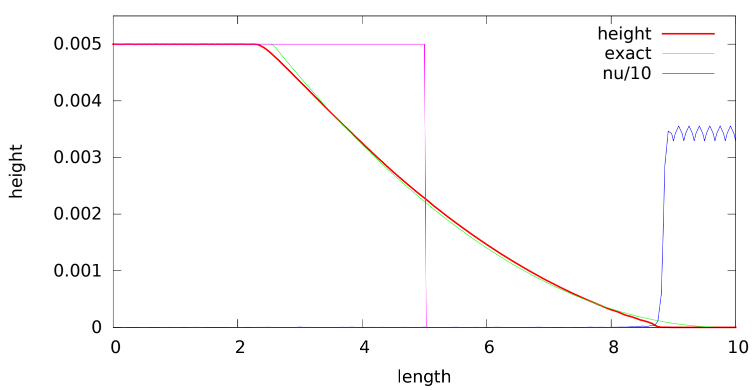

Fig. 1.4 Dam break problem: $D = (0, 10)$, $t_F = 120$, 60 elements, $N = 4$, $\alpha = 2$, $\beta = 20$, $h_{\text{thres}} = 10^{-6}$. EVM and exact solutions, entropy viscosity at $t_F = 11$. The initial condition is also shown.

if some slight differences can be observed at the upper left and bottom right parts of the expansion wave.

Dam break over bumps: We now solve the dam break problem on a dry domain with a sinusoidal topography. Fig. 1.5 shows a snapshot of the solution. At the end of the computation one recovers the situation met previously for the cup problem, i.e. the fluid oscillates between the two bumps and remains trapped therein indefinitely.

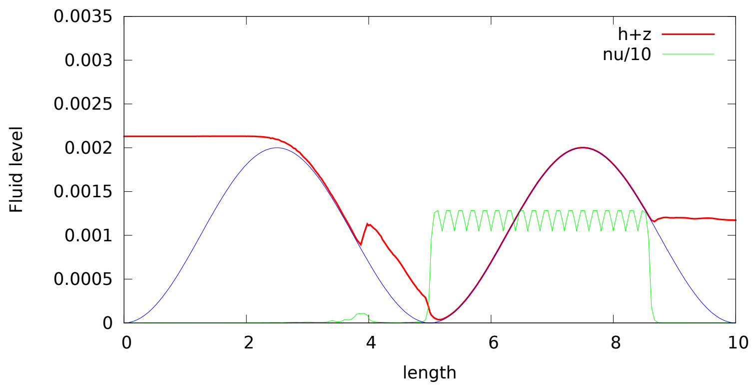

Fig. 1.5 Dam/bump problem: $D = (0, 10)$, $t_F = 600$, 60 elements, $N = 4$, $\alpha = 1.5$, $\beta = 30$, $h_{\text{thres}} = 10^{-5}$. EVM solution and entropy viscosity at $t = 129$. Initial condition: $h + z = 0.003$ if $x < 2$, $h = 0$ if $x > 2$.

Acknowledgements This work is partly supported by the National Science Foundation grants DMS-1015984 and DMS-1217262 and by the Air Force Office of Scientific Research, USAF, under grant/contract number FA99550-12-0358. It is also supported by grants of the french research agency (ANR project MEDIMAX).

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